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Traveling waves for a three-component reaction–diffusion model of farmers and hunter-gatherers in the Neolithic transition

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Abstract
The Neolithic transition began the spread of early agriculture throughout Europe through interactions between farmers and hunter-gatherers about 10,000 years ago. Archeological evidences indicate that the expanding velocity of farming into a region occupied by hunter-gatherers is roughly constant all over Europe. In the late twentieth century, from the contribution of the radiocarbon dating, it could be found that there are two types of farmers: one is the original farmer and the other is the converted farmer which is genetically hunter-gatherers but learned agriculture from neighbouring farmers. Then this raises the following questions: Which farming populations play a key role in the expansion of farmer populations in Europe? and what is the fate of hunter-gatherers (e.g., become extinct, or live in lower density, or live in agricultural life-style)? We consider a three-component reaction–diffusion system proposed by Aoki, Shida and Shigesada, which describes the interactions among the original farmers, the converted farmers, and the hunter-gatherers. In order to resolve these two questions, we discuss traveling wave solutions which give the information of the expanding velocity of farmer populations. The main result is that two types of traveling wave solutions exist, depending on the growth rate of the original farmer population and the conversion rate of the hunter-gatherer population to the converted farmer population. The profiles of traveling wave solutions indicate that the expansion of farmer populations is determined by the growth rate of the original farmer and the (maximal) carrying capacity of the converted farmer, and the fate of hunter-gatherers is determined by the growth rate of the hunter-gatherer and the conversion rate of the hunter-gatherer to the converted farmer. Thus, our results provide a partial answer to the above two questions.

Keywords Neolithic transition in Europe · Farmers and hunter-gatherer model. Population biology · Three-component system · Traveling wave · Minimal velocity

Mathematics Subject Classification 34A34 · 34A12 · 35K57

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1 Introduction

The Neolithic transition began the spread of early agriculture throughout Europe through interactions between farmers and hunter-gatherers about 10,000 years ago. Archeological evidences indicate that the expanding velocity of farming into a region occupied by hunter-gatherers is roughly constant all over Europe. In order to understand theoretically this expanding velocity, Ammerman and Cavalli-Sforza proposed a two-component reaction–diffusion system for farmers and hunter-gatherers (Ammerman and Cavalli-Sforza 1971, 1973, 1984)

$$\begin{cases} F_t = d_f F_{xx} + r_f\big(1 - F/K_f\big)F + eFH, \\ H_t = d_h H_{xx} + r_h\big(1 - H/K_h\big)H - eFH, \end{cases} \tag{1.1}$$

where the subscripts x and t denote partial differentiation with respect to x and t, respectively. Here $F(x,t)$ and $H(x,t)$ represent the densities of farmers and hunter-gatherers at position x and time t, respectively; d_f and d_h are the diffusion constants of farmers and hunter-gatherers, respectively; r_f and r_h are the intrinsic growth rates of farmers and hunter-gatherers, respectively; K_f and K_h are the carrying capacities of farmers and hunter-gatherers, respectively; and e is the conversion rate from hunter-gatherers to farmers. Consider the initial value problem for (1.1) in the whole interval $\mathbb{R}$ where the initial function $F(\cdot, 0)$ is non-negative and compactly supported and $H(\cdot, 0) = K_h$ on $\mathbb{R}$. Such an initial condition corresponds to the expansion of farmer into the region inhabited by hunter-gatherers at the level of the carrying capacity. Then numerical results (Kabir et al. 2018) indicate that the farmer F asymptotically expands into the region of $H(x,t)$ with constant velocity, and that the hunter-gatherer H goes extinct if $eK_f > r_h$, while lives in lower density if $eK_f < r_h$.

In the late twentieth century, the radiocarbon dating research (Mellars 1996) suggested that there were two types of farmers: one is the original farmer, and the other is the converted one which is genetically the same as hunter-gatherers but learned agriculture from neighbouring farmers directly and indirectly. It is already reported that farmers expanded into the region of hunter-gatherers. Then this raises the following two questions:

(Q1) Which farming population plays a key role of expanding farmer populations?
(Q2) What is the fate of the hunter-gatherers?

Particularly, the question (Q1) cannot be answered from the Ammerman–Cavalli-Sforza model (1.1). In order to answer these two questions, we consider a three-component reaction–diffusion system for two farming populations and a hunting-gathering one which was proposed by Aoki et al. (1996). The system, in the one-dimensional habitat, is described by

$$\begin{cases} F_t = d_f F_{xx} + r_f F\left(1 - (F + C)/K_f\right), \\ C_t = d_f C_{xx} + r_c C\left(1 - (F + C)/K_c\right) + e(F + C)H, \\ H_t = d_h H_{xx} + r_h H\left(1 - H/K_h\right) - e(F + C)H, \end{cases} \tag{1.2}$$

where $F(x,t)$, $C(x,t)$, and $H(x,t)$ represent the densities of original farmers, converted farmers, and hunter-gatherers at position x and time t, respectively; d_f and d_h

are the diffusion constants of farmers and hunter-gatherers, respectively; r_f, r_c, and r_h are the intrinsic growth rates of original farmers, converted farmers, and hunter-gatherers, respectively; $K_f = K_c = K$ and K_h are the carrying capacities of farmers and hunter-gatherers, respectively; and e is the conversion rate of hunter-gatherers to farmers. All parameters are assumed to be positive. By introducing the dimensionless quantities

$$\bar{t} = r_c t, \quad \bar{x} = \sqrt{\frac{r_c}{d_f}}x,$$
$$\bar{F}(x,t) = F(x,t)/K, \quad \bar{C}(x,t) = C(x,t)/K, \quad \bar{H}(x,t) = H(x,t)/K_h,$$
$$d = d_h/d_f, \quad a = r_f/r_c, \quad b = r_h/r_c, \quad s = eK_h/r_c, \quad g = eK/r_c,$$

and omitting the bars for notational simplicity, system (1.2) is transferred to the following

$$\begin{cases} F_t = F_{xx} + aF(1 - F - C), \\ C_t = C_{xx} + C(1 - F - C) + s(F + C)H, \\ H_t = dH_{xx} + bH(1 - H) - g(F + C)H. \end{cases} \tag{1.3}$$

As we will see, the dynamics of (1.3) strongly depends on the ratios $a/(1+s)$ and g/b. We remark that $d > 1$ (or $d_h > d_f$) should be assumed from the archeological point of view. But most of our analysis does not need this archeological constraint.

Now we address the questions (Q1) and (Q2). These questions are related to the asymptotical behavior of solutions of model (1.3). To do this, we consider the initial and boundary value problem for (1.3) in a finite interval $[0, L]$. Motivated by the requirement of farmers and hunter-gatherers at the early stage of the Neolithic transition, we impose the initial conditions as

$$(F, C, H)(x, 0) = (F_0, C_0, H_0)(x), \ 0 \le x \le L, \tag{1.4}$$

where F_0 is non-negative and compactly supported, the converted farmer C_0 is totally zero in the whole habitat, and the hunter-gather H_0 is uniform at the level of the carrying capacity of hunter-gathers, that is $H_0 \equiv 1$. We also impose the boundary conditions as the Neumann boundary conditions, i.e.

$$(F_x, C_x, H_x)(x, t) = 0, \ t > 0, \ x = 0 \text{ and } L. \tag{1.5}$$

Then the recent results by Eliaš et al. (2021) have answered to (Q2), that is, if $0 < g < b$, the solution $(F, C, H)(x, t)$ of (1.3)–(1.5) converges to $(0, c^*, h^*)$ as $t \to \infty$, where c^* and h^* are constants satisfying $1 - c^* + sh^* = 0$ and $b(1 - h^*) - gc^* = 0$ which is equivalent to $c^* = b(1+s)/(b+sg)$ and $h^* = (b-g)/(b+sg)$, and that if $g \ge b$, $(F, C, H)(x, t)$ converges to $(\lambda^*, 1 - \lambda^*, 0)$ as $t \to \infty$ for some constant λ^* satisfying $0 \le \lambda^* \le 1$. Archeologically speaking, when the conversion rate of hunter-gatherers to farmers is less than the growth rates of hunter-gatherers ($g < b$), the original farmers fade out, while the converted farmers exist and the hunter-gatherers do not fade out but exist in lower density. On the other hand, for the opposite case

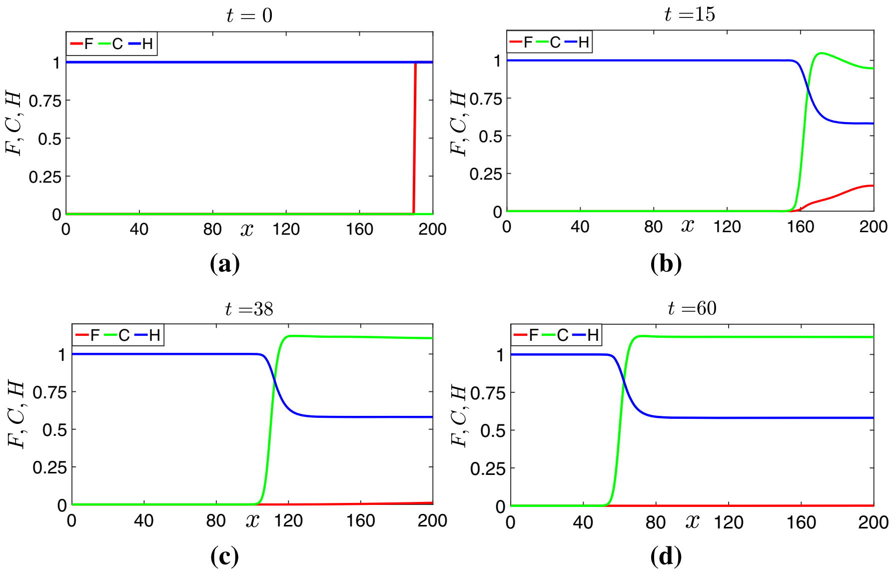

Fig. 1 Time-evolution of the solution (F, C, H) of system (1.3)–(1.5) with $L = 200$. The initial data is that $F_0(x) = 0(0 \leq x \leq 190)$ and $1(190 < x \leq 200)$, $C_0(x) = 0(0 \leq x \leq 200)$, and $H_0(x) = 1(0 \leq x \leq 200)$. Here the parameters are $d = 1.2$, $a = 1.0$, $s = 0.2$, $b = 0.8$, and $g = 0.3$

$(g \geq b)$, the hunter-gatherers fade out, while the total farmers $(F + C)(x, t)$ exist and tend to the carrying capacity 1 as $t \to \infty$.

Next we address the question (Q1). This question is related to the transient behavior of solutions of the model (1.3). For this, we first numerically consider the transient behavior of the solution (F, C, H) of (1.3)–(1.5). To begin with, note that the recent results by Mori and Xiao (2019) indicate that the spreading velocity of the solution (F, C, H) of (1.3)–(1.5) is given by $\max\{2\sqrt{a}, 2\sqrt{1+s}\}$. This implies that $a = 1+s$ is the critical value of spreading velocities. With this in mind, we first consider the case where $g < b$ for which the original farmer F tends to zero. Indeed, when $g = 0.3$ and $b = 0.8$, $a = 1.0$ and $s = 0.2$ $(a < 1 + s)$, and the initial conditions are specified as in Fig. 1a, b–d exhibit that the F-component fades out and the (C, H) propagates as if it were a traveling wave with constant speed and fixed profiles. For the same set of parameters except for $a = 1.5$ $(a > 1 + s)$, Fig. 2b–d exhibit that the propagating behavior for (F, C, H) appears as if it were a traveling wave solution (F, C, H) where the F-component possesses a pulse-like profile. These two cases suggest the existence of two different types of traveling waves. Second, consider the case where $g > b$ for which the hunter-gathers H tends to zero. When $g = 1.0$ and $b = 0.8$, and $a = 1.0$ and $s = 0.2$ $(a < 1 + s)$, Fig. 3b–d exhibit that the propagating behavior of (F, C, H) appears as if it were a traveling wave solution (F, C, H) with the F-component being identically zero. Although F is left at the right-hand side in Fig. 3c, d, it slowly fades out after large time. Finally, for the same set of parameters except for $a = 1.5$ $(a > 1 + s)$, Fig. 4b–d exhibit that the propagating behavior of (F, C, H) appears as if it were a traveling wave. The numerical results motivate us

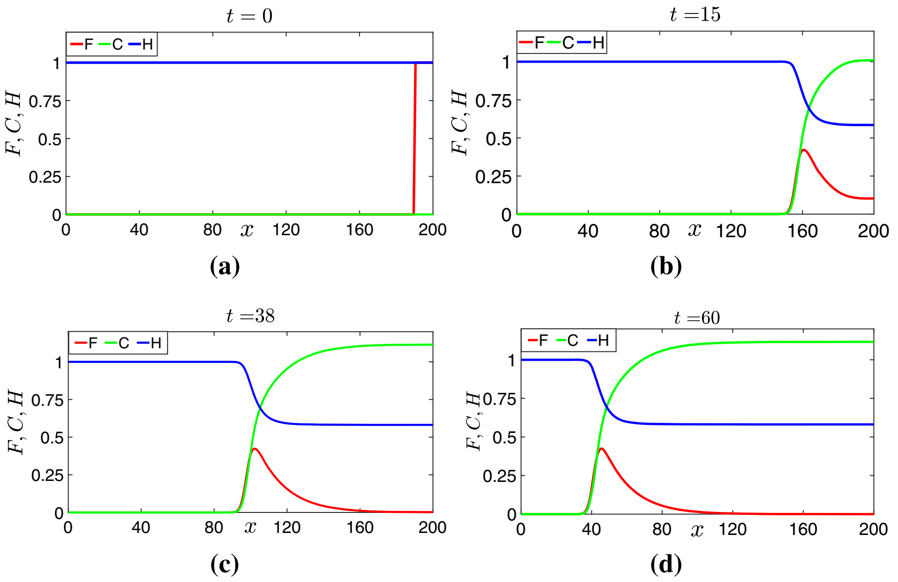

Fig. 2 Time-evolution of the solution (F, C, H) of system (1.3)–(1.5) with $L = 200$. The initial data (F_0, C_0, H_0) is the same as the one in Fig. 1. Here the parameters in (1.3) are $d = 1.2, a = 1.5, s = 0.2, b = 0.8$, and $g = 0.3$

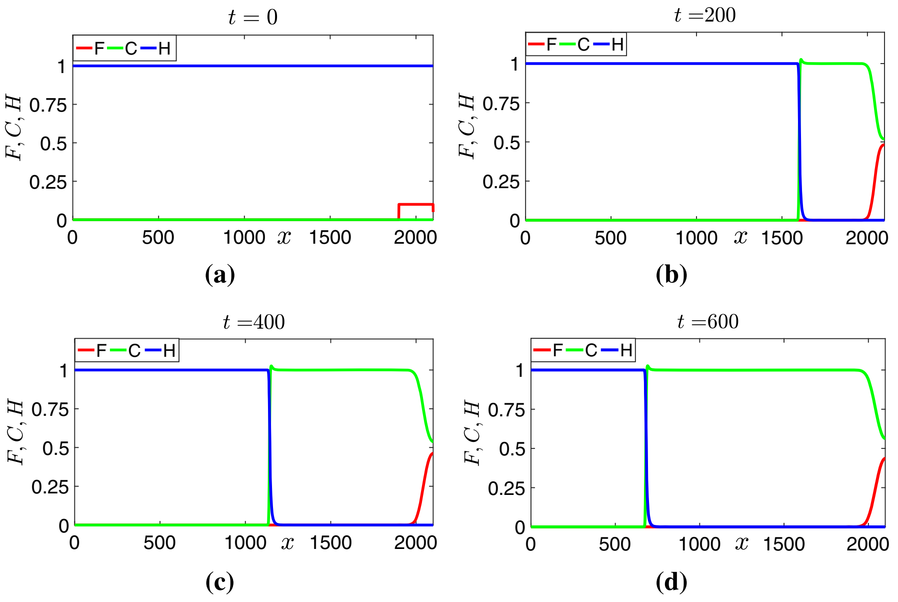

Fig. 3 Time-evolution of the solution (F, C, H) of system (1.3)–(1.5) with $L = 2000$. The initial data is that $F_0(x) = 0(0 \leq x \leq 1800)$ and $0.1(1800 < x \leq 200)$, $C_0(x) = 0(0 \leq x \leq 2000)$, and $H_0(x) = 1(0 \leq x \leq 2000)$. Here the parameters in (1.3) are $d = 1.2, a = 1, s = 0.2, b = 0.8$, and $g = 1.0$

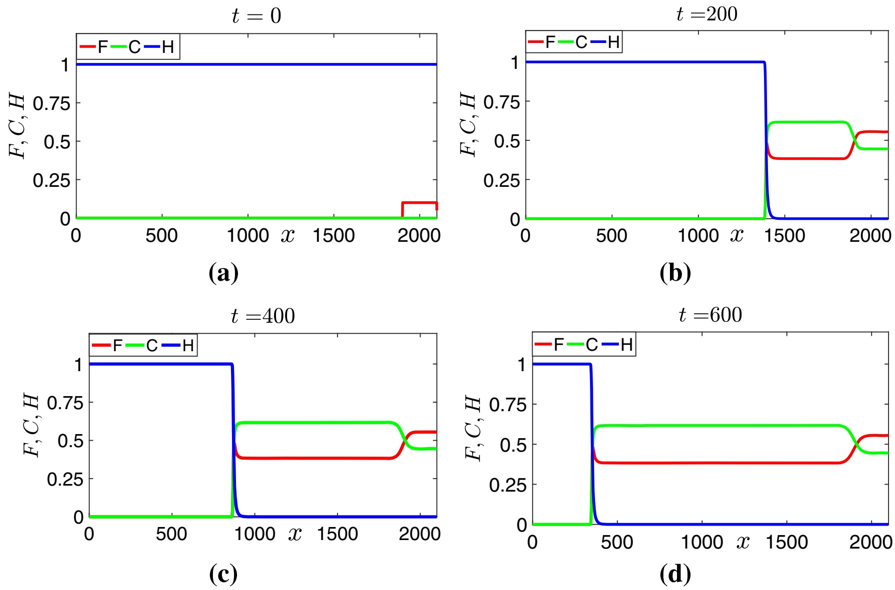

Fig. 4 Time-evolution of the solution (F, C, H) of system (1.3)–(1.5) with $L = 2000$. The initial data (F_0, C_0, H_0) is the same as the one in Fig. 3. Here the parameters in (1.3) are $d = 1.2, a = 1.5, s = 0.2, b = 0.8$, and $g = 1$

to show the existence of two types of traveling wave solutions of (1.3), depending on parameters, that is, one is the three-component traveling wave for (F, C, H) and the other is the (reduced) two-component one for (C, H) with $F \equiv 0$.

Traveling waves and main results

The aforementioned numerical results suggest that traveling wave solutions of (1.3) and their associated wave profiles can partially answer the questions (Q1) and (Q2). On the other hand, the main disadvantage of numerical studies is th,at one can never know how much the results are dependent solely on the particular values chosen for the parameters. Hence, we will employ analytical approaches to investigate traveling wave solutions of (1.3).

To begin with, we give the definition of traveling wave solutions of (1.3). Indeed, a traveling wave solution of system (1.3) is a nonnegative solution of system (1.3) of the form

$$(F(x,t), C(x,t), H(x,t)) = (f(z), c(z), h(z)), \quad z = x + vt,$$

with v being the velocity of the wave. Upon substituting the ansatz on (f, c, h) into (1.3), we are led to the governing system for (f, c, h) as follows:

$$f'' - vf' + af(1 - f - c) = 0, \tag{1.6a}$$
$$c'' - vc' + c(1 - f - c) + s(f + c)h = 0, \quad -\infty < z < \infty, \tag{1.6b}$$
$$dh'' - vh' + bh(1 - h) - g(f + c)h = 0, \tag{1.6c}$$

together with the boundary conditions

$$(f, c, h)(-\infty) = (0, 0, 1) \text{ and } (f, c, h)(+\infty) = E_\infty. \tag{1.7}$$

Here the prime indicates differentiation with respect to z, and the constant steady state E_∞ of (1.3) is

$$E_\infty = \begin{cases} (0, c^*, h^*), & \text{if } 0 < g < b, \\ (f^\natural, c^\natural, 0), & \text{if } g \geq b, \end{cases}$$

where

$$c^* := \frac{b(1+s)}{b+sg} \text{ and } h^* := \frac{b-g}{b+sg},$$

and $f^\natural$ and $c^\natural$ are some nonnegative constants satisfying $f^\natural + c^\natural = 1$. We remark that the boundary condition for (f, c, h) at $-\infty$ is motivated by the requirement of farmers and hunter-gatherers at the early stage of the Neolithic transition. We refer the readers to the discussion (Sect. 4) for the choice of initial conditions (1.4).

Note that if $f \equiv 0$ on $\mathbb{R}$, then the traveling problem (1.6)–(1.7) is reduced to the following traveling wave problem for (c, h):

$$\begin{cases} vc_z = c_{zz} + c(1 - c) + sch, \\ vh_z = dh_{zz} + bh(1 - h) - gch, \\ (c, h)(-\infty) = (0, 1), \quad (c, h)(\infty) = e_\infty, \end{cases} \tag{1.8}$$

where

$$e_\infty = \begin{cases} (1, 0), & \text{for } g \geq b, \\ (c^*, h^*), & \text{for } g < b. \end{cases} \tag{1.9}$$

Here (c^*, h^*) is defined as in the definition of E_∞. The traveling wave problem (1.8)–(1.9) exactly corresponds to that for the Ammerman–Cavalli-Sforza model (1.1). In fact, recently, Tsai et al. (2020) have shown that the traveling wave problem (1.8) and (1.9) admits a positive solution (c, h) iff $v \geq 2\sqrt{1+s}$. In addition, the solution satisfies $0 < c < 1 + s$, $0 < h < 1$ and $h' < 0$ on $\mathbb{R}$. It follows that, for all $a > 0$, the traveling problem (1.6) and (1.7) admits a nonnegative solution $(0, c, h)$ with $c > 0$ and $h > 0$ iff $v \geq 2\sqrt{1+s}$. Here, we call such a solution a trivial traveling wave solution.

Now we turn to the existence of nontrivial traveling wave solutions (i.e., positive traveling wave solutions). To proceed, consider the quadratic polynomial

$$P(\lambda) := \lambda^2 - v\lambda + a. \tag{1.10}$$

Let λ_1 and λ_2 denote two roots of $P(\lambda)=0$ with $\mathrm{Re}(\lambda_1)\le \mathrm{Re}(\lambda_2)$. If $v>v^*:=2\sqrt{a}$, then $0<\lambda_1<\lambda_2$ and $P(\lambda)<0$ when $\lambda\in(\lambda_1,\lambda_2)$. If $v=v^*$, then $\lambda_1=\lambda_2=v^*/2$. In the sequel, we retain the notations λ_1, λ_2, and v^*. Also we call the λ_1 (resp. λ_2) as the slow-rate (resp. fast-rate).

We first consider the case $0<a\le 1+s$. For this case, we recall that in general, traveling waves with non-minimal speed in monostable reaction–diffusion system always tend to the unstable equilibrium along the "slow-rate direction" (the eigendirection associated with the slow-rate), while traveling waves with minimal speed always tend to the unstable equilibrium along the "fast-rate direction" (the eigendirection associated with the fast-rate) . When applying to a nonnegative solution (f,g,h) of the traveling problem (1.6) and (1.7), this suggests that for non-minimal speed case, $f(z)e^{-\lambda_1 z}\sim C$ for some $C>0$ and for all z close to $-\infty$, while for minimal speed case, $f(z)|z|^{-1}e^{-\lambda_1 z}\sim C$ for some $C>0$ and for all z close to $-\infty$. However, our analysis from Lemma 2.1 suggests that there are no such positive wave solutions which tend to $(0,0,1)$ along the "slow-rate direction" near negative infinity. Thus, we may conclude that for $0<a\le 1+s$, there are no positive solutions (f,c,h) of the traveling problem (1.6) and (1.7). Furthermore, our numerical study (Figs. 1 and 3 for the case $a<1+s$) indicates that any nonnegative solution (f,c,h) of the traveling problem (1.6) and (1.7) must satisfy $f\equiv 0$ on $\mathbb{R}$ (i.e., only trivial traveling wave solutions exist).

We next consider the case $a>1+s$. For this case, we establish the existence of traveling waves for system (1.3) in the following theorem.

Theorem 1.1 *Assume $a>1+s$. Then problem* (1.6) *and* (1.7) *admits a positive solution (f,c,h) iff $v\ge v^*$. Moreover, $f^\natural$ and $c^\natural$ are positive if $g>b$, and (f,c,h) satisfies the following properties:*

(i) $0<f,h<1$, $c>0$, *and* $0<f+c<1+s$ *over* $\mathbb{R}$.
(ii) *As $z\to-\infty$, we have*

$$f(z)=\begin{cases}\mathcal{O}(e^{\lambda_1 z}), & \text{if } v>v^*,\\ \mathcal{O}(-ze^{\lambda_1 z}), & \text{if } v=v^*.\end{cases}$$

We remark that there are no assumptions imposed on the (dimensionless) diffusivity of the hunter-gathers for the conclusion of Theorem 1.1. Note also that for $g\ge b$, the exact expression of $(f^\sharp,c^\sharp)$ is not derived here. However, numerical evidences suggest that the value of $(f^\sharp,c^\sharp)$ depends on model parameters. In order to answer the questions (Q1) and (Q2), we need to gain more information about the wave profiles (f,c,h). For this, consider the total population of farmers $\mathbf{F}:=f+c$ (we will call $\mathbf{F}$ as the total farmer for short), and define the following two sets:

$$\begin{aligned}\mathcal{A}&:=\{\mathbf{F}\in C^2(\mathbb{R})|\ 0<\mathbf{F}<1 \text{ and } \mathbf{F}'>0 \text{ over } \mathbb{R},\ \text{and } \mathbf{F}(\infty)=1\},\\ \mathcal{B}&:=\{\mathbf{F}\in C^2(\mathbb{R})|\exists z_{\mathbf{F}}\in\mathbb{R} \text{ such that } \mathbf{F}(z_{\mathbf{F}})=1, \mathbf{F}'\\ &\qquad >0 \text{ in } (-\infty,z_{\mathbf{F}}],\ \text{and } \mathbf{F}>1 \text{ in } (z_{\mathbf{F}},\infty)\}.\end{aligned}$$

In the sequel, we retain the notation $z_{\mathbf{F}}$.

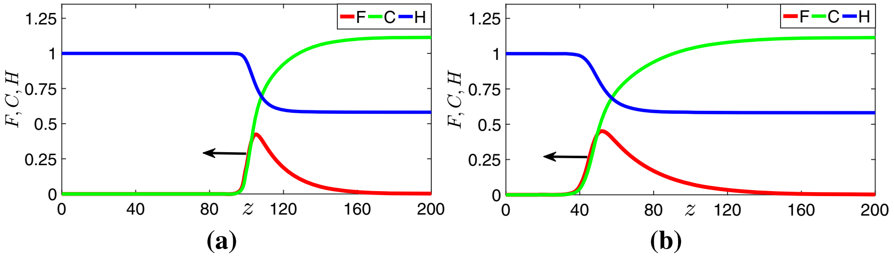

Fig. 5 Traveling wave profiles (F, C, H) of system (1.3) with $g \in (0, b)$ **a** minimal velocity ($c^* \approx 2.449$) and **b** non-minimal velocity ($c \approx 4.102$) where the parameters in (1.3) are $d = 1.2$, $a = 1.5$, $s = 0.2$, $b = 0.8$, and $g = 0.3$. The arrow indicates the propagating direction

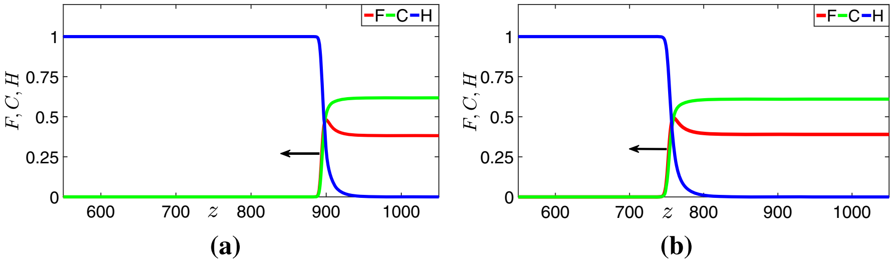

Fig. 6 Traveling wave profiles (F, C, H) of system (1.3) with $g \geq b$ and total farmer $\mathbf{F} \in \mathcal{B}$ **a** minimal velocity ($c^* \approx 2.449$) and **b** non-minimal velocity ($c \approx 3.200$) where the parameters in (1.3) are the same as the ones in Fig. 5 except $g = 1$. The arrow indicates the propagating direction

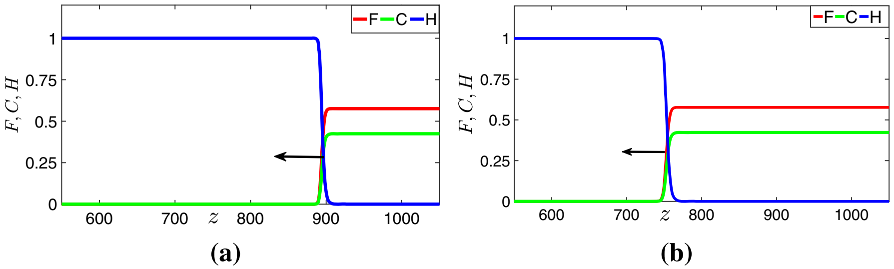

Fig. 7 Traveling wave profiles (F, C, H) of system (1.3) with $g \geq b$ and total farmer $\mathbf{F} \in \mathcal{A}$ **a** minimal velocity ($c^* \approx 2.449$) and **b** non-minimal velocity ($c \approx 3.200$) where the parameters in (1.3) are the same as the ones in Fig. 5 except $g = 2$. The arrow indicates the propagating direction

Proposition 1.1 *Assume that $a > 1 + s$. Let (f, c, h) be the solution of* (1.6) and (1.7) *given in Theorem* 1.1.

Then the following hold:

(i) *The total farmer $\mathbf{F}$ belongs to $\mathcal{A} \cup \mathcal{B}$. In addition, if either $g \in (0, b)$, or $g \in [b, 1 + b)$ and $d \geq 1$, then $\mathbf{F}$ lies in the set $\mathcal{B}$ (See Figs.* 5 *and* 6 *for an illustration).*

(ii) *If $\mathbf{F} \in \mathcal{A}$, then $f' > 0$ and $c' > 0$ over $\mathbb{R}$.*

(iii) *If $\mathbf{F} \in \mathcal{B}$, then there exists a $z_f < z_{\mathbf{F}}$ such that $f' > 0$ on $(-\infty, z_f)$ and $f' < 0$ on (z_f, ∞), where $z_{\mathbf{F}}$ is the point defined in the definition of the set $\mathcal{B}$.*

(iv) *If $g \geq b$, then $h' < 0$ over $\mathbb{R}$; if $g \in (0, b)$, then $h' < 0$ on $(-\infty, z_{\mathbf{F}})$.*

Traveling wave solutions with minimal velocity v^* and non-minimal velocity $v(> v^*)$ for (ii), (iii) and (iv) are numerically shown in Figs. 5, 6 and 7, respectively. We make five remarks. (i) According to Theorem 1.1, $\mathbf{F}(\infty) = f(\infty) + c(\infty) = b(1+s)/(b+sg) > 1$ when $g \in (0, b)$, and $\mathbf{F}(\infty) = f(\infty)+c(\infty) = 1$ when $g \geq b$. (ii) Numerical experiments indicate that $h' < 0$ over $\mathbb{R}$ for $0 < g < b$. However, due to technical difficulty, we just prove that $h' < 0$ on $(-\infty, z_{\mathbf{F}})$. (iii) We are unable to identify any sufficient condition under which $\mathbf{F}$ lies in the set $\mathcal{A}$. From numerical experiments, we conjecture that there exists a constant $g^* \geq b + 1$ such that $\mathbf{F} \in \mathcal{B}$ if $0 < g < g^*$ and $\mathbf{F} \in \mathcal{A}$ if $g \geq g^*$. (iv) The behavior of the back of the traveling waves indicates that for the case $g < b$, the hunter-gatherers can exist with lower density, while for the case $g \geq b$, the hunter-gatherers are converted into farmers completely. (v) Although the existence result of traveling waves of system (1.3) is similar to that of the well-known Fisher–KPP equations (Fisher 1937; Kolmogorov et al. 1937), we do not solve the uniqueness of the profiles of traveling waves. The main difficulty is due to the fact that system (1.3) does not enjoy the comparison principle. Also there are very few uniqueness results on wave profiles in non-cooperative Fisher–KPP systems, and nonlocal Fisher–KPP equations which could be reduced from a parabolic system (Bouin and Calvez 2014; Bouin et al. 2017). We will address the uniqueness of wave profiles of system (1.3) in our future work.

Finally, we outline the organization of the paper. The main difficulty of mathematical analysis for system (1.3) is due to the fact that it is a three-component reaction–diffusion system without comparison principle. The proof of Theorem 1.1 consists of two steps. For the first step which is given in Sect. 2, we first use the fixed point approach, motivated by Berestycki et al. (2005), to obtain the solution of a truncated problem whose boundary values are determined by a pair of upper/lower-solutions. Then with the use of a limiting argument, we can get a solution (f, c, h) of (1.6) with $(f, c, h)(-\infty) = (0, 0, 1)$. For (f, c, h) to be a traveling wave, one needs to verify that $(f, c, h)(\infty) = E_\infty$. Due to this reason, we call (f, c, h) as a semi-wave before the completion of the second step. For the second step which is given in Sect. 3, we verify $(f, c, h)(\infty) = E_\infty$. This step is divided into two cases: (i) $g \in (0, b)$, and (ii) $g \geq b$. In order to proceed with this step, we introduce the total farmer $\mathbf{F} := f + c$, and convert (1.6) into a system governed by $(f, \mathbf{F}, h)$ [see system (3.1)]. This idea of introducing the total farmers $\mathbf{F}$ is crucial for the analysis of the second step. This idea seems to be new for the analysis of a three-component system. Then we study the basic properties of the total farmer $\mathbf{F}$ in Sect. 3.2. Next, in Sect. 3.3 we construct a Lyapunov functional to verify $(f, c, h)(\infty) = E_\infty$ for the case $g \in (0, b)$, while the verification of $(f, c, h)(\infty) = E_\infty$ for the case $g \geq b$ is discussed in Sect. 3.4 via a direct analysis. Also some properties of traveling waves are shown in Sect. 3.4. Finally, a conclusion is given in Sect. 4.

2 Existence of semi-waves

2.1 Necessary conditions for existence of positive traveling wave solutions

In the following lemma, we will establish that positive traveling wave solutions of (1.3) exist only if $v \geq v^*$, and that for $a \in (0, 1 + s]$, there exist no positive traveling wave solutions of (1.3) tending to the unstable equilibrium (0, 0, 1) along the “slow-rate direction” (i.e., the eigendirection associated with the smaller eigenvalue λ_1). The later assertion suggests that for $a \in (0, 1+s]$, only trivial traveling wave solutions of (1.3) exist.

Lemma 2.1 *Recall that* $\lambda_1 = (v - \sqrt{v^2 - 4a})/2$ *and* $v^* = 2\sqrt{a}$. *Then the followings hold:*

(i) *For* $a > 0$ *and* $v < v^*$, *if* (f, c, h) *is a nonnegative bounded solution of* (1.6) *and* (1.7), *then* $f \equiv 0$ *on* $\mathbb{R}$.
(ii) *For* $a \in (0, 1+s]$ *and* $v \geq v^*$, *there exists no nonnegative bounded solution* (f, c, h) *of* (1.6) *and* (1.7) *with* $f(z)|z|^{-m}e^{-\lambda_1 z} \sim C$ *for some* $C > 0$ *and for all* z *close to* $-\infty$. *Here* $m = 0$ *if* $v > v^*$, *and* $m = 1$ *if* $v = v^*$.

Proof Let (f, c, h) be a nonnegative bounded solution of the traveling wave problem (1.6) and (1.7). Then by Hartman–Grobman theorem (Hartman 1982), the dynamical behavior of (f, c, h) around $z = -\infty$ is determined by a solution $(\hat{f}, \hat{c}, \hat{h})$ of the linearied system of (1.6) around (0, 0, 1) which is given by

$$\begin{cases} \hat{f}'' - v\hat{f}' + a\hat{f} = 0, \\ \hat{c}'' - v\hat{c}' + (1+s)\hat{c} + s\hat{f} = 0, \quad -\infty < z < \infty, \\ d\hat{h}'' - v\hat{h}' - g\hat{f} - g\hat{c} - b\hat{h} = 0. \end{cases} \tag{2.1a}$$

In the remainder of the proof, we will use the linear system (2.1) to deduce the assertions of this lemma.

First, we consider the assertion (i). Indeed, if $v \leq -v^*$ or $|v| < v^*$, then the $\hat{f}$-component of a solution $(\hat{f}, \hat{c}, \hat{h})$ of (2.1) is either unbounded or not of a single sign for z near $-\infty$ unless $\hat{f} = 0$. This proves the assertion (i).

Next, we turn to prove the assertion (ii). Note that $\lambda_c^{\pm} = (v \pm \sqrt{v^2 - 4(1+s)})/2$ are two zeros of $P_c(\lambda) := \lambda^2 - v\lambda + (1+s)$. We first consider the case where $a \in (0, 1+s)$ and $v \geq v^*$. For contradiction, assume that there is such a solution (f, c, h) of (1.6) and (1.7). Then due to $v \geq 2\sqrt{a}$, the $\hat{f}$-component of the associated solution $(\hat{f}, \hat{c}, \hat{h})$ of (2.1) is given by $\hat{f}(z) = k_1|z|^m e^{\lambda_1 z} + k_2 e^{\lambda_2 z}$ with $k_1 > 0$ and $k_2 \in \mathbb{R}$, and m being defined as in the assertion (ii). A direct analysis of (2.1a) gives

$$\hat{c}(z) = \frac{-sk_1}{P_c(\lambda_1)}|z|^m e^{\lambda_1 z} + \frac{-sk_2}{P_c(\lambda_2)} e^{\lambda_2 z} + l^- e^{\lambda_c^- z} + l^+ z^n e^{\lambda_c^+ z}, \ l^{\pm} \in \mathbb{R},$$

where $n = 0$ if $v > 2\sqrt{1+s}$, and $n = 1$ if $v > 2\sqrt{1+s}$. Due to $a < 1+s$ and $v \geq 2\sqrt{a}$, we have $P_c(\lambda_1) > 0$ and $\lambda_1 < \mathrm{Re}(\lambda_c^-) \leq \mathrm{Re}(\lambda_c^+) < \lambda_2$ where $\mathrm{Re}(\lambda)$ is the real part of $\lambda \in \mathbb{C}$. Thus, the first term in the expression of $\hat{c}(z)$ is negative and dominates the other three terms, and so $\hat{c}(z) < 0$ for z close to $-\infty$. This in turn implies $c(z) < 0$ for z close to $-\infty$. This is a contradiction, and thus establishing the assertion (ii) for the case where $a \in (0, 1+s)$ and $v \geq v^*$.

Now, we prove the assertion (ii) for the case where $a = 1+s$ and $v > v^*$. For contradiction, assume that there is such a solution (f, c, h) of (1.6) and (1.7). Then the $\hat{f}$-component of the associated solution $(\hat{f}, \hat{c}, \hat{h})$ of (2.1) is given by $\hat{f}(z) = k_1 e^{\lambda_1 z} + k_2 e^{\lambda_2 z}$ with $k_1 > 0$ and $k_2 \in \mathbb{R}$. Now a direct analysis of (2.1a) gives

$$\hat{c}(z) = \frac{sk_1}{\sqrt{v^2 - 4a}} z e^{\lambda_1 z} + \frac{sk_2}{\sqrt{v^2 - 4a}} z e^{\lambda_2 z} + l^- e^{\lambda_c^- z} + l^+ e^{\lambda_c^+ z}, \ l^\pm \in \mathbb{R}.$$

Note that $\lambda_1 = \lambda_c^- < \lambda_2 = \lambda_c^+$ since $a = 1+s$. Thus, the first term in the expression of $\hat{c}(z)$ is negative for all $z < 0$ and dominates the other three terms, and so $\hat{c}(z) < 0$ for z close to $-\infty$. This in turn implies $c(z) < 0$ for z close to $-\infty$. This is a contradiction, and thus proving the assertion (ii) for the case where $a = 1 + s$ and $v > v^*$.

Finally, we prove the assertion (ii) for the case where $a = 1 + s$ and $v = v^*$. For contradiction, assume that there is such a solution (f, c, h) of (1.6) and (1.7). Then the $\hat{f}$-component of the associated solution $(\hat{f}, \hat{c}, \hat{h})$ of (2.1) is given by $\hat{f}(z) = k_1 z e^{\lambda_1 z} + k_2 e^{\lambda_1 z}$ with $k_1 < 0$ and $k_2 \in \mathbb{R}$. Now a direct analysis of (2.1a) gives

$$\hat{c}(z) = -\frac{sk_1}{6} z e^{\lambda_1 z} - \frac{sk_2}{2} e^{\lambda_1 z} + l^- e^{\lambda_c^- z} + l^+ z e^{\lambda_c^+ z}, \ l^\pm \in \mathbb{R}.$$

Note that $\lambda_1 = \lambda_2 = \lambda_c^\pm$ since $a = 1 + s$ and $v = v^*$. Thus, the first term in the expression of $\hat{c}(z)$ is negative for $z < 0$ and dominates the other three terms, and so $\hat{c}(z) < 0$ for z close to $-\infty$. This is a contradiction, and thus proving the assertion (ii) for the case where $a = 1 + s$ and $v = v^*$. The proof of this lemma is thus completed. □

Remark By exactly the same phase-plane arguments as those in this lemma, one can show that there are no "trivial" traveling waves solutions of (1.3) with wave speed $v < 2\sqrt{1+s}$.

In the remainder of this paper, we always assume that $a > 1 + s$ and $v \geq v^*$.

2.2 Upper and lower solutions

In this subsection, we will construct a pair of upper and lower solutions of (1.6). To begin with, we define upper and lower solutions of (1.6) as follows:

Definition 2.1 $(\bar{f}, \bar{c}, \bar{h})$ and $(\underline{f}, \underline{c}, \underline{h})$ are called a pair of upper and lower solutions of (1.6) on $\mathbb{R}$ if $(\bar{f}, \bar{c}, \bar{h})$ and $(\underline{f}, \underline{c}, \underline{h})$ are continuous functions on $\mathbb{R}$ satisfying

$$\bar{f}'' - v\bar{f}' + a\bar{f}(1 - \bar{f}) - a\bar{f}\underline{c} \leq 0, \tag{2.2a}$$
$$\underline{f}'' - v\underline{f}' + a\underline{f}(1 - \underline{f}) - a\underline{f}\bar{c} \geq 0, \tag{2.2b}$$
$$\bar{c}'' - v\bar{c}' + \bar{c}(1 - \underline{f} - \underline{c}) + s(\bar{f} + \bar{c})\bar{h} \leq 0, \tag{2.2c}$$
$$\underline{c}'' - v\underline{c}' + \underline{c}(1 - \bar{f} - \bar{c}) + s(\underline{f} + \underline{c})\underline{h} \geq 0 \tag{2.2d}$$
$$d\bar{h}'' - v\bar{h}' + b\bar{h}(1 - \bar{h}) - g(\underline{f} + \underline{c})\bar{h} \leq 0, \tag{2.2e}$$
$$d\underline{h}'' - v\underline{h}' + b\underline{h}(1 - \underline{h}) - g(\bar{f} + \bar{c})\underline{h} \geq 0, \tag{2.2f}$$

except for finitely many points in $\mathbb{R}$.

2.2.1 The non-critical case $v > v^*$

To construct upper–lower solutions, we select positive constants A, η and σ such that

$$A > s/(a-1-s), \tag{2.3}$$
$$\eta < \min\{\lambda_1, \lambda_2 - \lambda_1\}, \tag{2.4}$$

and

$$\sigma < \min(v/d, \lambda_1, \eta). \tag{2.5}$$

Then we pick positive constants L and M such that $M > L > 1$,

$$-LP(\lambda_1+\eta) \geq a(A+1), \tag{2.6}$$

and

$$M\sigma(v-d\sigma) \geq g(A+1). \tag{2.7}$$

Now we define $(\bar{f}, \bar{c}, \bar{h})$ and $(\underline{f}, \underline{c}, \underline{h})$ by

$$\begin{aligned}
\bar{f}(z) &:= \begin{cases} e^{\lambda_1 z}, & z \leq z_0 := 0, \\ 1, & z > z_0, \end{cases} \\
\underline{f}(z) &:= \begin{cases} \left(1 - Le^{\eta z}\right)e^{\lambda_1 z}, & z \leq z_1 := -\ln L/\eta < 0 \\ 0, & z > z_1, \end{cases} \\
\bar{c}(z) &:= Ae^{\lambda_1 z}, \\
\underline{c}(z) &:= 0, \\
\bar{h}(z) &:= 1, \\
\underline{h}(z) &:= \begin{cases} 1 - Me^{\sigma z}, & z \leq z_2 := -\ln M/\sigma < 0, \\ 0, & z > z_2. \end{cases}
\end{aligned}$$

Note that $z_2 < z_1$ since $M > L$ and $\eta > \sigma$. It is obvious that $\bar{h}$ and $\underline{c}$ satisfy (2.2e) and (2.2d) respectively. In the following, we will show that $(\bar{f}, \bar{c}, \bar{h})$ and $(\underline{f}, \underline{c}, \underline{h})$ are a pair of upper and lower solutions of (1.6) on $\mathbb{R}$.

Lemma 2.2 *The function $\bar{f}$ satisfies (2.2a) for all $z \neq z_0$.*

Proof For $z > z_0$, $\bar{f}(z) = 1$ and (2.2a) holds obviously. For $z < z_0$, $\bar{f}(z) = e^{\lambda_1 z}$. Since $P(\lambda_1) = 0$, it follows that

$$\bar{f}'' - v\bar{f}' + a\bar{f}(1-\bar{f}) - a\bar{f}\underline{c} \leq \bar{f}'' - v\bar{f}' + a\bar{f} = P(\lambda_1)e^{\lambda_1 z} = 0.$$

This completes the proof of this lemma. □

Lemma 2.3 *The function $\bar{c}$ satisfies (2.2c) for all $z \in \mathbb{R}$.*

Proof Since $\bar{h} \equiv 1$ on $\mathbb{R}$ and $P(\lambda_1) = 0$, it follows that for each $z \in \mathbb{R}$,

$$\begin{aligned}
&\bar{c}'' - v\bar{c}' + \bar{c}(1 - \underline{f} - \underline{c}) + s(\bar{f} + \bar{c})\bar{h} \\
&\quad \le [\bar{c}'' - v\bar{c}' + \bar{c} + s\bar{c}] + s\bar{f} \\
&\quad \le [P(\lambda_1) - a + 1 + s]Ae^{\lambda_1 z} + se^{\lambda_1 z} \\
&\quad = [A(s + 1 - a) + s]e^{\lambda_1 z} \\
&\quad \le 0 \quad \text{(by (2.3))}.
\end{aligned}$$

The proof of this lemma is thus completed. □

Lemma 2.4 *The function $\underline{f}$ satisfies* (2.2b) *for all $z \neq z_1$.*

Proof For $z > z_1$, since $\underline{f} \equiv 0$ in (z_1, ∞), the inequality (2.2b) follows. For $z < z_1$, we can use the fact $P(\lambda_1) = 0$ and (2.6) to deduce that

$$\begin{aligned}
&\underline{f}'' - v\underline{f}' + a\underline{f}(1 - \underline{f}) - a\underline{f}\bar{c} \\
&\quad \ge \underline{f}'' - v\underline{f}' + a\underline{f} - a\bar{f}(\bar{f} + \bar{c}) \\
&\quad = e^{(\lambda_1+\eta)z}[-LP(\lambda_1 + \eta) - a(A + 1)e^{(\lambda_1-\eta)z}] \\
&\quad \ge 0 \quad \text{(by (2.4) and (2.6))}.
\end{aligned}$$

Hence (2.2b) holds. □

Lemma 2.5 *The function $\underline{h}$ satisfies* (2.2f) *for all $z \neq z_2$.*

Proof For $z > z_2$, (2.2f) holds immediately since $\underline{h} \equiv 0$ in (z_2, ∞). For $z < z_2$, we have

$$\begin{aligned}
&d\underline{h}'' - v\underline{h}' + b\underline{h}(1 - \underline{h}) - g(\bar{f} + \bar{c})\underline{h} \\
&\quad \ge d\underline{h}'' - v\underline{h}' - g(\bar{f} + \bar{c})\bar{h} \\
&\quad = e^{\sigma z}[M\sigma(v - d\sigma) - g(A + 1)e^{(\lambda_1-\sigma)z}] \\
&\quad \ge 0 \quad \text{(by (2.5) and (2.7))}.
\end{aligned}$$

The proof of this lemma is therefore completed. □

2.2.2 The critical case $v = v^*$

We select $0 < \nu < \lambda_1$ such that $P(\nu) < a - s - 1$, and choose $z_0 < -1$ such that

$$(P(\nu) + s + 1 - a) - sze^{(\lambda_1-\nu)z} \le 0 \quad \forall z \le z_0. \tag{2.8}$$

Then we set $\rho := -(1/z_0)e^{-\lambda_1 z_0}$. Pick A such that

$$A > \max\left\{\rho, se^{-\nu z_0}/(a - s - 1 - P(\nu))\right\}. \tag{2.9}$$

Noting that

$$\lim_{z\to-\infty}\left[a\rho^2(-z)^{7/2}e^{\lambda_1 z}-a\rho A(-z)^{5/2}e^{\nu z}\right]=0,$$

there exists a number $z_1 < z_0$ such that

$$a\rho^2(-z)^{7/2}e^{\lambda_1 z}+a\rho A(-z)^{5/2}e^{\nu z}<\frac{1}{4}\quad \forall z\le z_1. \tag{2.10}$$

Set $L := \rho\sqrt{-z_1} > 1$ and pick a positive number σ such that $\sigma < \min(\nu, v/d, \lambda_1)$. Since $ze^{(\lambda_1-\sigma)z} \to 0$ and $e^{(\nu-\sigma)z} \to 0$ as $z \to -\infty$, there exists $z_2 < z_1$ such that

$$\sigma(v-d\sigma)-g(-\rho z e^{(\lambda_1-\sigma)z}+Ae^{(\nu-\sigma)z})>0\quad \forall z\le z_2. \tag{2.11}$$

Set $M := e^{-\sigma z_2}$. Then $M > 1$ since $z_2 < 0$.

Now we define $(\bar f, \bar c, \bar h)$ and $(\underline{f}, \underline{c}, \underline{h})$ by

$$\begin{aligned}
\bar f(z) &:= \begin{cases} -\rho z e^{\lambda_1 z}, & z\le z_0,\\ 1, & z>z_0,\end{cases}\\
\underline{f}(z) &:= \begin{cases} \left[-\rho z-L(-z)^{1/2}\right]e^{\lambda_1 z}, & z\le z_1\\ 0, & z>z_1,\end{cases}\\
\bar c(z) &:= Ae^{\nu z},\\
\underline{c}(z) &:= 0,\\
\bar h(z) &:= 1,\\
\underline{h}(z) &:= \begin{cases} 1-Me^{\sigma z}, & z\le z_2,\\ 0, & z>z_2.\end{cases}
\end{aligned}$$

In the following, we will show that $(\bar f, \bar c, \bar h)$ and $(\underline{f}, \underline{c}, \underline{h})$ are a pair of upper and lower solutions of (1.6) on $\mathbb{R}$.

Lemma 2.6 *The function $\bar f$ satisfies (2.2a) for all $z \ne z_0$.*

Proof For $z > z_0$, $\bar f(z) \equiv 1$ and (2.2a) holds obviously. For $z < z_0$, $\bar f(z) = -\rho z e^{\lambda_1 z}$. Since $P(\lambda_1) = 0$ and $\lambda_1 = v/2$, it follows that

$$\begin{aligned}
&\bar f''-v\bar f'+a\bar f(1-\bar f)-a\bar f\underline{c}\le \bar f''-v\bar f'+a\bar f\\
&=-\rho z e^{\lambda_1 z}P(\lambda_1)e^{\lambda_1 z}+\rho(v-2\lambda_1)e^{\lambda_1 z}=0.
\end{aligned}$$

The proof of this lemma is thus completed. □

Lemma 2.7 *The function $\bar c$ satisfies (2.2c) for all $z \in \mathbb{R}$.*

Proof Since $\bar h \equiv 1$ on $\mathbb{R}$, $\nu < \lambda_1$, and $P(\nu) < a - s - 1$, it follows that for $z < z_0$,

$$\bar c''-v\bar c'+\bar c(1-\underline{f}-\underline{c})+s(\bar f+\bar c)\bar h$$

$$
\begin{aligned}
&\le [\bar{c}'' - v\bar{c}' + \bar{c} + s\bar{c}] + s\bar{f}\\
&= [P(\nu) - a + 1 + s]Ae^{\nu z} - s\rho z e^{\lambda_1 z}\\
&\le [(P(\nu) + s + 1 - a)\rho - s\rho z e^{(\lambda_1 - \nu)z}]e^{\nu z} \quad (\text{Using } A > \rho)\\
&\le 0 \quad (\text{by } (2.8)),
\end{aligned}
$$

and for $z > z_0$,

$$
\begin{aligned}
&\bar{c}'' - v\bar{c}' + \bar{c}(1 - \underline{f} - \underline{c}) + s(\bar{f} + \bar{c})\bar{h}\\
&\quad \le [\bar{c}'' - v\bar{c}' + \bar{c} + s\bar{c}] + s\bar{f}\\
&\quad = [P(\nu) - a + 1 + s]Ae^{\nu z} + s\\
&\quad \le [P(\nu) + s + 1 - a]Ae^{\nu z_0} + s\\
&\quad \le 0 \quad (\text{by } (2.9)),
\end{aligned}
$$

which completes the proof of this lemma. □

Lemma 2.8 *The function $\underline{f}$ satisfies (2.2b) for all $z \neq z_1$.*

Proof For $z > z_1$, since $\underline{f} \equiv 0$ in (z_1, ∞), the inequality (2.2b) follows. For $z < z_1$, $\underline{f}(z) = \bar{f}(z) - L(-z)^{1/2}e^{\lambda_1 z}$. A simple computation gives that

$$
\underline{f}'(z) = \bar{f}'(z) + Le^{\lambda_1 z}\left[\frac{1}{2}(-z)^{-1/2} - \lambda_1(-z)^{1/2}\right], \tag{2.12}
$$

and

$$
\underline{f}''(z) = \bar{f}''(z) + Le^{\lambda_1 z}\left[\frac{1}{4}(-z)^{-3/2} + \lambda_1(-z)^{-1/2} - (\lambda_1)^2(-z)^{1/2}\right]. \tag{2.13}
$$

Thus for $z < z_1$,

$$
\begin{aligned}
&\underline{f}'' - v\underline{f}' + a\underline{f}(1 - \underline{f}) - a\underline{f}\bar{c}\\
&\quad \ge \underline{f}'' - v\underline{f}' + a\underline{f} - a\bar{f}(\bar{f} + \bar{c})\\
&\quad = (-z)^{-3/2}e^{\lambda_1 z}\left[\frac{L}{4} - a\rho^2(-z)^{7/2}e^{\lambda_1 z} - a\rho A(-z)^{5/2}e^{\nu z}\right]\\
&\quad \ge 0 \quad (\text{Using } L > 1 \text{ and } (2.10)).
\end{aligned}
$$

Hence (2.2b) holds. The proof of this lemma is therefore completed. □

Lemma 2.9 *The function $\underline{h}$ satisfies (2.2f) for all $z \neq z_2$.*

Proof For $z > z_2$, the inequality (2.2f) holds immediately since $\underline{h} \equiv 0$ in (z_2, ∞). For $z < z_2$, we have

$$
d\underline{h}'' - v\underline{h}' + b\underline{h}(1 - \underline{h}) - g(\bar{f} + \bar{c})\underline{h}
$$

$$
\begin{aligned}
&\geq d\underline{h}'' - v\underline{h}' - g(\bar{f} + \bar{c})\bar{h} \\
&= e^{\sigma z}\left[M\sigma(v - d\sigma) - g(-\rho z e^{(\lambda_1-\sigma)z} + Ae^{(\nu-\sigma)z})\right] \\
&\geq 0 \quad \text{(Using } M > 1 \text{ and (2.11)).}
\end{aligned}
$$

The proof of this lemma is therefore completed. □

Remark 2.10 Although the values of z_0, z_1, and z_2 for $v > v^*$ are different from those for $v = v^*$, the use of the same notations z_0, z_1, and z_2 makes that the constructed upper/lower solutions satisfies the same set of inequalities for each $v \geq v^*$, as shown in the lemmas of this subsection. We will retain the notations of z_0, z_1, and z_2 in the remainder of this section.

2.3 A truncated problem

In this subsection, we consider the following truncated problem:

$$
\begin{aligned}
&f'' - vf' + af(1 - f - c) = 0 \quad \text{in } I_l, &(2.14a)\\
&c'' - vc' + c - c(f + c) + s(f + c)h = 0 \quad \text{in } I_l, &(2.14b)\\
&dh'' - vh' + bh(1 - h) - g(f + c)h = 0 \quad \text{in } I_l, &(2.14c)\\
&f(z) = \underline{f}(z),\ c(z) = \underline{c}(z),\ h(z) = \underline{h}(z) \quad \text{in } (-\infty, -l] \cup [l, \infty), &(2.14d)
\end{aligned}
$$

where $a > 1 + s$, $v \geq v^*$ and $I_l := (-l, l)$ with $l > \max\{|z_1|, |z_2|\}$.

For convenience, setting

$$X := C(\mathbb{R}) \times C(\mathbb{R}) \times C(\mathbb{R}) \text{ and } Y := C^2(I_l) \times C^2(I_l) \times C^2(I_l),$$

then X is a Banach space equipped with the norm $\|(\phi_1, \phi_2, \phi_3)\|_X = \|\phi_1\|_{C(\mathbb{R})} + \|\phi_2\|_{C(\mathbb{R})} + \|\phi_3\|_{C(\mathbb{R})}$. We will apply the Schauder fixed point theorem to show that there exists a triple of functions $(f, c, h) \in X \cap Y$ satisfying (2.14). To this end, let

$$E := \{(f, c, h) \in X |\ \underline{f} \leq f \leq \bar{f}, \underline{c} \leq c \leq \bar{c}, \text{ and } \underline{h} \leq h \leq \bar{h} \text{ in } \mathbb{R}\},$$

which is a closed convex set in X. Define the mapping $\mathcal{F} : E \to X$ as follows: given $(f_0, c_0, h_0) \in E$,

$$\mathcal{F}(f_0, c_0, h_0) := (f, c, h),$$

where (f, c, h) is the unique triple of functions $(f, c, h) \in X \cap Y$ satisfying

$$
\begin{aligned}
&f'' - vf' + af_0(1 - f) - ac_0 f = 0 \quad \text{in } I_l, &(2.15a)\\
&c'' - vc' + c_0 - (f_0 + c_0)c + s(f_0 + c_0)h_0 = 0 \quad \text{in } I_l, &(2.15b)\\
&dh'' - vh' + bh_0(1 - h) - g(f_0 + c_0)h = 0 \quad \text{in } I_l, &(2.15c)\\
&f(z) = \underline{f}(z),\ c(z) = \underline{c}(z),\ h(z) = \underline{h}(z) \quad \text{in } (-\infty, -l] \cup [l, \infty). &(2.15d)
\end{aligned}
$$

Since (2.15) is a decoupled system and the equations for f, c, and h are inhomogeneous linear equations, the existence and uniqueness of (f, c, h) can be directly obtained by Theorem 3.1 of Chapter 12 in the book of Hartman (1982). Obviously, any fixed point of $\mathcal{F}$ is a triple of functions $(f, c, h) \in X \cap Y$ satisfying (2.14). In the following, we will verify that the mapping $\mathcal{F}$ satisfies the conditions of the Schauder fixed point theorem.

Lemma 2.11 *$\mathcal{F}E \subset E$; that is, for a given $(f_0, c_0, h_0) \in E$, the solution (f, c, h) of (2.15) satisfies $\underline{f} \le f \le \bar{f}$, $\underline{c} \le c \le \bar{c}$, and $\underline{h} \le h \le \bar{h}$ in $\mathbb{R}$.*

Proof First, we note that by the maximum principle, we have that f, c, and h are positive in I_l. Since $\underline{c} \equiv 0$ and c is nonnegative, $\underline{c} \le c$ is obvious. Since $\hat{f} := 1$ and $\bar{h} \equiv 1$ satisfy

$$\begin{aligned} d\bar{h}'' - v\bar{h}' + bh_0(1 - \bar{h}) - g\bar{h}(f_0 + c_0) \le 0, \\ \hat{f}'' - v\hat{f}' + af_0(1 - \hat{f}) - a\hat{f}c_0 \le 0 \end{aligned}$$

in I_l, it follows from the comparison principle that $f \le \hat{f} \equiv 1$ and $h \le \bar{h} \equiv 1$ over $\mathbb{R}$.

Due to $(f_0, c_0, h_0) \in E$, we can use (2.15) to deduce that

$$f'' - vf' + a\bar{f}(1 - f) - af\underline{c} \ge 0, \tag{2.16}$$

$$f'' - vf' + a\underline{f}(1 - f) - af\bar{c} \le 0, \tag{2.17}$$

$$c'' - vc' + \bar{c} - c(\underline{f} + \underline{c}) + s(\bar{f} + \bar{c})\bar{h} \ge 0, \tag{2.18}$$

$$dh'' - vh' + b\underline{h}(1 - h) - gh(\bar{f} + \bar{c}) \le 0 \tag{2.19}$$

in I_l. Now we claim that $\underline{f} \le f$ over $\mathbb{R}$. Recall that f is positive in I_l. Together with (2.15d) and the fact that $\underline{f} \equiv 0$ in $[z_1, \infty)$ and $z_1 \in I_l$, we see that

$$\underline{f} \le f \text{ in } (-\infty, -l] \cup [z_1, \infty). \tag{2.20}$$

So it suffices to show that $\underline{f} \le f$ in $(-l, z_1)$. By (2.2b) and (2.17), the function $\Psi := f - \underline{f}$ satisfies

$$\Psi'' - v\Psi' - a\underline{f}\Psi - a\bar{c}\Psi \le 0$$

in $(-l, z_1)$. In addition, using (2.20), we have $\Psi(-l) \ge 0$ and $\Psi(z_1) \ge 0$. Therefore it follows from the maximum principle that $\Psi \ge 0$ in $(-l, z_1)$. This gives that $\underline{f} \le f$ in $(-l, z_1)$. Similarly, one can easily show that $f \le \bar{f}$, $c \le \bar{c}$, and $\underline{h} \le h$ over $\mathbb{R}$. Hence the proof of this lemma is completed. ◻

Finally, since it is standard to show that the mapping $\mathcal{F}$ is continuous and precompact [e.g., see the proofs of Lemma 4.4 and Lemma 4.5 of Fu (2014)], we omit the proofs. Now we can apply the Schauder fixed point theorem to conclude that $\mathcal{F}$ has a fixed point $(f_l, c_l, h_l) \in X \cap Y$, which is a triple of functions satisfying (2.14) and $\underline{f} \le f \le \bar{f}$,

$\underline{c} \le c \le \bar{c}$, and $\underline{h} \le h \le \bar{h}$ over $\mathbb{R}$. From the above discussion, we have the following existence result for the truncated problem (2.14).

Lemma 2.12 *Let $v \ge v^*$. For each $l > \max\{|z_1|, |z_2|\}$, there exists a triple of functions $(f_l, c_l, h_l) \in X \cap Y$ satisfying* (2.14). *Moreover,*

$$0 \le \underline{f} \le f_l \le \bar{f} \le 1,\ 0 = \underline{c} \le c_l \le \bar{c},\ 0 \le \underline{h} \le h_l \le \bar{h} \equiv 1 \quad on\ \mathbb{R}. \tag{2.21}$$

2.4 Proof of existence of semi-waves

In this subsection, we will establish the existence of semi-waves of system (1.3) by using the solution (f_l, c_l, h_l) of the truncated problem (2.14) and a limiting argument.

Lemma 2.13 *Assume that $a > 1 + s$. If $v \ge v^*$, then system* (1.6) *admits a positive solution (f, c, h) satisfying*

$$(f, c, h)(-\infty) = (0, 0, 1) \tag{2.22}$$

and

$$0 \le \underline{f} \le f \le \bar{f} \le 1,\ 0 \le \underline{c} \le c \le \bar{c},\ 0 \le \underline{h} \le h \le \bar{h} \equiv 1 \tag{2.23}$$

on $\mathbb{R}$. Moreover, $c > 0$ and $0 < f, h < 1$ over $\mathbb{R}$, and as $z \to -\infty$, we have

$$f(z) = \begin{cases} \mathcal{O}(e^{\lambda_1 z}), & if\ \ v > v^*, \\ \mathcal{O}(-ze^{\lambda_1 z}), & if\ \ v = v^*. \end{cases} \tag{2.24}$$

Proof Let $\{l_n\}_{n\in\mathbb{N}}$ be an increasing sequence in $\mathbb{R}$ such that $l_1 > \max\{|z_1|, |z_2|\}$ and $l_n \to \infty$ as $n \to \infty$, and let (f_n, c_n, h_n) be a triple of functions in $X \cap Y$ satisfying (2.14) with $l = l_n$ and (2.21) on $\mathbb{R}$. For any fixed $N \in \mathbb{N}$, since the function $\bar{c}$ is bounded above in $[-l_N, l_N]$, it follows from (2.21) that the sequences $\{f_n\}_{n\ge N}$, $\{c_n\}_{n\ge N}$, $\{h_n\}_{n\ge N}$, $\{af_n(1 - f_n - c_n)\}_{n\ge N}$, $\{c_n - c_n(f_n + c_n) + s(f_n + c_n)h_n\}_{n\ge N}$, and $\{bh_n(1 - h_n) - g(f_n + c_n)h_n\}_{n\ge N}$ are uniformly bounded in $[-l_N, l_N]$. Then we can Lemma 3.3 of Fu (2014) to infer that the sequences

$$\{f_n'\}_{n\ge N}, \{c_n'\}_{n\ge N} \text{ and } \{h_n'\}_{n\ge N}$$

are also uniformly bounded in $[-l_N, l_N]$. Using (2.14), we can express f_n'', c_n'', and h_n'' in terms of f_n, c_n, h_n, f_n', c_n', and h_n'. Differentiating (2.14), we can use the resulting equations to express f_n''', c_n''', and h_n''' in terms of f_n, c_n, h_n, f_n', c_n', h_n', f_n'', c_n'', and h_n''. Consequently, the sequences

$$\{f_n''\}_{n\ge N}, \{c_n''\}_{n\ge N}, \{h_n''\}_{n\ge N}, \{f_n'''\}_{n\ge N}, \{c_n'''\}_{n\ge N}, \text{ and } \{h_n'''\}_{n\ge N}$$

are uniformly bounded in $[-l_N, l_N]$. With the aid of Arzela–Ascoli theorem and a diagonal process, we can get a subsequence of $\{(f_n, c_n, h_n)\}_{n\in\mathbb{N}}$ which converges uniformly to a triple of functions $(f, c, h) \in C^2(\mathbb{R}) \times C^2(\mathbb{R}) \times C^2(\mathbb{R})$ in any compact interval of $\mathbb{R}$ as $n \to \infty$. Then it is easy to see that (f, c, h) is a nonnegative solution of system (1.6) and satisfies (2.23) over $\mathbb{R}$. Together with definitions of $(\bar{f}, \bar{c}, \bar{h})$ and $(\underline{f}, \underline{c}, \underline{h})$, we get (2.22) and (2.24).

Furthermore, we claim that $0 < f, h < 1$ and $c > 0$ over $\mathbb{R}$. For contradiction, we assume that $f(\tilde{z_1}) = 0$ for some $\tilde{z_1} \in \mathbb{R}$. Then $f'(\tilde{z_1}) = 0$. Therefore the uniqueness theory of ordinary differential equations gives that $f \equiv 0$, which contradicts the fact that $f \geq \underline{f} > 0$ on $(-\infty, z_1)$. Hence $f > 0$ over $\mathbb{R}$. Similarly, we also have $h > 0$ over $\mathbb{R}$. Now suppose $c(\tilde{z_2}) = 0$ for some $\tilde{z_2} \in \mathbb{R}$. Then $c'(\tilde{z_2}) = 0$ and $c''(\tilde{z_2}) \geq 0$, which contradicts (1.6b) with $z = \tilde{z_2}$. Hence $c > 0$ over $\mathbb{R}$. To prove $f < 1$ over $\mathbb{R}$, we also use a contradictory argument and assume that $f(\tilde{z_3}) = 1$ for some $\tilde{z_3} \in \mathbb{R}$. In this case, $f'(\tilde{z_3}) = 0$ and $f''(\tilde{z_3}) \leq 0$. This contradicts (1.6a) with $z = \tilde{z_3}$. Hence $f < 1$ over $\mathbb{R}$. By a similar way, we also have $h < 1$ over $\mathbb{R}$. This completes the proof of this lemma. □

3 Existence of traveling waves

In this section, we show that semi-waves established in Lemma 2.13 are actually traveling waves. To see this, let (f, c, h) be a semi-wave of system (1.6) obtained in Lemma 2.13. We will verify that the limit $(f, c, h)(\infty)$ exists and $f + c < 1 + s$ on $\mathbb{R}$. To this end, we set $\mathbf{F} := f + c$. Then (1.6) becomes

$$f'' - vf' + a(1 - \mathbf{F})f = 0, \tag{3.1a}$$
$$\mathbf{F}'' - v\mathbf{F}' + \mathbf{F}(1 + sh - \mathbf{F}) + (a - 1)(1 - \mathbf{F})f = 0, \tag{3.1b}$$
$$dh'' - vh' + bh(1 - h) - g\mathbf{F}h = 0. \tag{3.1c}$$

Throughout this section, we will retain the notations f, c, $\mathbf{F}$, and h.

3.1 Auxiliary lemmas

In this section, we will use the Barbălat's Lemma (Barbălat 1959) to show the convergence of semi-waves. For readers' convenience, we state the Barbălat's lemma below.

Lemma 3.1 [Barbălat's Lemma of Barbălat (1959)] *Suppose $w \in C^1(b, \infty)$ and $\lim_{t\to\infty} w(t)$ exists. If w' is uniformly continuous, then $\lim_{t\to\infty} w'(t) = 0$.*

The following *a priori* estimates for the second-order differential equations are useful in the remainder of the proof.

Lemma 3.2 *Let B be a positive number and $G \in C(\mathbb{R})$. Suppose that $w \in C^2(\mathbb{R})$ is a solution of*

$$w'' - Bw' = G(z) \tag{3.2}$$

in $\mathbb{R}$. If w and G are bounded in $\mathbb{R}$, then so are w' and w''. Moreover,

$$\|w'\|_{L^\infty(\mathbb{R})} \leq \frac{\|G\|_{L^\infty(\mathbb{R})}}{B} \tag{3.3}$$

and

$$\|w''\|_{L^\infty(\mathbb{R})} \leq 2\|G\|_{L^\infty(\mathbb{R})}. \tag{3.4}$$

Proof We claim that (3.3) holds. For contradiction, we assume that there exists $\tilde{z_0} \in \mathbb{R}$ such that

$$|w'(\tilde{z_0})| > \frac{\|G\|_{L^\infty(\mathbb{R})}}{B}.$$

Multiplying (3.2) by e^{-Bz} and then integrating both sides of the resulting equation from $\tilde{z_0}$ to z, we get that for all $z \geq \tilde{z_0}$,

$$e^{-Bz}w'(z) - e^{-B\tilde{z_0}}w'(\tilde{z_0}) = \int_{\tilde{z_0}}^{z} e^{-B\xi}G(\xi)d\xi$$

and therefore,

$$\begin{aligned}
|w'(z)| &\geq e^{Bz}\left(e^{-B\tilde{z_0}}|w'(\tilde{z_0})| - \int_{\tilde{z_0}}^{z} e^{-B\xi}|G(\xi)|d\xi\right)\\
&\geq e^{Bz}\left(e^{-B\tilde{z_0}}|w'(\tilde{z_0})| - \|G\|_{L^\infty(\mathbb{R})}\int_{\tilde{z_0}}^{z} e^{-B\xi}d\xi\right)\\
&\geq e^{B(z-\tilde{z_0})}\left(|w'(\tilde{z_0})| - \frac{\|G\|_{L^\infty(\mathbb{R})}}{B}\right)\\
&\geq |w'(\tilde{z_0})| - \frac{\|G\|_{L^\infty(\mathbb{R})}}{B} > 0 \quad \forall z \geq \tilde{z_0},
\end{aligned}$$

which contradicts the boundedness of w. Hence (3.3) holds. Finally, (3.4) follows from (3.2) and (3.3). The proof of the lemma is thus completed. □

To gain more details about the profile of **F**, we need the following estimate.

Lemma 3.3 *Let D, v and k be positive constants. Suppose that $u : \mathbb{R} \to \mathbb{R}$ is of class C^2 in (z_u, ∞), and satisfies*

$$Du_{zz} - vu_z - ku \leq 0 \quad on\, [z_u, \infty)$$

for some $z_u \in \mathbb{R}$ and $u(z) \to 0$ as $z \to \infty$. Then we have

$$u(z) \geq u(z_u)e^{\frac{v-\sqrt{v^2+4Dk}}{2D}(z-z_u)} \quad for\, z \geq z_u.$$

Proof The proof is motivated by the argument of (Berestycki et al. 2009, Proposition 4.1). For any $l > 0$, we consider the boundary value problem

$$\begin{cases} D\phi^l_{zz} - v\phi^l_z - k\phi^l = 0, & z \in (z_u, z_u + l),\\ \phi^l(z_u) = u(z_u),\ \phi^l(z_u + l) = u(z_u + l). \end{cases}$$

A direct calculation gives

$$\phi^l(z) = \Big(\frac{u(z_u + l) - u(z_u)e^{\rho^- l}}{e^{\rho^+ l} - e^{\rho^- l}}\Big)e^{\rho^+(z-z_u)} + \Big(\frac{u(z_u)e^{\rho^+ l} - u(z_u + l)}{e^{\rho^+ l} - e^{\rho^- l}}\Big)e^{\rho^-(z-z_u)}$$

with $\rho^\pm = (v \pm \sqrt{v^2 + 4Dk})/2D$. From the u-equation and comparison principle, it follows that

$$u(z) \geq \phi^l(z) \quad \text{for } z \in [z_u, z_u + l].$$

Note that $u(z_u + l) \to 0$ as $l \to \infty$. Then, by fixing $z > z_u$ and taking the limit of the above inequality as $l \to \infty$, $u(z) \geq u(z_u)e^{\rho^-(z-z_u)}$ holds for $z \geq z_u$. This completes the proof. □

3.2 Basic properties of total farmers $\mathbf{F}(\cdot)$

Lemma 3.4 $0 < \mathbf{F}(z) < 1 + s$ *for all* $z \in \mathbb{R}$.

Proof By (3.1b) and the fact that $h(z) \in (0, 1)$ for $z \in \mathbb{R}$, we see that $\mathbf{F}'' > 0$ as long as $\mathbf{F}' \geq 0$ and $\mathbf{F} \geq 1 + s$. Recall that $\mathbf{F}(-\infty) = 0$. For contradiction, we assume that $\tilde{z_1} := \inf\{z \in \mathbb{R}|\ \mathbf{F}(z) \geq 1 + s\}$ exists. Then, by checking the standard first-order and second-order derivatives condition at any critical point, we find that the first critical point on the right of $\tilde{z_1}$ is necessarily a strict local minimum. Since this is impossible, $\mathbf{F} > 1 + s$ and $\mathbf{F}' > 0$ in $(\tilde{z_1}, \infty)$.

We claim that

$$\mathbf{F}' < \frac{v}{2}\mathbf{F} \tag{3.5}$$

in $(\tilde{z_1}, \infty)$. To see this, we consider the function $\psi := \mathbf{F}' - (v/2)\mathbf{F}$. For contradiction, suppose that there exists $\tilde{z_2} > \tilde{z_1}$ such that $\psi(\tilde{z_2}) \geq 0$. Recall that $h(z) \in (0, 1)$ for $z \in \mathbb{R}$. Then since

$$\begin{aligned}
\psi' - \frac{v}{2}\psi &= \mathbf{F}'' - v\mathbf{F}' + \frac{v^2}{4}\mathbf{F} \\
&= \mathbf{F}(\mathbf{F} - 1 - sh) + (a - 1)(\mathbf{F} - 1)f + \frac{v^2}{4}\mathbf{F} \\
&\geq \mathbf{F}(\mathbf{F} - 1 - s) + (a - 1)(\mathbf{F} - 1)f + \frac{v^2}{4}\mathbf{F},
\end{aligned}$$

it follows that $\psi'(z) - \frac{v}{2}\psi(z) > 0$ for all $z \geq \tilde{z_2}$. Together with the fact that $\psi(\tilde{z_2}) \geq 0$, one can easily verify that $\psi(z) > 0$ for all $z > \tilde{z_2}$. So we have $\mathbf{F}'(z) > \frac{v}{2}\mathbf{F}(z)$ for all $z > \tilde{z_2}$, which implies that $\mathbf{F}(z) > e^{\frac{v}{2}(z-\tilde{z_2})}\mathbf{F}(\tilde{z_2})$ for all $z > \tilde{z_2}$. On the other hand, if $v > v^*$, then it follows from (2.23) and definition of $\bar{f}$ and $\bar{c}$ that $\mathbf{F}(z) \leq \bar{f}(z) + \bar{c}(z) \leq (A + 1)e^{\lambda_1 z}$ over $\mathbb{R}$. Since $\lambda_1 < \frac{v}{2}$, we get a contradiction. Similarly, if $v = v^*$, then $\mathbf{F}(z) \leq \bar{f}(z) + \bar{c}(z) \leq (A + 1)e^{\nu z}$ for all $z > 0$. Since $\nu < \lambda_1 = v/2$, we get a contradiction again. Hence (3.5) holds.

Now we pick a number $\tilde{z_3} > \tilde{z_1}$. Then $\mathbf{F}'(\tilde{z_3})/\mathbf{F}(\tilde{z_3}) > 0$. By (3.1b) and the fact that $\mathbf{F}(z) > 1 + s$ for $z > \tilde{z_1}$ and $h(z) \in (0, 1)$ for $z \in \mathbb{R}$, we know that $p := \mathbf{F}'/\mathbf{F}$ satisfies

$$p' = vp + (\mathbf{F} - 1 - sh) - (a - 1)(1/\mathbf{F} - 1)f - p^2 > vp - p^2$$

in $(\tilde{z_1}, \infty)$. Then $q := 1/p$ satisfies the first-order linear differential inequality $q' + vq < 1$ in $(\tilde{z_1}, \infty)$. Hence we deduce that

$$q(z) < q(\tilde{z_3})e^{v(\tilde{z_3}-z)} + \frac{1}{v}\left(1 - e^{v(\tilde{z_3}-z)}\right)$$

and so, by $q > 0$,

$$p(z) > \frac{p(\tilde{z_3})ve^{vz}}{ve^{v\tilde{z_3}} + p(\tilde{z_3})(e^{vz} - e^{v\tilde{z_3}})}$$

for all $z > \tilde{z_3}$. Therefore, $\liminf_{z\to\infty} p(z) \geq v$, which contradicts (3.5). Hence the proof of this lemma is completed. □

Lemma 3.5 *The nth derivatives $f^{(n)}$, $\mathbf{F}^{(n)}$, and $h^{(n)}$ are bounded in $\mathbb{R}$ for all $n \in \mathbb{N}$.*

Proof Since f, $\mathbf{F}$, and h are bounded in $\mathbb{R}$, it follows from Lemma 3.2 that the derivatives f', $\mathbf{F}'$, h', f'', $\mathbf{F}''$, and h'' are bounded in $\mathbb{R}$. Differentiating (3.1a)–(3.1c), we see that f''', $\mathbf{F}'''$, and h''' are also bounded in $\mathbb{R}$. Then by induction, we can conclude that the nth derivatives $f^{(n)}$, $\mathbf{F}^{(n)}$, and $h^{(n)}$ are bounded in $\mathbb{R}$ for all $n \in \mathbb{N}$. This completes the proof of this lemma. □

For the readers' convenience, we recall that

$$\mathcal{A} := \{\mathbf{F} \in C^2(\mathbb{R}) |\ 0 < \mathbf{F} < 1 \text{ and } \mathbf{F}' > 0 \text{ over } \mathbb{R},\ \text{ and } \mathbf{F}(\infty) = 1\},$$
$$\mathcal{B} := \{\mathbf{F} \in C^2(\mathbb{R}) | \exists z_{\mathbf{F}} \in \mathbb{R} \text{ such that } \mathbf{F}(z_{\mathbf{F}}) = 1, \mathbf{F}' > 0 \text{ in } (-\infty, z_{\mathbf{F}}],\ \text{ and } \mathbf{F} > 1 \text{ in } (z_{\mathbf{F}}, \infty)\}.$$

We will show that $\mathbf{F} \in \mathcal{A} \cup \mathcal{B}$ in the following lemma.

Lemma 3.6 $\mathbf{F} \in \mathcal{A} \cup \mathcal{B}$.

Proof From (3.1b), we see that $\mathbf{F}$ cannot take a local minimum whenever $\mathbf{F} \leq 1$. Further, we claim that $\mathbf{F}$ also cannot take a local maximum whenever $\mathbf{F} \leq 1$. To see this, we will use a contradictory argument, and so assume that $\mathbf{F}$ takes a local maximum at $\tilde{z_1}$ and $\mathbf{F}(\tilde{z_1}) \leq 1$. By (3.1b) and the fact that $h > 0$ over $\mathbb{R}$, we have $\mathbf{F}''(\tilde{z_1}) < 0$. Observe that $\mathbf{F}''(z) < 0$ as long as $\mathbf{F}'(z) \leq 0$ and $0 < \mathbf{F}(z) \leq 1$. Taken together, it follows from (3.1b) that $\mathbf{F}''(z) < 0$ for all $z \geq \tilde{z_1}$, which contradicts the positivity of $\mathbf{F}$. This proves the assertion of the claim.

Since $\mathbf{F}(-\infty) = 0$ and $\mathbf{F}$ cannot take local extrema whenever $\mathbf{F} \leq 1$, it holds that either $\mathbf{F}' > 0$ and $0 < \mathbf{F} < 1$ over $\mathbb{R}$, or there exists $z_{\mathbf{F}} \in \mathbb{R}$ such that $\mathbf{F}(z_{\mathbf{F}}) = 1$, and $\mathbf{F}' > 0$ in $(-\infty, z_{\mathbf{F}})$. For the former case, $\mathbf{F}(\infty)$ exists and $\mathbf{F}(\infty) \in (0, 1]$. Moreover, by Lemma 3.5, $\mathbf{F}'$, $\mathbf{F}''$, and $\mathbf{F}'''$ are bounded over $\mathbb{R}$. Then it follows from Barbălat's lemma that $\mathbf{F}'(\infty) = \mathbf{F}''(\infty) = 0$. Thus, by (3.1b) and the fact that $h > 0$ over $\mathbb{R}$, we see that $\mathbf{F}(\infty) \notin (0, 1)$. Hence $\mathbf{F}(\infty) = 1$ and $\mathbf{F} \in \mathcal{A}$. For the latter case, we claim that $\mathbf{F} > 1$ in $(z_{\mathbf{F}}, \infty)$. For contradiction, suppose that there exists $\tilde{z_{\mathbf{F}}} \geq z_{\mathbf{F}}$ such that $\mathbf{F}(\tilde{z_{\mathbf{F}}}) = 1$ and $\mathbf{F}'(\tilde{z_{\mathbf{F}}}) \leq 0$. Then using the argument for the claim in the first paragraph, we can infer that $\mathbf{F}''(z) < 0$ for all $z \geq \tilde{z_{\mathbf{F}}}$, which contradicts the positivity of $\mathbf{F}$. Further, the argument also gives that $\mathbf{F}'(z_{\mathbf{F}}) > 0$. Thus $\mathbf{F} \in \mathcal{B}$. The proof of this lemma is thus completed. □

The following two lemmas characterize the wave profiles (f, c), which give the proofs of Propositions 1.1(ii) and 1.1(iii).

Lemma 3.7 *If* $\mathbf{F} \in \mathcal{A}$, *then* $f' > 0$ *and* $c' > 0$ *over* $\mathbb{R}$.

Proof Since $0 < \mathbf{F} < 1$ over $\mathbb{R}$, it follows from (3.1a) that

$$f'' - vf' < 0 \tag{3.6}$$

over $\mathbb{R}$. We claim that $f' > 0$ over $\mathbb{R}$. For contradiction, we assume that $f'(\hat{z_0}) \leq 0$ for some $\hat{z_0} \in \mathbb{R}$. For the case $f'(\hat{z_0}) < 0$, since (3.6) gives that $(e^{-vz} f'(z))' < 0$ over $\mathbb{R}$, it follows that for $z > \hat{z_0}$, $e^{-vz} f'(z) < e^{-v\hat{z_0}} f'(\hat{z_0})$, and therefore

$$f'(z) < e^{v(z-\hat{z_0})} f'(\hat{z_0}). \tag{3.7}$$

So $f'(z) \to -\infty$ as $z \to \infty$, which contradicts the boundedness of f'. For the case $f'(\hat{z_0}) = 0$, since (3.7) yields that $f'(z) < 0$ for $z > \hat{z_0}$, we can get a contradiction from the assertion of the previous case. Hence $f' > 0$ over $\mathbb{R}$. Similarly, since (1.6b) gives that $c'' - vc' < 0$ over $\mathbb{R}$, it implies that $c' > 0$ over $\mathbb{R}$. □

Lemma 3.8 *If* $\mathbf{F} \in \mathcal{B}$, *then there exists a* $z_f < z_{\mathbf{F}}$ *such that* $f' > 0$ *on* $(-\infty, z_f)$ *and* $f' < 0$ *on* (z_f, ∞).

Proof First, by (3.1a) and the fact that $\mathbf{F} > 1$ in $(z_{\mathbf{F}}, \infty)$, we deduce that $f'' - vf' > 0$ in $(z_{\mathbf{F}}, \infty)$. Then using the argument of Lemma 3.7, we get $f' < 0$ in $(z_{\mathbf{F}}, \infty)$.

Next, we claim that f cannot take an extreme value at $z_{\mathbf{F}}$. Suppose that this is not true. Then by (3.1a) and the fact that $\mathbf{F}(z_{\mathbf{F}}) = 1$, we have $f'(z_{\mathbf{F}}) = f''(z_{\mathbf{F}}) = 0$. From the definition of the set $\mathcal{B}$, we have $\mathbf{F}'(z_{\mathbf{F}}) > 0$. Then by differentiating (3.1a), we get $f'''(z_{\mathbf{F}}) = af(z_{\mathbf{F}})\mathbf{F}'(z_{\mathbf{F}}) > 0$. Thus, $f'(z) > 0$ for $z > z_{\mathbf{F}}$ and z close to $z_{\mathbf{F}}$. This is a contradiction to the fact that $f' < 0$ in $(z_{\mathbf{F}}, \infty)$. Thus f cannot take an extreme value at $z_{\mathbf{F}}$.

Finally, we recall that $f(-\infty) = 0$. Note that, by (3.1a), f cannot take a local minimal value at any $\bar{z} < z_{\mathbf{F}}$. Taken together, the assertion of this lemma holds. □

The following lemma provides a sufficient condition under which $\mathbf{F}$ lies in the set $\mathcal{B}$, which, together with Lemma 3.6, gives the proof of Proposition 1.1(i).

Lemma 3.9 *Suppose one of the following conditions holds:*

(i) $g \in (0, b)$;
(ii) $g \in [b, 1 + b)$ *and* $d \geq 1$.

Then $\mathbf{F}$ *must belong to the set* $\mathcal{B}$.

Proof For contradiction, we assume that $\mathbf{F} \in \mathcal{A}$. Then, by definition of the set $\mathcal{A}$, we see that $\mathbf{F}(\infty) = 1$. So it follows from Lemma 3.5 and Barbălat's lemma that $\mathbf{F}'(\infty) = 0$ and $\mathbf{F}''(\infty) = 0$. Together with (3.1b), we get that $h(\infty) = 0$.

Suppose $g \in (0, b)$. Choose a $\tilde{g} \in (g/b, 1)$. Since $\mathbf{F}(\infty) = 1$ and $h(\infty) = 0$, we can find a $z_h \gg 1$ such that $h'(z_h) < 0$ and $(g/b)\mathbf{F}(z) + h(z) - 1 < \tilde{g} - 1 < 0$ for all $z \geq z_h$. Then, by (3.1c), we have $h''(z_h) < 0$. Moreover, if $z \geq z_h$, then $h''(z) < 0$ as long as $h'(z) \leq 0$. Taken together, it follows that $h''(z) < 0$ for all $z \geq z_h$, which contradicts the positivity of h. Hence $\mathbf{F} \in \mathcal{B}$.

Suppose $g \in [b, 1+b)$ and $d \geq 1$. Since $\mathbf{F} \in \mathcal{A}$, it follows that $\mathbf{F}(z) \in (0, 1)$ for $z \in \mathbb{R}$. Set $f_0 := \lim_{z\to\infty} f(z)$ and $w(z) := sh(z) - (\mathbf{F}(z) - 1) - (a-1)(f(z) - f_0)$ for $z \in \mathbb{R}$. Fix a $\epsilon_0 \in (0, \frac{1+b-g}{1+b})$. From $h(\infty) = 0$, $\mathbf{F}(\infty) = 1$, and $g \in [b, 1+b)$, we can choose a large z_h such that the inequalities

$$\begin{cases} |h(z) + (g/b)(\mathbf{F}(z) - 1)| < \epsilon_0, \ |w(z)| < \epsilon_0, \\ b(\frac{g}{b} - 1) - 1 + (1+b)\epsilon_0 < 0, \end{cases} \tag{3.8}$$

hold for $z \geq z_h$.

Set $\tilde{F} := \mathbf{F} - 1$ on $\mathbb{R}$. Then using (3.1c) and (3.8), h satisfies the inequality

$$dh'' - vh' - b((g/b) - 1 + \epsilon_0)h = b(h + (g/b)\tilde{F} - \epsilon_0)h \leq 0$$

for $z \geq z_h$. Then an application of Lemma 3.3 to the above inequality yields

$$h(z) \geq h(z_h)e^{\rho_h(z-z_h)} \quad \text{for } z \geq z_h, \tag{3.9}$$

where $\rho_h = (v - \sqrt{v^2 + 4db[(g/b) - 1 + \epsilon_0]})/2d < 0$.

Note that $\tilde{F}(z) < 0$ for $z \in \mathbb{R}$. Then using (3.1b) and the second inequality of (3.8), $\tilde{F}$ satisfies the inequality

$$\tilde{F}'' - v\tilde{F}' - (1 + (a-1)f_0 - \epsilon_0)\tilde{F} = -sh - (w(z) - \epsilon_0)\tilde{F} < -sh$$

for $z \geq z_h$, which, together with (3.9), yields

$$\tilde{F}'' - v\tilde{F}' - (1 + (a-1)f_0 - \epsilon_0)\tilde{F} + sh(z_h)e^{\rho_h(z-z_h)} < 0 \tag{3.10}$$

for $z \geq z_h$. Now set

$$k = -\frac{sh(z_h)}{\rho_h^2 - v\rho_h - (1 + (a-1)f_0 - \epsilon_0)} \quad \text{and} \quad \rho_{\tilde{F}} = \frac{v - \sqrt{v^2 + 4(1 + (a-1)f_0 - \epsilon_0)}}{2}.$$

In view of $d \geq 1$, it follows from the third inequality of (3.8) that

$$\begin{aligned} \rho_h^2 - v\rho_h - (1 + (a-1)f_0 - \epsilon_0) &\leq d\rho_h^2 - v\rho_h - (1 + (a-1)f_0 - \epsilon_0) \\ &= [b(\frac{g}{b} - 1) - 1 + (1+b)\epsilon_0] - (a-1)f_0 < 0. \end{aligned}$$

Then $k > 0$, and, due to $d \geq 1$, we have $0 > \rho_h > \rho_{\tilde{F}}$. Now $P(z) = ke^{\rho_h(z-z_h)}$ satisfies the equation

$$P'' - vP' - (1 + (a-1)f_0 - \epsilon_0)P = 0$$

for $z \geq z_h$, which, together with (3.10), yields

$$(\tilde{F} - P)'' - v(\tilde{F} - P)' - (1 + (a-1)f_0 - \epsilon_0)(\tilde{F} - P) < 0$$

for $z \geq z_h$. It then follows from Lemma 3.3 that

$$(\tilde{F} - P)(z) \geq (\tilde{F} - P)(z_h)e^{\rho_{\tilde{F}}(z-z_h)}$$

holds for $z \geq z_h$. This in turn implies that $\mathbf{F}(z) \geq 1 + ke^{\rho_h(z-z_h)} + (\tilde{F} - P)(z_h)e^{\rho_{\tilde{F}}(z-z_h)}$ for $z \geq z_h$. Since $k > 0$ and $0 > \rho_h > \rho_{\tilde{F}}$, we have $\mathbf{F}(z) > 1$ for all $z \geq \bar{z}$ and for some large $\bar{z} \geq z_h$. This contradicts the assumption that $\mathbf{F}(z) < 1$ for $z \in \mathbb{R}$. The proof of the lemma is thus completed. □

3.3 Traveling waves for the case $g \in (0, b)$

Recall that $c^* = b(1+s)/(b+sg)$ and $h^* = (b-g)/(b+sg)$. In order to show that $(f, c, h)(\infty) = (0, c^*, h^*)$, we define the Lyapunov functional $\mathcal{L}$ by $\mathcal{L}(f, \mathbf{F}, h)(z) := \mathcal{L}_1(f, \mathbf{F}, h)(z) + (s/g) \cdot \mathcal{L}_2(f, \mathbf{F}, h)(z)$, where

$$\begin{aligned}\mathcal{L}_1(f, \mathbf{F}, h)(z) &:= -\mathbf{F}' + v\mathbf{F} + c^*\left(\frac{\mathbf{F}'}{\mathbf{F}}\right) - c^* v \ln\left(\frac{\mathbf{F}}{c^*}\right)\\ &\quad - (a-1)\int_0^z \left(1 - \frac{c^*}{\mathbf{F}(\theta)}\right) f(\theta)(1 - \mathbf{F}(\theta))d\theta\end{aligned}$$

and

$$\mathcal{L}_2(f, \mathbf{F}, h)(z) := -dh' + vh + h^*d\left(\frac{h'}{h}\right) - h^*v\ln\left(\frac{h}{h^*}\right).$$

Along the solution $\chi(z) := (f(z), \mathbf{F}(z), h(z))$ of (3.1), we use (3.1b) and (3.1c) to deduce

$$\begin{aligned}\frac{d}{dz}\mathcal{L}_1(\chi(z)) &= -(\mathbf{F}'' - v\mathbf{F}') + c^*\left[\frac{\mathbf{F}'' - v\mathbf{F}'}{\mathbf{F}} - \left(\frac{\mathbf{F}'}{\mathbf{F}}\right)^2\right] - \left(1 - \frac{c^*}{\mathbf{F}}\right)(a-1)f(1-\mathbf{F})\\ &= \left(\frac{c^*}{\mathbf{F}} - 1\right)(\mathbf{F}'' - v\mathbf{F}') - c^*\left(\frac{\mathbf{F}'}{\mathbf{F}}\right)^2 - \left(1 - \frac{c^*}{\mathbf{F}}\right)(a-1)f(1-\mathbf{F})\\ &= \left(1 - \frac{c^*}{\mathbf{F}}\right)[\mathbf{F}(1 + sh - \mathbf{F}) + (a-1)f(1-\mathbf{F})] - c^*\left(\frac{\mathbf{F}'}{\mathbf{F}}\right)^2\\ &\quad - \left(1 - \frac{c^*}{\mathbf{F}}\right)(a-1)f(1-\mathbf{F})\end{aligned}$$

$$= s(\mathbf{F} - c^*)(h - h^*) - (\mathbf{F} - c^*)^2 - c^* \left(\frac{\mathbf{F}'}{\mathbf{F}}\right)^2,$$

and

$$\begin{aligned}\frac{d}{dz}\mathcal{L}_2(\chi(z)) &= -(dh'' - vh') + h^* \left[\frac{dh'' - vh'}{h} - d\left(\frac{h'}{h}\right)^2\right] \\ &= \left(\frac{h^*}{h} - 1\right)(dh'' - vh') - h^* d\left(\frac{h'}{h}\right)^2 \\ &= b(h - h^*)(1 - h - (g/b)\mathbf{F}) - h^* d\left(\frac{h'}{h}\right)^2 \\ &= -g(\mathbf{F} - c^*)(h - h^*) - b(h - h^*)^2 - h^* d\left(\frac{h'}{h}\right)^2.\end{aligned}$$

Thus, we have

$$\frac{d}{dz}\mathcal{L}(\chi(z)) = -(\mathbf{F} - c^*)^2 - c^* \left(\frac{\mathbf{F}'}{\mathbf{F}}\right)^2 - \frac{sb}{g}(h - h^*)^2 - \frac{sh^* d}{g}\left(\frac{h'}{h}\right)^2 \leq 0.$$

In the following, we will use the Barbălat's Lemma to show that $\frac{d}{dz}\mathcal{L}(\chi(z)) \to 0$ as $z \to \infty$, from which $\mathbf{F}(\infty) = c^*$ and $h(\infty) = h^*$ follow.

Lemma 3.10 *Suppose $g \in (0, b)$. Then $(f, c, h)(\infty) = (0, c^*, h^*)$.*

Proof Since $g \in (0, b)$, it follows from Lemma 3.9 that $\mathbf{F} \in \mathcal{B}$. We divide the proof into five steps.
Step 1: we claim that $f(\infty) = 0$. By Lemmas 3.8 and 3.9, $f(\infty)$ exists. So it follows from Lemma 3.5 and Barbălat's Lemma that $f'(\infty) = f''(\infty) = 0$. By (3.1a), we have $f(\infty) = 0$ or $\mathbf{F}(\infty) = 1$. From the proof of Lemma 3.9 for the case $g < b$, we see that $\mathbf{F}(\infty) \neq 1$. Hence $f(\infty) = 0$.
Step 2: we claim that h'/h is bounded over $\mathbb{R}$. Since $0 < \mathbf{F} < 1 + s$ and $0 < h < 1$ on $\mathbb{R}$, it follows that $g\mathbf{F} + b(h - 1) < C_0$, where $C_0 := g(1 + s)$. Let $q := h'/h$. Then by (3.1c), we have

$$dq' = vq - dq^2 + g\mathbf{F} + b(h - 1) < vq - dq^2 + C_0 = -d(q - q^-)(q - q^+), \quad (3.11)$$

where $q^- := (v - \sqrt{v^2 + 4C_0 d})/(2d) < 0$ and $q^+ := (v + \sqrt{v^2 + 4C_0 d})/(2d) > 0$.

We claim that $q < q^+$ over $\mathbb{R}$. Note that $q(-\infty) = 0$ due to the fact that $h(-\infty) = 1$ and $h'(-\infty) = 0$. If the claim is not true, then $\check{z}_q := \inf\{z \in \mathbb{R} | \ q(z) \geq q^+\}$ exists. Obviously, $q(\check{z}_q) = q^+$ and $q'(\check{z}_q) \geq 0$. On the other hand, by (3.11), we have that $q'(\check{z}_q) < -(q(\check{z}_q) - q^-)(q(\check{z}_q) - q^+) = 0$ which is a contradiction. Hence $q < q^+$ over $\mathbb{R}$.

Now we fix a $q_0 < q^-$ such that

$$v\xi - d\xi^2 + C_0 < -d\xi^2/2 \quad \forall \xi \leq q_0. \quad (3.12)$$

We claim that $q > q_0$ over $\mathbb{R}$. For contradiction, we assume that there exists $z_q \in \mathbb{R}$ such that $q(z_q) \le q_0 < q^-$. Since (3.11) gives that $q' < 0$ as long as $q < q^-$, it follows that $q < q_0$ in (z_q, ∞). Using (3.11) and (3.12), we get $q' < -(1/2)q^2$, which leads to

$$q(z) < \tilde{q}(z) := \frac{2q(z_q)}{q(z_q)(z - z_q) + 2}$$

for $z > z_q$. Note that $\tilde{q}(z) \to -\infty$ as $z \to \tilde{z_q}^-$, where $\tilde{z_q} := z_q - 2/q(z_q) > z_q$. This contradicts the global existence of q. Hence $q_0 < q$.

Finally, since q is bounded over $\mathbb{R}$, the assertion of the claim is established.

Step 3: we claim that $\int_0^z (1 - c^*/\mathbf{F}(\theta))\, f(\theta)(1 - \mathbf{F}(\theta))d\theta$ *is bounded in* $(z_{\mathbf{F}}, \infty)$. Integrating both sides of (3.1a) from $z_{\mathbf{F}}$ to z and rearranging the resulting equation, we get

$$a \int_{z_{\mathbf{F}}}^{z} f(\eta)(\mathbf{F}(\eta) - 1)d\eta = f'(z) - f'(z_{\mathbf{F}}) - v(f(z) - f(z_{\mathbf{F}})).$$

Since $f(\mathbf{F} - 1)$ is positive in $(z_{\mathbf{F}}, \infty)$, and f' and f are bounded in $(z_{\mathbf{F}}, \infty)$, the above equality gives that $f(\mathbf{F} - 1)$ lies in $L^1(z_{\mathbf{F}}, \infty)$. Finally, since $|1 - c^*/\mathbf{F}(\theta)| \le 1 + c^*/\mathbf{F}(\theta) \le 1 + c^*$ for $\theta \ge z_{\mathbf{F}}$, it follows that

$$|\int_{z_{\mathbf{F}}}^{z} \left(1 - \frac{c^*}{\mathbf{F}(\theta)}\right) f(\theta)(1 - \mathbf{F}(\theta))d\theta| \le (1 + c^*) \int_{z_{\mathbf{F}}}^{\infty} f(\theta)(\mathbf{F}(\theta) - 1)d\theta < \infty.$$

Hence $\int_0^z (1 - c^*/\mathbf{F}(\theta))\, f(\theta)(1 - \mathbf{F}(\theta))d\theta$ is bounded in $(z_{\mathbf{F}}, \infty)$.

Step 4: we claim that h has a positive lower bound. For contradiction, we assume that $\inf_{z > z_{\mathbf{F}}} h(z) = 0$. Recall from Lemmas 3.4 and 3.5 that $1 < \mathbf{F} < 1 + s$ and $\mathbf{F}'$ is bounded in $(z_{\mathbf{F}}, \infty)$. It follows that $\mathbf{F}'/\mathbf{F}$ and $\ln(\mathbf{F}/c^*)$ are bounded in $(z_{\mathbf{F}}, \infty)$. Recall that $h, h', h'/h$, and $\int_0^z (1 - c^*/\mathbf{F}(\theta))\, f(\theta)(1 - \mathbf{F}(\theta))d\theta$ are also bounded in $(z_{\mathbf{F}}, \infty)$. Taken together, we see that all terms of $\mathcal{L}(\chi(z))$ except $-h^* v \ln\left(\frac{h}{h^*}\right)$ are bounded in $(z_{\mathbf{F}}, \infty)$. Therefore $\inf_{z > z_{\mathbf{F}}} h(z) = 0$ implies that $\sup_{z > z_{\mathbf{F}}} \mathcal{L}(\chi(z)) = \infty$, which contradicts the fact that $\mathcal{L}(\chi(z)) \le \mathcal{L}(\chi(z_{\mathbf{F}})) < \infty$ in $(z_{\mathbf{F}}, \infty)$.

Step 5: we claim that $c(\infty) = c^*$ and $h(\infty) = h^*$. Since $h < 1$, it follows that $-h^* v \ln\left(\frac{h}{h^*}\right)$ is bounded below in $(z_{\mathbf{F}}, \infty)$. So we conclude that $\mathcal{L}(\chi(z))$ is bounded below in $(z_{\mathbf{F}}, \infty)$. Together with the fact that $\mathcal{L}(\chi(z))$ is nonincreasing in z, we see that $\lim_{z \to \infty} \mathcal{L}(\chi(z))$ exists. Besides, Lemma 3.5, step 2, step 4, and the boundedness of $(f, \mathbf{F}, h)$ yield that $\frac{d}{dz}\mathcal{L}(\chi(z))$ and $\frac{d^2}{dz^2}\mathcal{L}(\chi(z))$ are bounded in $(z_{\mathbf{F}}, \infty)$. Hence, by applying the Barbălat's Lemma, we get $\frac{d}{dz}\mathcal{L}(\chi(z)) \to 0$ as $z \to \infty$, which leads to $\mathbf{F}(\infty) = c^*$ and $h(\infty) = h^*$. Hence $c(\infty) = \mathbf{F}(\infty) - f(\infty) = c^*$. The proof of this lemma is completed. □

Now Lemmas 2.13, 3.4 and 3.10 together give the following lemma, which also provides the proofs of Theorem 1.1 for $g \in (0, b)$.

Lemma 3.11 *Suppose $g \in (0, b)$. If $v \geq v^*$, then system (1.6) and (1.7) admits a positive solution (f, c, h) with $E_\infty = (0, c^*, h^*)$. Moreover, (f, c, h) has the following properties:*

(i) $0 < f, h < 1$, $c > 0$, *and* $0 < f + c < 1 + s$ *over* $\mathbb{R}$.
(ii) *As* $z \to -\infty$, (2.24) *holds.*

3.4 Traveling waves for the case $g \geq b$

The following lemma and Lemma 2.13 together give the proofs of Theorem 1.1 for $g \geq b$.

Lemma 3.12 *Suppose $g \geq b$. Then $(f, c, h)(\infty)$ exists with $f(\infty) + c(\infty) = 1$ and $h(\infty) = 0$. Moreover, if $g > b$, then $f(\infty)$ and $c(\infty)$ are positive.*

Proof Recall that $\mathbf{F} \in \mathcal{A} \cup \mathcal{B}$. Suppose $\mathbf{F} \in \mathcal{A}$. Since $\mathbf{F}(\infty) = 1$, it follows from Barbălat's lemma and Lemma 3.5 that $\mathbf{F}'(\infty) = 0$ and $\mathbf{F}''(\infty) = 0$. Together with (3.1b), we get that $h(\infty) = 0$. By Lemma 3.7, $f' > 0$ and $c' > 0$ over $\mathbb{R}$ and so $f(\infty)$ and $c(\infty)$ exist. Further, due to $\mathbf{F}(\infty) = 1$ and $\mathbf{F} = f + c$, we have $f(\infty) + c(\infty) = 1$.

Suppose $\mathbf{F} \in \mathcal{B}$. By (3.1c) and the fact that $h \in (0, 1)$ and $\mathbf{F} > 1$ in $(z_\mathbf{F}, \infty)$, and $g \geq b$, we have $dh'' - vh' > 0$ in $(z_\mathbf{F}, \infty)$. Then using the argument of Lemma 3.7, we get $h' < 0$ in $(z_\mathbf{F}, \infty)$ and therefore $h(\infty)$ exists. Thus by Barbălat's lemma and Lemma 3.5, we have $h'(\infty) = 0$ and $h''(\infty) = 0$. Together with (3.1c), we get that either $h(\infty) = 0$ or $1 - h(\infty) - (g/b)\mathbf{F}(\infty) = 0$. Indeed, since $g \geq b$, $(g/b)\mathbf{F}(\infty) \geq 1$ and then $1 - h(\infty) - (g/b)\mathbf{F}(\infty) \leq -h(\infty)$. Hence $h(\infty) \neq 0$ implies $h(\infty) < 0$, which is obviously contradictory, so that $h(\infty) = 0$.

Since $f'' - vf' > 0$ in $(z_\mathbf{F}, \infty)$, which implies that $f' < 0$ in $(z_\mathbf{F}, \infty)$ and therefore $f(\infty)$ exists. Then by Barbălat's lemma and Lemma 3.5, we have $f'(\infty) = 0$ and $f''(\infty) = 0$. Together with (3.1a), we get $f(\infty) = 0$ or $\mathbf{F}(\infty) = 1$. We claim that $\mathbf{F}(\infty) = 1$. For contradiction, we assume $\limsup_{z\to\infty} \mathbf{F}(z) > 1$. Then $f(\infty) = 0$ and there are two cases for $\mathbf{F}$: either $\mathbf{F}(\infty)$ exists with $\mathbf{F}(\infty) > 1$ or $1 \leq \liminf_{z\to\infty} \mathbf{F}(z) < \limsup_{z\to\infty} \mathbf{F}(z)$. For the former case, we can use Barbălat's lemma and Lemma 3.5 to get that $\mathbf{F}'(\infty) = 0$ and $\mathbf{F}''(\infty) = 0$. Together with (3.1b), we get a contradiction. For the latter case, we set $\zeta := \limsup_{z\to\infty} \mathbf{F}(z) > 1$. Then there exists a sequence of numbers $\{\hat{z_n}\} \nearrow \infty$ such that $\mathbf{F}'(\hat{z_n}) = 0$, $\mathbf{F}''(\hat{z_n}) \leq 0$, and $\lim_{n\to\infty} \mathbf{F}(\hat{z_n}) = \zeta$. Together with (3.1b) and the fact that $f(\infty) = h(\infty) = 0$, we get a contradiction again. Hence $\mathbf{F}(\infty) = 1$. Recall that $f(\infty)$ exists. Taken together, we see that $c(\infty)$ also exists and $f(\infty) + c(\infty) = \mathbf{F}(\infty) = 1$.

Now we claim that $f(\infty)$ and $c(\infty)$ are positive if $g > b$. Suppose $\mathbf{F} \in \mathcal{A}$. Since $f' > 0$ and $c' > 0$ over $\mathbb{R}$, it follows that $f(\infty)$ and $c(\infty)$ are positive. Suppose $\mathbf{F} \in \mathcal{B}$. We first claim that $\mathbf{F} - 1 \in L^1(z_\mathbf{F}, \infty)$. By (3.1c), we have

$$dh'' - vh - (g - b)h = bh^2 + gh(\mathbf{F} - 1) > 0$$

in $[z_{\mathbf{F}}, \infty)$. Using the argument of Lemma 3.3, we deduce that

$$h(z) \leq h(z_{\mathbf{F}}) \exp\left(\frac{v - \sqrt{v^2 + 4d(g-b)}}{2d} \cdot (z - z_{\mathbf{F}})\right)$$

for all $z \in [z_{\mathbf{F}}, \infty)$. Hence $h \in L^1(z_{\mathbf{F}}, \infty)$. Now we integrate (3.1a) from $z_{\mathbf{F}}$ to ∞ to get

$$a \int_{z_{\mathbf{F}}}^{\infty} f(1 - \mathbf{F}) dz = f'(z_{\mathbf{F}}) - vf(z_{\mathbf{F}}).$$

By (3.1b) and Lemma 3.4, we have

$$\mathbf{F}(\mathbf{F} - 1) \leq \mathbf{F}'' - v\mathbf{F}' + s(1+s)h + (a-1)f(1 - \mathbf{F}).$$

Integrating the above inequality from $z_{\mathbf{F}}$ to ∞ and noting $\mathbf{F}(z_{\mathbf{F}}) = \mathbf{F}(\infty) = 1$, we get

$$\begin{aligned}\int_{z_{\mathbf{F}}}^{\infty} (\mathbf{F} - 1) dz \leq \int_{z_{\mathbf{F}}}^{\infty} \mathbf{F}(\mathbf{F} - 1) dz \leq -\mathbf{F}'(z_{\mathbf{F}}) + s(1+s) \int_{z_{\mathbf{F}}}^{\infty} h dz \\ + (a-1) \int_{z_{\mathbf{F}}}^{\infty} f(1 - \mathbf{F}) dz < \infty.\end{aligned}$$

Hence $\mathbf{F} - 1 \in L^1(z_{\mathbf{F}}, \infty)$ holds.

Next, we claim that $f'/f > -L$ for some positive constant L sufficiently large such that $f'(z_{\mathbf{F}}) + Lf(z_{\mathbf{F}}) > 0$, $vL > as$, and $L^2 + vL - as > 0$.

Let

$$\Phi(z) := f'(z) + Lf(z).$$

It suffices to show that $\Phi(z) > 0$ for all $z \geq z_{\mathbf{F}}$. Note that $\Phi(z_{\mathbf{F}}) > 0$. For contradiction, we assume that there exists $\hat{z_1} > z_{\mathbf{F}}$ such that $\Phi(\hat{z_1}) = 0$ and $\Phi'(\hat{z_1}) \leq 0$. Then there are two possibilities: either

$$\Phi(z) \leq 0 \quad \forall z \geq \hat{z_1} \tag{3.13}$$

or

$$\Phi(\hat{z_2}) = 0 \text{ and } \Phi'(\hat{z_2}) \geq 0, \tag{3.14}$$

for some $\hat{z_2} \geq \hat{z_1}$. For the first case, (3.13) gives

$$f'(z) \leq -Lf(z) \quad \forall z \geq \hat{z_1}.$$

Together with the fact that $\mathbf{F} < 1 + s$, we deduce from (1.6a) that

$$f''(z) = vf'(z) + a(\mathbf{F}(z) - 1)f(z) < (as - vL)f(z) < 0 \quad \forall z \geq \hat{z_1},$$

which implies that f' is decreasing in $[\hat{z_1}, \infty)$. Hence $f'(z) \le f'(\hat{z_1}) \le -Lf(\hat{z_1}) < 0$ for all $z \ge \hat{z_1}$, which contradicts the boundedness of f. For the second case, (3.14) yields that

$$f'(\hat{z_2}) = -Lf(\hat{z_2}) < 0 \ \text{ and } \ f''(\hat{z_2}) \ge -Lf'(\hat{z_2}) > 0. \tag{3.15}$$

Using (1.6a), we deduce that

$$\begin{aligned} 0 &= f''(\hat{z_2}) - vf'(\hat{z_2}) + a(1 - \mathbf{F}(\hat{z_2}))f(\hat{z_2}) \\ &\ge (L^2 + vL - as)f(\hat{z_2}) \quad (\text{by } (3.15), \text{ and the fact that } \mathbf{F} < 1 + s) \\ &> 0 \quad (\text{by definition of } L), \end{aligned}$$

which is a contradiction again. Hence $f'/f > -L$.

Finally, by dividing (1.6a) by f and then integrating the resulting equation from $z_{\mathbf{F}}$ to z, we get

$$\frac{f'(z)}{f(z)} - \frac{f'(z_{\mathbf{F}})}{f(z_{\mathbf{F}})} + \int_{z_{\mathbf{F}}}^{z} \left(\frac{f'}{f}\right)^2 dz - v \ln f(z) + v \ln f(z_{\mathbf{F}}) = a \int_{z_{\mathbf{F}}}^{z} (\mathbf{F} - 1) dz.$$

If $f(\infty) = 0$, then $\int_{z_{\mathbf{F}}}^{z} (\mathbf{F} - 1)dz = \infty$, a contradiction. Hence $f(\infty) > 0$. On the other hand, by Lemma 3.8, we infer that $f(\infty) < 1$. Together with the fact that $f(\infty) + c(\infty) = 1$, we get $c(\infty) > 0$. The proof is thus completed. $\square$

Now Lemmas 2.13, 3.12 and 3.4 together give the following lemma, which also provides the proofs of Theorem 1.1 for $g \ge b$.

Lemma 3.13 *Suppose $g \ge b$. If $v \ge v^*$, then system (1.6) and (1.7) admits a positive solution (f, c, h) with $E_\infty = (f^\natural, c^\natural, 0)$, where $f^\natural$ and $c^\natural$ are nonnegative constants with $f^\natural + c^\natural = 1$. In addition, $f^\natural$ and $c^\natural$ are positive if $g > b$, and (f, c, h) satisfies the following properties:*

(i) *$0 < f, h < 1$, $c > 0$, and $0 < f + c < 1 + s$ over $\mathbb{R}$.*
(ii) *As $z \to -\infty$, (2.24) holds.*

Finally, by the following two lemmas we show the monotonicity of h, which completes the proof of Proposition 1.1(iv).

3.5 Monotonicity of *h*

Lemma 3.14 *Suppose $g \ge b$. Then $h' < 0$ over $\mathbb{R}$.*

Proof First, since $0 < h < 1$ on $\mathbb{R}$ and $h(-\infty) = 1$, we have $h'(z) < 0$ for $z < 0$ and large $-z$. We claim that $h' \le 0$ over $\mathbb{R}$. To establish the claim, we use a contradictory argument. Thus we assume $\tilde{z_0} := \inf\{z \in \mathbb{R} | \ h'(z) > 0\}$ exists. Then $h'(\tilde{z_0}) = 0$ and $h''(\tilde{z_0}) \ge 0$, and from (3.1c), it follows that $g\mathbf{F}(\tilde{z_0}) + bh(\tilde{z_0}) \ge b$. On the other hand, since $h(\infty) = 0$, we can find a $\tilde{z_1} > \tilde{z_0}$ such that $h' > 0$ on $(\tilde{z_0}, \tilde{z_1})$ and $h'(\tilde{z_1}) = 0$. Thus, we have $h''(\tilde{z_1}) \le 0$, and due to (3.1c), $g\mathbf{F}(\tilde{z_1}) + bh(\tilde{z_1}) \le b$. Since $g \ge b$, $\mathbf{F}(\tilde{z_1}) \in (0, 1)$ holds. Recall that $\mathbf{F}(z)$ is increasing as long as $\mathbf{F}(z) \in (0, 1)$. Thus we can deduce that $\mathbf{F}(\tilde{z_1}) > \mathbf{F}(\tilde{z_0})$. Together with the fact that $h(\tilde{z_1}) > h(\tilde{z_0})$, if follows that $g\mathbf{F}(\tilde{z_1}) + bh(\tilde{z_1}) > g\mathbf{F}(\tilde{z_0}) + bh(\tilde{z_0}) \geq b$. This is a contradiction, and thus $h' \leq 0$ over $\mathbb{R}$.

Further, we claim that $h' < 0$ over $\mathbb{R}$. For contradiction, we assume that $h'(\tilde{z_2}) = 0$ for some $\tilde{z_2} \in \mathbb{R}$. Then $h''(\tilde{z_2}) = 0$. Recall that $\mathbf{F} \in \mathcal{A} \cup \mathcal{B}$. Suppose $\mathbf{F} \in \mathcal{A}$. Then $\mathbf{F}' > 0$ over $\mathbb{R}$. Differentiating (3.1c) and using the fact $h'(\tilde{z_2}) = h''(\tilde{z_2}) = 0$ and $\mathbf{F}'(\tilde{z_2}) > 0$, we get $h'''(\tilde{z_2}) > 0$, which gives that h'' is increasing in $(\tilde{z_2} - \delta, \tilde{z_2})$ for some $\delta > 0$. Together with $h''(\tilde{z_2}) = 0$, we obtain that $h'' < 0$ in $(\tilde{z_2} - \delta, \tilde{z_2})$ and so h' is decreasing in $(\tilde{z_2} - \delta, \tilde{z_2})$. Together with $h'(\tilde{z_2}) = 0$, we conclude that $h' > 0$ in $(z_0 - \delta\, \tilde{z_2})$, a contradiction. Suppose $\mathbf{F} \in \mathcal{B}$. Recall that $\mathbf{F}' > 0$ in $(-\infty, z_{\mathbf{F}})$ and $\mathbf{F} \geq 1$ in $[z_{\mathbf{F}}, \infty)$. Suppose $\tilde{z_2} < z_{\mathbf{F}}$. Then following the proof for the case $\mathbf{F} \in \mathcal{A}$, we get a contradiction. Suppose $\tilde{z_2} \geq z_{\mathbf{F}}$. Then using (3.1c) and the fact that $h > 0$ and $h'(\tilde{z_2}) = h''(\tilde{z_2}) = 0$, we get that $b = bh(\tilde{z_2}) + g\mathbf{F}(\tilde{z_2})$. Together with the fact that $\mathbf{F}(\tilde{z_2}) \geq 1$ and $h(\tilde{z_2}) > 0$, we get $b > g$, which contradicts the assumption that $g \geq b$. So the proof of this lemma is completed. □

Lemma 3.15 *Suppose $g < b$. Then $h' < 0$ in $(-\infty, z_{\mathbf{F}})$.*

Proof First, we claim that $h' \leq 0$ in $(-\infty, z_{\mathbf{F}})$. For contradiction, we assume $\tilde{z_0} := \inf\{z < z_{\mathbf{F}} |\, h'(z) > 0\}$ exists. Arguing as the proof of Lemma 3.14, we have $g\mathbf{F}(\tilde{z_0}) + bh(\tilde{z_0}) \geq b$. Together with $\mathbf{F}(\tilde{z_0}) \leq 1$, we get $h(\tilde{z_0}) \geq (b - g)/b > h^*$. In addition, since $h(\infty) = h^*$, we can find a $\tilde{z_1} > \tilde{z_0}$ such that $h' > 0$ on $(\tilde{z_0}, \tilde{z_1})$ and $h'(\tilde{z_1}) = 0$, and $g\mathbf{F}(\tilde{z_1}) + bh(\tilde{z_1}) \leq b$. So $g\mathbf{F}(\tilde{z_1}) + bh(\tilde{z_1}) \leq g\mathbf{F}(\tilde{z_0}) + bh(\tilde{z_0})$. Together with $h(\tilde{z_0}) < h(\tilde{z_1})$, we get $\mathbf{F}(\tilde{z_1}) < \mathbf{F}(\tilde{z_0}) \leq 1$, which contradicts the fact that $\mathbf{F}(z)$ is increasing as long as $\mathbf{F}(z) \in (0, 1)$. Hence $h' \leq 0$ in $(-\infty, z_{\mathbf{F}})$.

Further, following the proof of Lemma 3.14 for the case $\mathbf{F} \in \mathcal{A}$, we can show that $h' < 0$ in $(-\infty, z_{\mathbf{F}})$. The proof of this lemma is therefore completed. □

4 Conclusion

We have shown that (1.3) possesses two different types of traveling wave solutions $(f, c, h)(z)$ $(z = x + vt)$ with velocity v under the boundary conditions

$$(f, c, h)(-\infty) = (0, 0, 1) \text{ and } (f, c, h)(+\infty) = E_\infty,$$

defined by (1.7).

For any $a > 0$, there exists a traveling wave solution $(0, c, h)$ of (1.3) where (c, h) satisfies

$$\begin{aligned} vc_z &= c_{zz} + c(1 - c) + sch, \\ vh_z &= dh_{zz} + bh(1 - h) - gch, \end{aligned} \quad -\infty < z < +\infty$$

subject to the boundary conditions $(c, h)(-\infty) = (0, 1)$ and $(c, h)(+\infty) = e_\infty$, as in (1.8). In fact, it is shown by Tsai et al. (2020) that the above traveling wave problem admits positive solutions for arbitrary $v \geq 2\sqrt{1 + s}$ where $2\sqrt{1 + s}$ is called the minimal velocity. When $a < 1 + s$, Figs. 1, 2, 3 and 4 suggest that the expansion of farmers is given by the converted farmers propagating with velocity $2\sqrt{1 + s}$.

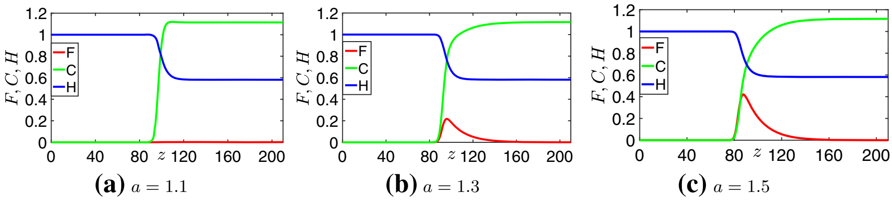

Fig. 8 Traveling wave solution (F, C, H) of system (1.6) and (1.7) with the minimal velocity, where the parameters are $d = 1.2$, $s = 0.2$, $b = 0.8$, and $g = 0.3$ and a varies and is given in the caption

On the other hand, when $a > 1 + s$, there exists another traveling wave solution of (1.3) and (1.7) for $v \geq 2\sqrt{a}$ which is a triple of functions (f, c, h) with nonzero f-component. Here we note that for $a > 1 + s$, although this type of traveling wave solutions consist of three components (f, c, h), the associated minimal velocity (or spreading velocity) is given by $2\sqrt{a}$. Surprisingly, this velocity is the same as the minimal velocity of traveling wave solutions of the Fisher–KPP equation (Fisher 1937; Kolmogorov et al. 1937)

$$F_t = F_{xx} + a(1 - F)F.$$

Therefore, this suggests that the propagating velocity of the farmers is dominated by the original farmers.

In the abstract, two questions (Q1) and (Q2) are raised. For the question (Q1), we can infer the following from the wave profiles of traveling wave solutions of the model (1.3):

(1) When $a < 1+s$, the original farmer F fades out, and so only the converted farmer C dominates the expansion into the region occupied by hunter-gatherers.
(2) When $a > 1 + s$, both of the original and converted farmers are involved in the expansion of farming populations into the region occupied by hunter-gatherers.

Next, for the question (Q2), we showed the two possibilities on the asymptotical behavior of hunter-gatherer populations: (i) The population becomes extinct completely when $g \geq b$; and (ii) it continues to live in lower density when $g < b$.

We finally note that the family of traveling wave solutions $(0, c, h)$ with the minimal velocity $2\sqrt{1 + s}$ exists for any $a > 0$, while the other family of traveling wave solutions (f, c, h) with the minimal velocity $2\sqrt{a}$ exists for $a > 1 + s$. We may say that $(0, c, h)$ is a trivial traveling wave solution, while (f, c, h) is a non-trivial one. Now, we numerically study the stability property of the trivial traveling wave solutions. We consider (1.3) in a finite but rather long interval with Neumann boundary conditions. The initial function (F_0, C_0, H_0) is specified in the way that F_0 is a small and compactly supported perturbation of $F = 0$, and (C_0, H_0) is almost approximated by the trivial traveling wave solutions. Then, for $a < 1 + s$, the F-component of the solution (F, C, H) tends to zero and (C, H) tends to the trivial traveling wave solution $(c, h)(x + vt)$ with $v = 2\sqrt{1 + s}$, while for $a > 1 + s$, the solution (F, C, H) tends to the non-trivial traveling wave solution with velocity $v = 2\sqrt{a}$, as if the non-trivial traveling wave solution with minimal velocity was bifurcated from the trivial one with minimal velocity, as shown in Figure 4.1. Of course, this conjecture is obtained by numerical speculation. So, we should discuss it analytically. It is left for our future work.

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