國立政治大學風險管理與保險學系

博士學位論文

資產負債管理之研究分析

Essays on Asset and Liability Management Analysis

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中文摘要

本研究由三篇關於保險業資產負債管理議題的論文所構成。本文第二章檢視在台灣地區銷售之典型利率變動型壽險之公平定價問題。假設資產過程滿足 Heston 隨機變動模型、利率過程爲 CIR 模型，保險給付將爲一系列遠期起點期權之總和。本文就台灣財務市場之資料進行模型參數估計，再利用蒙地卡羅法計算契約公平價格，同時計算風險值 (VaR, ES)。本文第三章闡述國際板債券評價系統的實作細節。台灣保險業總資產近兩成之國際板債券在 IFRS-9 會計準則下非爲純債務工具，必須以公允價值衡量。在此我們敘述以美國固定期限公債收益率或美元 LIBOR 及 ICE 利率交換率校正的利率期限結構，配合芝加哥期貨交易所的歐式利率交換選擇權隱含波動度資料估計 Hull-White 短期利率模型之評價理論細節，並使用開放原始碼程式語言 Python 與函式庫 QuantLib 及三元樹演算法實作國際板債券評價系統。除與櫃買中心系統價格輸出結果相比較外，我們展示本系統在給定利率期限結構與市場現有商品規格下可贖回債券期初價值與隱含年利率、不可贖回期間與可贖回頻率關係之計算。本文第四章探討 copula-GARCH 模型在變額年金保證價值計算上的應用。有效的風險管理前提在於推估各種資產間的機率關係，並計算反映系統狀態的各種定量指標的能力。現代計算技術的進步使得更符合實際、不須過份簡化的多變量機率模型運用變爲可能，而 copula 正是如此的多變量機率模型。結合 GARCH 時間序列模型，我們利用一系列基於無母數統計與經驗過程理論的穩健統計檢定方法，針對給定 S&P500 與 S&P600 指數時間序列選擇並匹配最適 copula-GARCH 模型，進而推估變額年金保證價值。

關鍵詞： 利率變動型壽險、隨機變動模型、蒙地卡羅模擬、國際板債券、變額年金、copula-GARCH。

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Abstract

This study focuses on the management of the three most challenging topics life insurers in Taiwan currently face, namely the life insurance policy with the highest annual gross premium income, the dominating asset on the life insurer's balance sheet, and the development of a model which faithfully captures the dependency between multiple underlying assets within the life insurer's portfolio. We first examine the fair pricing of interest rate sensitive life insurance policies that are commonly sold in Taiwan. With the reference portfolio following Heston's stochastic volatility process, the payoff function of these policies consists of a series of forward-start options. Although the option to surrender are standard features of these policies, policyholders incur heavy penalties should they exercise such option. Given certain policyholder behaviour, we study the impact of the minimum guaranteed interest rate, and the annually declared bonus rate on the issuing company's solvency. The need for pricing transparency and a reliable source of reference is of utmost importance in view of the sheer volume of the international bonds listed on the Taipei Exchange that the life insurers in Taiwan hold and the lack of a liquid secondary market. We provide the life insurers the means to evaluate the mark-to-market value of these callable bonds without having to rely on third parties to do so. We are able to collate publicly available data and make use of open source software to construct a bespoke system that can independently price the international bonds. The copula concept with its multivariate time-series model generalization, namely the copula-GARCH model, and robust statistical inference procedures based on the empirical processes theory are investigated in depth. A vast majority of existing literature on applications of copula often makes assumptions without justification or conducts inadequate statistical tests for verifications. Here we demonstrate what we believed to be the preferred way of using copula for financial and risk management applications by the detailed valuation of guarantees embedded in variable annuities with multiple underlying assets.

Keywords: interest sensitive life insurance, stochastic volatility, Monte-Carlo simulation, international bond, variable annuities, copula-GARCH

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1 Introduction

This study focuses on the management of the three most challenging topics life insurers in Taiwan currently face, namely the life insurance policy with the highest annual gross premium income, the dominating asset on the life insurer's balance sheet, and the development of a model which faithfully captures the dependency between multiple underlying assets within the life insurer's portfolio.

Following the discontinuation of participating policies as instructed by the Financial Supervisory Commission (FSC) of Taiwan, the new generation of alternative products such as the interest rate sensitive life (ISL) policy has emerged. Due to the quick rollout of ISLs, their pricing are based on the already discontinued participating policies. However, the life insurers did not anticipate it to take up over 40% of their product line, the need to properly price the product has become a main concern for life insurers. Traditional actuarial approach of policy design and evaluation is fundamentally deterministic, therefore it is no longer adequate when encountered with rapidly changing economic climate. We thus adopt a stochastic interest rate model which follows the Cox-Ingersoll-Ross (CIR) process, and the stock model obeys the Heston stochastic volatility process. Based on these premises, our ISL policy pricing employ Monte Carlo methodology to simulate and assess whether the bonus declaration mechanism or product structure is sustainable.

As required by current legislation, assets that are not classified as pure debt instrument must be reported at fair value by life insurers in Taiwan. Over one fifth of the life insurer's assets are allocated to international bonds which are listed on the Taipei Exchange (TPEx). As a result, the fair pricing of these international bonds are of utmost importance. To the best of our knowledge, the few available publications regarding the pricing of these international bonds are less than transparent and lack technical details, furthermore, there exists anomalies in the published price. In this part of the research, we collate and summarize the theories pertaining to the fair pricing of typical international bonds, and develop a software based on these principles.

Developments of modern computing technologies have enabled the transition from traditional simplistic models to full-fledged stochastic ones with real-world considerations;

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multivariate probability models that can faithfully characterize their elements are in ever greater need. Given that the life insurer's portfolio consists of an array of various underlying assets, it is crucial to be able to find a model best suited to quantify the dependency between them and effectively manage such risks. The copula is such a multivariate probability distribution for which the marginal probability distribution of each variable is uniform. The copula approach could be useful to high-dimensional statistical applications as one is allowed to estimate the distribution of random vectors by estimating marginals and copula separately. For time series applications the copula concept could be extended to reflect the model dynamics. In this part of the research, we review the essence of the copula-GARCH model and the associated statistical tests; as an illustration, we show that the co-movement of the monthly S&P500 and S&P600 indices is best described by a certain copula-GARCH model and applying said probability model for the evaluation of corresponding variable annuity product.

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Interest Sensitive Life Policy

∗ This chapter is based on Hsuan & Chang (2018b).

2.1 Introduction

The sharp decline of interest rates and the most recent 2008 financial crisis had elicited discussions and attention on the management, fair valuation, and default risks of interest sensitive life insurance policies, where most of these type of policies offer an explicit interest rate guarantee that the policyholder's account will be credited on an annual basis, together with any excess return from the reference portfolio. In the case of participating life policies, this excess return may be in the form of a certain percentage, say 70% of the performance on the reference portfolio or in the case of interest sensitive life policies, the positive difference between the return on the reference portfolio and that of the guaranteed interest rate.

The study of this chapter will focus on the life insurance policy with cliquet-style guarantees that has the highest annual gross premium income in Taiwan, namely, the interest sensitive life (ISL) insurance policy, under which the minimum surrender conditions are dictated by law and its guidelines; however, our analysis can also be applied to similar insurance markets that offer life insurance policies with minimum interest rate guarantees and / or with an initial surrender restriction period with penalties. Given the exceptional growth in new businesses written on ISLs and their share with respect to the entire product line of Taiwanese life insurers of 5%, 19%, and 44% in 2012, 2013, and 2014 respectively [1], it is imperative that the financial risks on these ISLs are fairly assessed.

The traditional life insurance policies in Taiwan had predominantly been participating or with-profits policies. Following the stock market drop after the financial crisis caused by the burst of the dot-com bubble and the sharp decline in the interest rate environment, from over 6% pre-2001 to under 2% in 2004 on the 2-year fixed deposit rate, market interest rates have sunk to new lows. Life insurers were unable to meet the mandatory

[1] According to the data from the Life Insurance Association of the Republic of China (LIAROC).

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requirements, such as the 70%[2] participating rate on excess profits to be distributed over the years on the participating policies. The Financial Supervisory Commission (FSC) of Taiwan then instructed the life insurers to discontinue any sales of participating policies in 2004. Given the generally low interest rates and the investors' search for yield, the status quo in the life insurance market is that the life insurers started offering alternative products, such as ISL policies which has a guaranteed minimum interest rate and returns that are much more competitive than fixed term deposits offered by the banks. The minimum guaranteed interest rate offered on the ISLs are usually on par with the market interest rates. However, these life insurers offering products with minimum guarantees are also under additional pressure due to the regulation imposing the limit and restriction on the percentage and type of asset class it can invest in[3].

Extensive studies has been done on the fair valuation and risk management of life insurance policies with minimum interest rate guarantees, for example, Briys & de Varenne (1994, 1997), Bacinello (2001); Bacinello & Ortu (1996), Grosen & Jørgensen (2002); Jensen et al. (2001), Miltersen & Persson (2003), Tanskanen & Lukkarinen (2003), Barbarin & Devolder (2005), Bauer et al. (2006), Bernard et al. (2006), Gatzert & Kling (2007), Kling et al. (2007), and Graf et al. (2011) to cite a few.

To the best of our knowledge, with the ISLs being a relatively new product that was launched in 2012, the life insurers had relied on past experiences on the participating life policies and the actuarial method to price the ISLs. In this study we apply the arbitrage free pricing methodology to the ISL policies that are currently in-force in Taiwan. In particular, our interest rate structure follows the framework of Cox et al. (1985) and the reference portfolio follow a stochastic volatility process as described in Heston (1993). As illustrated in Wilmott (2002), one would potentially underestimate the cost of the underlying when only constant volatility is considered. The parameters of both the Cox et al. (1985) and Heston (1993) are derived using Taiwan market data.

2.2 The interest sensitive life policy

Although similar to participating policies at first glance, ISLs in Taiwan can generally be characterized as an endowment, whole life, or pension (retirement) with a guaranteed interest rate, namely the "headline rate" and the potential to earn in excess of the guaranteed rate. The shortest maturity currently seen on the endowment and retirement plans are 13 years, and the longest is 30 years. Currently these ISL policies also offer maturity or living benefits, death and funeral expense benefit, and disability benefit as standard policy packages.

There is also an imposed vesting period of six years on the ISLs where the policyholder

[2]This percentage was established by the Financial Supervisory Commission (FSC) of Taiwan.
[3]See Article 146 of the Insurance Act of the Republic of China.

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incur penalties on the early surrendering of the contract. The cost of the early surrender is determined by Article 119 of the Insurance Act [4] where the minimum surrender value *"... may not be less than three-quarters of the non-forfeiture value that the proposer is entitled to receive"*, together with the minimum surrender charge levied on the ISLs being at least 1% per annum for a minimum of six years, as set forth by the applicable Orders, and Directions of the FSC.

The guaranteed interest rate is the minimum annual return set at the inception of the contract. This contractual guaranteed interest rate would remain fixed for the coverage period or the term of the contract. Any reference portfolio return with excess over the guaranteed interest rate would be credited to the policyholder's account. As such, these products with interest rate guarantees are not only sensitive to interest rate movements, but also the returns achievable under the prevailing market conditions.

2.3 The model framework

More recently, Gatzert & Schmeiser (2013) has provided a comprehensive overview of the different forms of traditional and innovative new life insurance products and Graf et al. (2012) in the methodologies in assessing these products that are commonly found in old-age provision products in practice.

Although the participating policies are not in issue anymore, however, they also offer minimum interest rate guarantees and are similar in nature to the ISLs. Given that the majority of the life insurance policies currently sold in Taiwan are ISLs, for the purpose of simplicity, we will categorise the existing participating policies as ISLs and assume that the balance sheet of a life insurance company would be a reflection of the assets and liabilities of the ISL. It is a statutory requirement that life insurance policies in Taiwan offer products with 100% guarantee of the contributions made. Hence, we do not consider products offering less or more than 100% guarantee.

Furthermore, we also assume that the policyholder will survive till the end of the contract's term with certainty, thus we do not consider mortality risk in our model.

2.3.1 The asset model

Policies underwritten by life insurers are often of a long-term nature, and it is the life insurer's obligation to manage its assets adequately to ensure its ability to meet future policyholder claims. It is not uncommon for life insurers to hold certain positions in fixed-income type assets and other risky assets on their balance sheet. We thus use the following three classic asset classes, bonds, stocks, and money market account in

[4] `http://law.tii.org.tw/Eng/FLAWDAT0201.asp?No=1A0020001&lsid=FL006746&hasChar=True&btnType=0&rlType=`

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constructing a generic asset portfolio for the purpose of this study.

It is implied that the models and frameworks adopted in this chapter are functions of time, t, unless otherwise stated. For example, we will drop the time index subscript and write $r = r_t$, and $Z_r = Z_{r,t}$ for notational convenience.

The value of the money market account M grows according to

$$\mathrm{d}M = rM\mathrm{d}t, \tag{2.3.1}$$

where r is the risk-free interest rate. The interest rate r follows the Cox et al. (1985) (CIR) process, and under the risk neutral probability measure, the interest rate dynamics of the CIR process is read as

$$\mathrm{d}r = \kappa_r(\theta_r - r)\,\mathrm{d}t + \sigma_r\sqrt{r}\,\mathrm{d}Z_r, \tag{2.3.2}$$

where $\kappa_r, \theta_r > 0$ represents the speed of adjustment and the long-run mean of the interest rate respectively, σ_r denotes the interest rate volatility, and Z_r denotes a Wiener process.

The bond price $B(t,T)$ at time t with maturity T in the CIR model is

$$B(t,T) = b_1(t,T)\exp\{-b_2(t,T)r\},$$

where (cf. Brigo & Mercurio (2006)(3.25))

$$\begin{aligned}
b_1(t,T) &= \left(\frac{2he^{((\kappa_r+h)(T-t))/2}}{2h + (\kappa_r + h)\left(e^{h(T-t)} - 1\right)}\right)^{\frac{2\kappa_r\theta_r}{\sigma_r^2}}, \\
b_2(t,T) &= \frac{2\left(e^{h(T-t)} - 1\right)}{2h + (\kappa_r + h)\left(e^{h(T-t)} - 1\right)}, \\
h &= \sqrt{\kappa_r^2 + 2\sigma_r^2}.
\end{aligned}$$

The differential form is written as

$$\frac{\mathrm{d}B(t,T)}{B(t,T)} = r\,\mathrm{d}t - b_2(t,T)\sigma_r\sqrt{r}\,\mathrm{d}Z_r. \tag{2.3.3}$$

We let the stock price follow a stochastic volatility process as described in Heston (1993). Then under the risk-neutral measure, the process can be express as

$$\mathrm{d}S = \mu S\,\mathrm{d}t + \sqrt{\nu}S\,\mathrm{d}Z_S, \tag{2.3.4}$$

$$\mathrm{d}\nu = \kappa_\nu\left(\theta_\nu - \nu\right)\,\mathrm{d}t + \sigma_\nu\sqrt{\nu}\,\mathrm{d}Z_\nu, \tag{2.3.5}$$

where μ is the long-term mean or the drift process of the asset price, ν is the variance of the underlying asset price, which is a random variable. κ_ν is the mean reverting speed for

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the variance. θ_ν is the mean reversion level for the variance. The correlation coefficient between Z_S and Z_ν is ρ, while Z_r is independent of Z_S and Z_ν.

Following from the above, the reference asset portfolio A_t thus consists of bonds, stocks and money market account. We assume the life insurer invests a constant proportion of w_B in bonds, w_S in stocks, and the balance w_M in the money market account. These proportions are kept constant by continuous rebalancing, and $w_B + w_S + w_M = 1$. Let ϕ_B denote the number of units the life insurer holds in bonds, ϕ_S be the number of units held in stock and ϕ_M the number of units held in the money market account. This yields, $w_B = \frac{\phi_B B(t,T)}{A}$, $w_S = \frac{\phi_S S}{A}$ and $w_M = \frac{\phi_M M}{A}$. Thus we get $A = \phi_B B(t,T) + \phi_S S + \phi_M M$.

We further assume the life insurer's reference asset portfolio is self-financing, thus we obtain $\mathrm{d}A = \phi_B \mathrm{d}B(t,T) + \phi_S \mathrm{d}S + \phi_M \mathrm{d}M$. The dynamics of the reference asset portfolio, which is also the portfolio return r_A of the ISL can then be written as

$$r_A = \frac{\mathrm{d}A}{A} = w_B \frac{\mathrm{d}B(t,T)}{B(t,T)} + w_S \frac{\mathrm{d}S}{S} + w_M \frac{\mathrm{d}M}{M}. \tag{2.3.6}$$

2.3.2 The liability model

Let us consider a generic set up of an ISL product. The life insurer provides an annual minimum guaranteed interest rate for the term of the policy at inception; furthermore, at the policy's annual anniversary any excess return generated from its reference portfolio is distributed to its policyholders at the discretion of the life insurer's management. Thus, the annual portfolio return credited to the policyholders' account cannot be less than the guaranteed interest rate as stated in the contract. Any living or death benefit received would be the higher of the account value or a predetermined multiple of the initial premium paid.

In practice, there is a lock-in period for these ISLs, and ranges between 6 to 10 years, where the surrender charge during this period is much higher than the portfolio return credited, which alternatively acted as a deterrent for early surrenders. This is not dissimilar to an European cliquet option with a maturity of T-years. The use of options to price corporate liabilities or life insurance contracts are not of a foreign nature, as can be seen in Black & Scholes (1973), Brennan & Schwartz (1976), Grosen & Jørgensen (1997, 2000, 2002), Bacinello (2001), and Bauer et al. (2006) etc. We thus use the fair valuation of a European cliquet option as our point of departure for the valuation of our liabilities as opposed to the actuarial approach.

Let P be the lump sum premium paid at the inception of the contract or policy and L be the liability at time t, such that the life insurer's liability is equivalent to the premium paid by the policyholder $P = L_0$. r_P is the policy interest rate credited to the policy account in year t; it is determined at each of the valuation dates i, $i =$

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$1, 2, \ldots, n$, as the lesser of the policy portfolio's asset return, r_A, in additional to any market related adjustments, ζ, and the policy portfolio's performance benchmark, K, plus any benchmark adjustments ξ. Define $r_{P,t} = \max\left(\min\left(K + \xi,\, r_{A,t} - \zeta\right),\, r_G\right)$ or $r_{P,t} = r_G + \max\left(\min\left(K + \xi,\, r_{A,t} - \zeta\right) - r_G, 0\right)$, then the policy interest rate and its relation with the liability can be expressed as

$$L_T = L_0 \left\{ 1 + \sum_{t=1}^{n} r_{P,t} \right\}. \tag{2.3.7}$$

$r_{P,t}$ is guaranteed to never fall below r_G, the guaranteed interest rate, which is specified in the policy contracts, and that $r_G \in [0.75\%, 1.50\%]$ as seen of the guaranteed interest rates currently declared amongst the life insurers in Taiwan. Depending on the life insurer's investment strategy, the benchmark K of these ISL policies can be as short-dated as the 2-year fixed deposit rate to one with a longer term such as Taiwan's 10-year Government Bond. The market and benchmark adjustors $\zeta \in [0.00\%, 7.00\%]$ and $\xi \in [-3.00\%, 3.00\%]$ can vary considerably and are exercised at the management's discretion.

2.3.2.1 The liability reserve

In practice, the ISLs are subject to monthly valuations and bonus declarations, we thus implement this in our liability model set up. Let the value of the liability reserve (or the cost of writing an ISL) be p per unit dollar insured. Under the risk neutral pricing principle, this value of the bonus option at time 0 can be expressed as (cf. Graf et al. (2012))

$$\begin{aligned} p &= \mathsf{E}^Q \left\{ \exp\left(-\int_0^T r(\tau)\,\mathrm{d}\tau \right) \cdot \max\left[\min\left(K + \xi,\, r_{A,t} - \zeta\right),\, r_G\right] \right\} \\ &= \sum_{t=1}^{12T} \left\{ \mathsf{E}^Q \left[\exp\left(-\int_0^{\frac{t}{12}} r(\tau)\,\mathrm{d}\tau \right) \cdot r_{P,\frac{t}{12}} \right] \right\}. \end{aligned} \tag{2.3.8}$$

2.3.2.2 The bonus stabilization reserve

Since the value of the bonus option is directly dependent on the investment performance in the capital market, the policyholder has the right to claim from the life insurer at maturity, the life insurer would also need to assess its ability of providing for such terms. Introducing

$$r_{s,t} = \begin{cases} r_A - r_P & r_A \geqslant r_P, \\ r_A - r_G & r_A \leqslant r_G, \\ 0 & \text{otherwise} \end{cases}$$

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we define p_s, the bonus stabilization reserve (BSR), as

$$p_s = \sum_{t=1}^{12T} \left\{ \mathsf{E}^Q \left[\exp\left(- \int_0^{\frac{t}{12}} r(\tau)\,\mathrm{d}\tau \right) \cdot r_{S,\frac{t}{12}} \right] \right\}. \tag{2.3.9}$$

For positive p_s, there exists a surplus after the distribution of the policy interest rate r_P given the level of r_G as set at the inception of the policy; for negative p_s the bonus stabilization reserve is in deficit.

2.4 Numerical simulation and illustration

2.4.1 Parameter estimation

All parameters appear in our models, namely the CIR process of interest rate and the Heston model of stock, should be estimated from actual market data before proceeding. The maximum likelihood estimation (MLE) method is applied to the parameter estimation problem of CIR process; the loss function approach is used for the Heston model.

2.4.1.1 Parameter estimation of the CIR process

The MLE method is taken from Iacus (2008), Kladıvko (2007). Given the n observations of interest rate time series $\{r_{t_i}\}, i = 1, 2, \ldots, n$ at observation time t_i with equally spaced interval Δt, the likelihood function $F(\vartheta)$, $\vartheta \equiv (\hat{\kappa_r}, \hat{\theta_r}, \hat{\sigma_r})$ is formed as

$$F(\vartheta) = \prod_{i=1}^{n} p(r_{t_{i+1}} | r_{t_i}; \vartheta), \tag{2.4.1}$$

where the conditional density $p(\cdot|\cdot)$ is (cf. Feller (1951))

$$p(r_{t_{i+1}} | r_{t_i}; \vartheta) = c\, e^{-u-v} \left(\frac{v}{u} \right)^2 I_q(2\sqrt{uv}) \tag{2.4.2}$$

with

$$c = \frac{2\hat{\kappa_r}}{\hat{\sigma_r}^2 (1 - e^{-\hat{\kappa_r}\Delta t})} \qquad u = c\, r_{t_i} e^{-\kappa_r \Delta t}$$

$$v = c\, r_{t_{i+1}} \qquad q = \frac{2\hat{\kappa_r}\hat{\theta_r}}{\hat{\sigma_r}^2} - 1$$

and I_q, the modified Bessel function of the first kind and order q. The maximizer of $F(\vartheta)$ is the sought-after parameter set.

In practice one often use the iterative Newton-type algorithm to numerically optimize $\log F(\vartheta)$; the problems of inherent overflow in the modified Bessel function implementation

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and the proper selection of the initial value must be addressed. For the latter, Kladıvko (2007) suggests the ordinary least square method which runs as follows. Discretize the CIR process as

$$r_{t_{i+1}} - r_{t_i} = \kappa_r(\theta_r - r_{t_i})\Delta t + \sigma_r \sqrt{r_{t_i}}\, \epsilon_{t_i},$$

where ϵ is normally distributed with mean 0 and variance Δt. The above can be written as

$$\frac{r_{t_{i+1}} - r_{t_i}}{\sqrt{r_{t_i}}} = \frac{\kappa_r\, \theta_r}{\sqrt{r_{t_i}}}\, \Delta t - \kappa_r \sqrt{r_{t_i}}\Delta t + \sigma_r\, \epsilon_{t_i},$$

then the initial value $(\hat{\kappa}, \hat{\theta})$ is determined by

$$(\hat{\kappa}, \hat{\theta}) = \underset{\kappa,\theta}{\operatorname{argmin}} \sum_{i=1}^{n-1} \left(\frac{r_{t_{i+1}} - r_{t_i}}{\sqrt{r_{t_i}}} - \frac{\kappa\, \theta}{\sqrt{r_{t_i}}}\, \Delta t + \kappa \sqrt{r_{t_i}}\Delta t \right) \tag{2.4.3}$$

and the initial value of $\hat{\sigma}$ is derived as the standard deviation of the residuals after substituting $(\hat{\kappa}, \hat{\theta})$.

The data used in the calibration of the CIR process consist of 3,179 available daily observations on the 10-year Government Bond of Taiwan over the period of August 2002 to January 2015, as quoted from the Taiwan Economic Journal (TEJ) DataBank.

2.4.1.2 Parameter estimation of the Heston process

We use the loss function approach to estimate the parameters in the Heston process. The loss function is defined as the error between the market prices quoted and the prices that are computed from the model; model parameters are determined to minimize the value of this loss function, so that the model price are as close as possible to the ones as observed in the market. The loss function approach is demonstrated in e.g. Gilli & Schumann (2010), chapter 6 of Rouah (2013), which we follow closely hereafter.

To be precise, the price C of the European call option under the Heston process is (cf. Bakshi & Madan (2000))

$$C = e^{-q\tau} S_0 P_1 - e^{-r\tau} X P_2 \tag{2.4.4}$$

where S_0, X, r, q and τ are the spot price, the strike price, the risk free rate, the dividend

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yield and the time to expiration, respectively; P_1, P_2 are given as

$$P_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left(\frac{e^{-iy \log X} \varphi(y-i)}{i\, y\, \varphi(i)} \right) \mathrm{d}y \tag{2.4.5}$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left(\frac{e^{-iy \log X} \varphi(y)}{i\, y} \right) \mathrm{d}y \tag{2.4.6}$$

In the above integrals $\Re(\cdot)$ denotes the real part function, and the characteristic function $\varphi(y)$ is (cf. Heston (1993))

$$\varphi(y) = \exp\left\{A_1(y) + A_2(y) + A_3(y)\right\} \tag{2.4.7}$$

where

$$A_1(y) = iyS_0 + iy(r-q)\tau \tag{2.4.8}$$

$$A_2(y) = \frac{\theta_\nu \kappa_\nu}{\sigma_\nu^2} \left(\left(\kappa_\nu - i\rho\sigma_\nu y - d\right)\tau - 2\log\left(\frac{1-ge^{-d\tau}}{1-g}\right)\right) \tag{2.4.9}$$

$$A_3(y) = \frac{\frac{\nu_0}{\sigma_\nu^2}\left(\kappa_\nu - i\rho\sigma_\nu y - d\right)\left(1-e^{-d\tau}\right)}{1-ge^{-d\tau}} \tag{2.4.10}$$

and

$$d = \sqrt{\left(i\rho\sigma_\nu y - \kappa_\nu\right)^2 + \sigma_\nu^2(iy + y^2)} \tag{2.4.11}$$

$$g = \frac{\kappa_\nu - i\rho\sigma_\nu y - d}{\kappa_\nu - i\rho\sigma_\nu y + d} \tag{2.4.12}$$

Now suppose we have a set of known market prices $C_{\mathrm{m}}(\tau_t, X_k)$ with the corresponding maturities τ_t, $t = 1, 2, \ldots, N_T$ and strikes X_k, $k = 1, 2, \ldots N_X$. Theoretical prices according to formula (2.4.4) with corresponding maturity-strike combinations and parameters to be determined $\kappa_\nu, \theta_\nu, \sigma_\nu, \nu_0, \rho$ are denoted by $C(\tau_t, X_k)$. The sought-after parameter estimation of the underlying Heston process is obtained via the minimization problem

$$\underset{\kappa_\nu, \theta_\nu, \sigma_\nu, \nu_0, \rho}{\operatorname{argmin}} \sum_{t,k} \frac{\left|C_{\mathrm{m}}(\tau_t, X_k) - C(\tau_t, X_k)\right|}{C_{\mathrm{m}}(\tau_t, X_k)} \tag{2.4.13}$$

The objective function appeared above is called the loss function. Apparently the loss function is complicated and not convex with respect to its arguments, hence traditional gradient-based methods will have difficulties finding the global minimizer. Here we adopt the differential evolution (Storn & Price (1997), Price et al. (2005)) heuristic to solve this problem. The differential evolution heuristic is a stochastic search strategy which updates the candidates by creating a new member according to some random scheme, compares the resulting objective values and selects the best among the candidates; the

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process repeats until some stopping criteria is met.

For the estimation of the Heston process, we use the data of TAIEX options [5]. The data consist of 48,938 entries, 248 trading days in 2014.

Table 2.1: Parameter definition and base values

Parameters	**Descriptions**	**Values**
r_0	Initial instantaneous interest rate	0.020
κ_r	Drift term of interest rate	0.032
θ_r	Mean reverting speed of interest rate	1.540
σ_r	Interest rate volatility	0.038
μ	Drift term of stock price	0.020
κ_ν	Long-run mean of the stock price variation	0.607
θ_ν	Mean reverting speed of the stock price variation	0.070
σ_ν	Variation of ν	0.293
ν_0	Estimated initial value of ν	0.065
ρ	Correlation coefficient of stock price and volatility	-0.757
w_B	Weight of bond	0.700
w_S	Weight of stock	0.200

2.4.2 Risk measures

The two most common risk measures are the value at risk (VaR) and the expected shortfall (ES). VaR asks the question "How bad can things get?", expected shortfall asks "If things do get bad, how much can the company expect to lose?". ES is the expected loss at the end of the period conditional on the loss being worse than the VaR loss Hull (2018). The ES is used since it includes both the probability and the severity of adverse losses in the surplus distribution. The α value at risk for a random variable Y is the α quantile of the cumulative distribution function F_Y:

$$\mathrm{VaR}_\alpha(Y) = \inf\{Y : F_Y(\mathrm{VaR}_\alpha) \geq \alpha\}.$$

The corresponding expected shortfall is given by

$$\mathrm{ES}_\alpha(Y) = \mathsf{E}\left(Y \mid Y < \mathrm{VaR}_\alpha\right).$$

[5] http://www.taifex.com.tw/enl/eng5/optIndxFSP

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2.4.3 Simulation

Based on the discrete approximations of the continuous solution of the underlying stochastic differential equations, simulation methods try to depict the process trajectory and facilitate the computation of the expected value of certain functionals of the process. Iacus (2008) provides a detailed overview of this topic.

For the simulation of the CIR process, the Euler scheme (cf. section 2.1 of Iacus (2008)) is used. However, issues arise when one tries the same approach to the simulation of the Heston process: the slow convergence of the scheme and the occurrence of negative variances. Several dedicated schemes have been developed to mitigate the problem; chapter 7 of Rouah (2013) is an in-depth survey. Here we adopt the Quadratic Exponential sampling scheme of Andersen (2008) for the Heston process.

The simulation time step Δt is $\frac{1}{250}$, and with policy maturity $T = 10$ years, $2,500$ terms are computed for each scenario; 5×10^4 scenarios are generated and stored for subsequent Monte Carlo computation, the total file size is 3.5 Gb after compression.

2.4.4 Numerical illustrations

In this section we present the results from the numerical analysis of our model, obtained via Monte Carlo simulation. Consider a policyholder that paid $P_0 = 100 = L_0$ units in premium when entering into an ISL contract with an annual guaranteed interest rate of $r_G = 1.50\%$ and a maturity of $T = 10$ years. We set these parameters as the base case of our study, in line with the ISL products on offer in Taiwan. Current r_P declared by the life insurers are in the range of $[2.65\%, 2.89\%]$. Thus, for ease of comparison and without loss of generality, the market and benchmark adjustors ζ and ξ are set to 0% and $r_P \in [2.00\%, 6.00\%]$. The effect of benchmark K had been taken into account in r_P. According to the report by the Taiwan Insurance Institute (TII), the average stock and bond holdings of life insurers are approximately 20% and 70% respectively.

The preliminary results indicate that for an ISL policy with a minimum guarantee rate of $r_G = 1.50\%$ and declaring an annual policy interest rate that is capped at $r_P = 3.00\%$ over the life of the policy, the cost of issuing such a product or the value of the liability reserve p is 18.4298 per 100 units of premium received (cf. Table 2.3 and Table 2.2); and the bonus stabilization reserve p_s of the life insurer at maturity T would be in deficit of 1.1999 (Table 2.4). In other words, for an ISL policy with a high guaranteed interest rate r_G (upper bound) and declaring a relatively high policy interest rate r_P can easily turn the life insurer into an under-funded position, which is equivalent of a loss for the issuance of such products; whereas by lowering the policy interest rate r_P by 50 basis points, say from 3.00% to 2.50%, not only does it decreases the cost of liability from 18.4298 to 16.5756 by more than 11% but also turned the insurer's bonus stabilization reserve p_s from an under-funded position into a surplus (profit) of 0.6543 per 100 units of

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premium received.

In Figure 2.1 and Figure 2.2, both the guaranteed interest rate r_G and the policy interest rate r_P appears to have a positive relation with respect to the value of liability reserves p. This is not counter-intuitive as given a certain level of guaranteed interest rate, say $r_G = 1.50\%$, one would expect the option price of the liability reserve to cost more for a policy that declares a higher policy interest rate than one that declares a lower one. This is to say, that the cost of the guarantee is higher for those ISL policies offering higher guaranteed interest rates, such as $r_G = 1.50\%$, compared to those offering no or minimal guaranteed interest rates ($r_G = 0\%$ and 0.50% respectively). The same is true for a given level of policy interest rate and different levels of guaranteed interest rate. Figure 2.3 and Figure 2.4 shows r_G and r_P are decreasing functions of the bonus stabilization reserve p_s.

Given the guaranteed interest rate of ISL products in Taiwan are in the range of 0.75% to 1.50%, and their corresponding policy interest rates to be in the range of 2.65% to 2.85%, one can also interpret Table 2.4 as a guideline to a life insurer's policy rate declaration strategy. The results suggest that for an ISL product without any interest rate guarantee ($r_G = 0\%$), the life insurer can declare up to 3.50% in policy interest without running the risk of being under-funded. However, as r_G increase, it is prudent not to over declare and over distribute its portfolio returns. Table 2.5 shows that for a life insurer offering ISL products to stay afloat, one should not over declare r_P for the specific levels of r_G stipulated in the contract.

The $\mathrm{VaR}_{95\%}$ and $\mathrm{ES}_{95\%}$ were computed and represented in Table 2.6 and Table 2.7.

Table 2.2: Liability reserve p for various levels of guaranteed interest rate r_G, and policy interest rate r_P. Unit: %.

r_P	$r_G = 0.00$	0.50	1.00	1.50	2.00
2.00	10.3908	11.5881	12.9482	14.4917	16.2434
2.50	12.4748	13.6720	15.0321	16.5756	18.3273
3.00	14.3289	15.5262	16.8863	18.4298	20.1815
3.50	15.9609	17.1582	18.5183	20.0618	21.8135
4.00	17.3832	18.5804	19.9405	21.4840	23.2357
4.50	18.6089	19.8062	21.1663	22.7098	24.4615
5.00	19.6558	20.8531	22.2132	23.7567	25.5084
5.50	20.5447	21.7420	23.1021	24.6455	26.3973
6.00	21.2931	22.4904	23.8504	25.3939	27.1456

2.5 Concluding remarks

Empirical studies have shown that an asset's log-return distribution is non-Gaussian. The fact that many popular models are still based on the assumption of normality is because of the simplicity that the Gaussian model presents. However, the use of Gaussian

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Table 2.3: Liability reserve p for various levels of policy interest rate r_P, and guaranteed interest rate r_G. Unit: %.

r_G	$r_P = 2.00$	3.00	4.00	5.00	6.00
0.00	10.3908	14.3289	17.3832	19.6558	21.2931
0.10	10.6178	14.5559	17.6101	19.8828	21.5200
0.20	10.8509	14.7889	17.8432	20.1158	21.7531
0.30	11.0903	15.0284	18.0826	20.3553	21.9926
0.40	11.3361	15.2742	18.3284	20.6011	22.2384
0.50	11.5881	15.5262	18.5804	20.8531	22.4904
0.60	11.8467	15.7848	18.8390	21.1117	22.7490
0.70	12.1118	16.0499	19.1041	21.3768	23.0140
0.80	12.3836	16.3216	19.3759	21.6485	23.2858
0.90	12.6623	16.6004	19.6546	21.9273	23.5646
1.00	12.9482	16.8863	19.9405	22.2132	23.8504
1.10	13.2413	17.1794	20.2336	22.5063	24.1435
1.20	13.5420	17.4801	20.5343	22.8070	24.4443
1.30	13.8504	17.7884	20.8427	23.1153	24.7526
1.40	14.1668	18.1049	21.1591	23.4318	25.0690
1.50	14.4917	18.4298	21.4840	23.7567	25.3939
1.60	14.8246	18.7627	21.8170	24.0896	25.7269
1.70	15.1661	19.1042	22.1584	24.4311	26.0684
1.80	15.5164	19.4545	22.5087	24.7814	26.4186
1.90	15.8754	19.8135	22.8677	25.1404	26.7777
2.00	16.2434	20.1815	23.2357	25.5084	27.1456

Table 2.4: Bonus stabilization reserve p_s for various levels of guaranteed interest rate r_G, and policy interest rate r_P. Positive reserve values (profit) are indicated in blue. Unit: %.

r_P	$r_G = 0.00$	0.50	1.00	1.50	2.00
2.00	6.8390	5.6417	4.2817	2.7382	0.9865
2.50	4.7551	3.5578	2.1977	0.6543	-1.0975
3.00	2.9009	1.7037	0.3436	-1.1999	-2.9516
3.50	1.2689	0.0716	-1.2884	-2.8319	-4.5836
4.00	-0.1533	-1.3506	-2.7106	-4.2541	-6.0059
4.50	-1.3791	-2.5764	-3.9364	-5.4799	-7.2316
5.00	-2.4260	-3.6232	-4.9833	-6.5268	-8.2785
5.50	-3.3149	-4.5121	-5.8722	-7.4157	-9.1674
6.00	-4.0632	-5.2605	-6.6206	-8.1641	-9.9158

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Table 2.5: Bonus stabilization reserve p_s for various levels of policy interest rate r_P, and guaranteed interest rate r_G. Unit: %.

r_G	$r_P = 2.00$	3.00	4.00	5.00	6.00
0.00	6.8390	2.9009	-0.1533	-2.4260	-4.0632
0.10	6.6121	2.6740	-0.3803	-2.6529	-4.2902
0.20	6.3790	2.4409	-0.6133	-2.8860	-4.5232
0.30	6.1395	2.2014	-0.8528	-3.1255	-4.7627
0.40	5.8937	1.9557	-1.0986	-3.3712	-5.0085
0.50	5.6417	1.7037	-1.3506	-3.6232	-5.2605
0.60	5.3831	1.4450	-1.6092	-3.8819	-5.5191
0.70	5.1181	1.1800	-1.8743	-4.1469	-5.7842
0.80	4.8463	0.9082	-2.1460	-4.4187	-6.0560
0.90	4.5675	0.6294	-2.4248	-4.6975	-6.3347
1.00	4.2817	0.3436	-2.7106	-4.9833	-6.6206
1.10	3.9886	0.0505	-3.0038	-5.2764	-6.9137
1.20	3.6878	-0.2503	-3.3045	-5.5772	-7.2144
1.30	3.3795	-0.5586	-3.6128	-5.8855	-7.5227
1.40	3.0631	-0.8750	-3.9292	-6.2019	-7.8392
1.50	2.7382	-1.1999	-4.2541	-6.5268	-8.1641
1.60	2.4052	-1.5329	-4.5871	-6.8598	-8.4970
1.70	2.0637	-1.8744	-4.9286	-7.2013	-8.8385
1.80	1.7135	-2.2246	-5.2788	-7.5515	-9.1888
1.90	1.3544	-2.5837	-5.6379	-7.9106	-9.5478
2.00	0.9865	-2.9516	-6.0059	-8.2785	-9.9158

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Table 2.6: Value-at-Risk ($\mathrm{VaR}_{95\%}$) on the per unit issuance of an ISL policy with a 10-year maturity.

r_P	$r_G = 0.0\%$	0.5%	1.0%	1.5%	2.0%
2.0%	0.0871	0.1071	0.1302	0.1567	0.1875
2.5%	0.1025	0.1224	0.1452	0.1715	0.2020
3.0%	0.1140	0.1338	0.1565	0.1826	0.2127
3.5%	0.1226	0.1424	0.1650	0.1909	0.2208
4.0%	0.1294	0.1492	0.1717	0.1976	0.2272
4.5%	0.1350	0.1548	0.1773	0.2030	0.2325
5.0%	0.1399	0.1598	0.1823	0.2079	0.2371
5.5%	0.1447	0.1646	0.1870	0.2124	0.2414
6.0%	0.1492	0.1691	0.1915	0.2167	0.2454

Table 2.7: Expected shortfall ($\mathrm{ES}_{95\%}$) on the per unit issuance of an ISL policy with a 10-year maturity.

r_P	$r_G = 0.0\%$	0.5%	1.0%	1.5%	2.0%
2.0%	0.1305	0.1516	0.1755	0.2027	0.2339
2.5%	0.1464	0.1674	0.1911	0.2179	0.2486
3.0%	0.1593	0.1801	0.2035	0.2300	0.2602
3.5%	0.1694	0.1900	0.2132	0.2394	0.2691
4.0%	0.1767	0.1971	0.2200	0.2460	0.2755
4.5%	0.1808	0.2011	0.2240	0.2499	0.2794
5.0%	0.1817	0.2021	0.2251	0.2513	0.2809
5.5%	0.1800	0.2007	0.2241	0.2507	0.2807
6.0%	0.1769	0.1981	0.2221	0.2490	0.2795

models when the distributions are not normal, could lead to the underestimation of extreme losses and hugely mispriced derivative products, see Jondeau et al. (2007). We incorporated the Cox et al. (1985) interest rate model and Heston (1993)'s non-Gaussian stochastic volatility. We also empirically derived the parameters in our models fitted from Taiwanese data, and subsequently numerically computed the value of the liability reserve, i.e. the cost of issuance, the expected surplus / deficit of the bonus stabilization reserve and standard risk measures such as VaR and ES through Monte Carlo simulations.

Our numerical results show that they are consistent with that of financial option pricing, in the sense that by offering an ISL policy with both higher guaranteed interest rate and policy interest rates, its liability reserves would also cost relatively more than one that does not, which is to say the cost of guarantee is higher for those offering higher rates. Although our model is constructed to the specifications of the most popular life insurance policy in Taiwan and its parameters estimated using Taiwanese market data, it can be easily adopted by other markets with similar products and our results be of interest to practitioners and the regulatory authorities.

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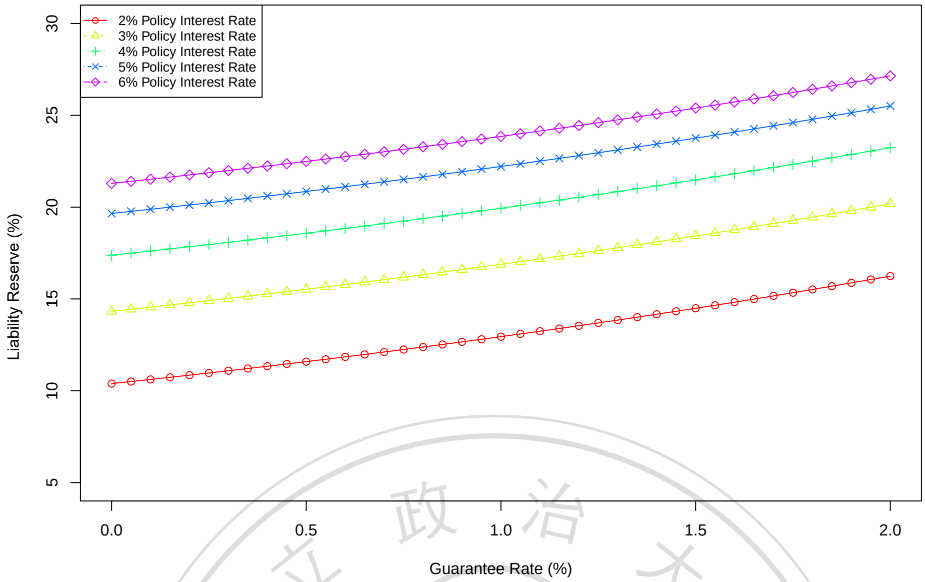

Figure 2.1: Liability reserve p for various levels of guaranteed interest rate r_G, given fixed level of policy interest rate r_P.

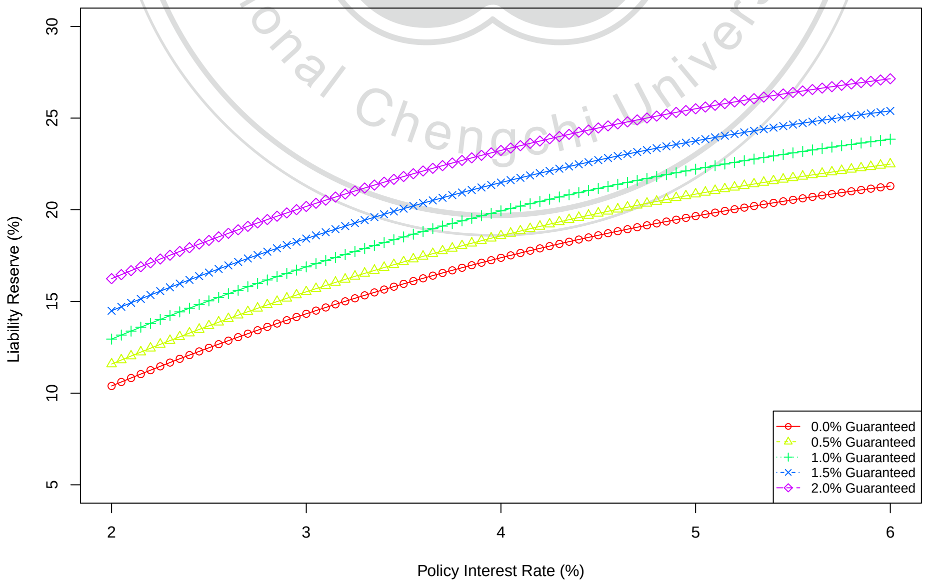

Figure 2.2: Liability reserve p for various levels of policy interest rate r_P given fixed level of guaranteed interest rate r_G.

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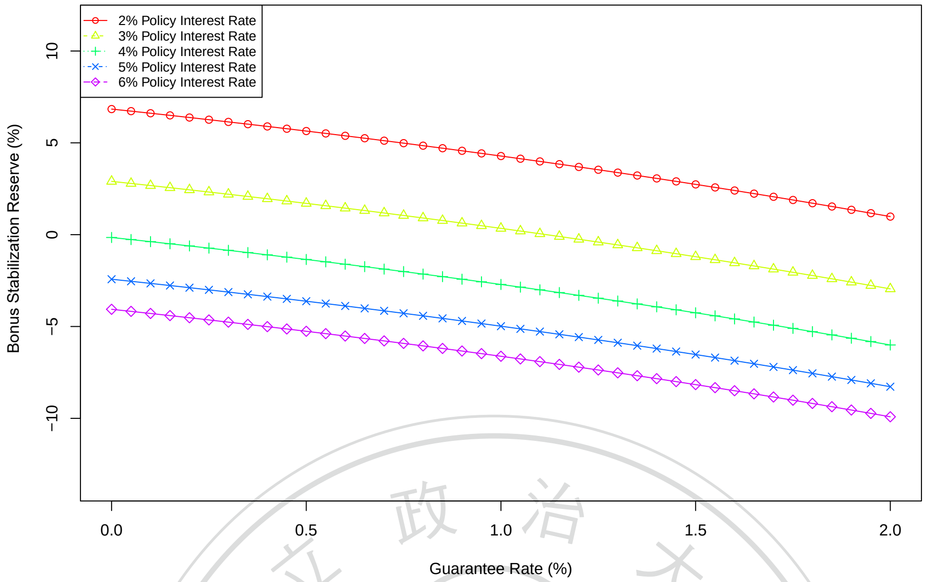

Figure 2.3: Bonus stabilization reserve p_s for various levels of guaranteed interest rate r_G, given fixed level of policy interest rate r_P.

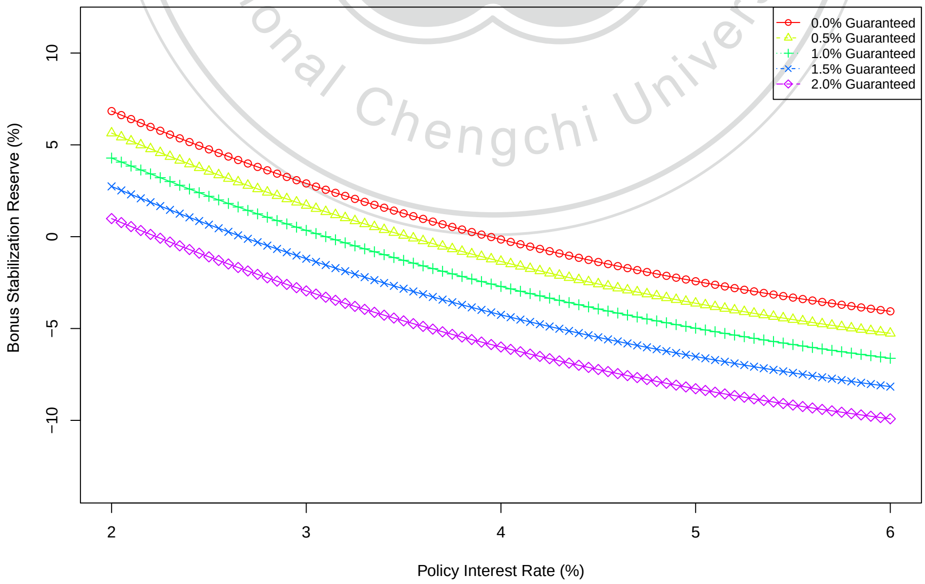

Figure 2.4: Bonus stabilization reserve p_s for various levels of policy interest rate r_P, given fixed level of guaranteed interest rate r_G.

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3

International Bond Valuation: Theory and Implementation

∗ This chapter is based on Hsuan & Chang (2019).

3.1 Introduction

The amendment of Article 146-4 of the Insurance Act of the Republic of China on 20 May 2014 stated that,

- The foreign investment value, which shall be approved by the competent authority, may not exceed 45% of the funds of each of the said insurance enterprises.
- The value for foreign currency denominated listed or over-the-counter certificates of domestic stocks or bonds that are invested in by insurance enterprises in accordance with provisions of the Act may not be included as part of the overseas investment ceiling.

As a result, the amendment prompted the insurance companies' allocation to foreign or offshore assets from 43.5%, almost to the statutory cap of 45% in 2013, to 65% in 2017. Of the increased position in offshore investments, all of which are directed to the international bonds that are listed on the Taipei Exchange (TPEx), and over 80% are USD-denominated callable zero coupon bonds (ZCBs) with long term maturities. To emphasise the importance of asset-liability management of an insurance company, the Financial Supervisory Commission (FSC) had amended Article 10 of the Regulations Governing Foreign Investments by Insurance Companies by stipulating that, for investment in bonds with covenants that the issuer may redeem the bond after a certain period of time, that period of time shall not be less than 5 years from the date of issue. In order to further contain the total amount of assets invested into these international bonds by the insurance companies, on 21 November 2018, the FSC ruled that the total investment amount plus

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the foreign investment amount included in the limit for foreign investments shall not exceed 145% of the insurer's approved foreign investment limit.

While the latest International Financial Reporting Standards (IFRS) 9 retained the concept of fair value option from the International Accounting Standards (IAS) 39, they revised the criteria of financial assets and liabilities - meaning most financial liabilities are held at amortised cost i.e. the cash flows from the instrument must consists of only principal and interest, else the instrument must be reported at fair value. To determine whether a financial liability is classified as a pure debt instrument, it is dependent on two tests, a contractual cash flow test (Solely Payments of Principal and Interest, SPPI) and a business model assessment. Unless the asset meets the requirements of both tests, it is measured at fair value with all changes in fair value reporting in profit and loss (FVPL). As such, any interests that are linked to options or instrument value derived from them, contains leveraging, can be converted into shares or counter floating rate financial instruments does not meet the requirements of the test. Thus, accounting for almost 20% of the insurance company's total assets, these international bonds does not classify as pure debt instrument and must be reported at fair value.

When faced with calculating the fair value of international bonds, one is most likely to encounter two fundamental problems — one being constructing a term structure of interest rates that reflect current conditions, the other is the application of risk neutral evaluation principles, i.e. the calculation of expected values under risk neutral measures. Due to the complex nature and importance of fixed income financial products, there exist many texts from basic theory by Blyth (2014); Corb (2012); Fabozzi & Mann (2010); Hayre (2001); Homer et al. (2013); Hull (2018); Jha (2011); Sundaresan (2009); Tuckman & Serrat (2012); Veronesi (2010) to more advanced reference books such as Andersen & Piterbarg (2010a,b,c); Brigo & Mercurio (2006). However, there are only a handful of domestic literatures that combine actual model parameter estimation with theoretical value calculation in the same context. In this chapter we set out to do exactly that. We further the research by Chang & Wu (2016), and applied actual market data to the theoretical models to calculate the implied option value, thus allowing us to more accurately determine the fair value of the international bonds.

To the best of our knowledge, TPEx had made available the daily theoretical price files of these callable ZCBs for download (Dai (2017)) since 18 December 2017. Another commercialized international bond price calculation system is PRIS of Taiwan Economic Newspaper Cultural Enterprise Co., Ltd. (TEJ).

In this chapter we provide practitioners with a self-contained solution that merges theoretical model with open source data and software. We start off by applying the U.S. Treasury Constant Maturity Rate, the USD LIBOR, and the Intercontinental Exchange (ICE) Swap Rate to the Hull-White short interest rate model (Hull & White (1990, 1993c)) to construct the term structure of interest rates. Next we use the implied volatility

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matrix of the European swaption published by the Chicago Mercantile Exchange (CME) to calibrate the model parameters. Lastly, the theoretical value of the callable ZCBs can be computed through the trinomial tree method as in Hull & White (1994a, 2001). One can implement this process through the use of the open source software QuantLib and its Python binding (Balaraman & Ballabio (2017); Ballabio (2017)). Our main contribution is by marrying existing theoretical model with real world data to provide a pricing system that is based on free, open sourced data and software, such that one would be able to conduct verification independent of third parties.

The rest of the chapter is organized as follows. Section 3.2 introduces some base definitions and theorems pertinent to the pricing of bonds in general. Section 3.3 allow us to illustrate the methodology used to derive the value(s) at each phase, while section 3.4 discusses how one would calibrate the model parameters; how factors such as the internal rate of return, non-callable period, and the redemption frequency affect a typical callable ZCB that is listed on the TPEx. Finally, section 3.5 concludes.

3.2 Preliminaries

3.2.1 Zero coupon bonds

Zero coupon bond prices are the basic quantities in interest rate theory, and all interest rates can be defined in terms of ZCB prices. Thus, we define the following:

Definition 3.2.1 (Zero coupon bond, ZCB). *A zero coupon bond with maturity T is a contract that guarantees its holder the payment of 1 unit of currency at time T, with no intermediate payments. Let $Z(t,T)$ be the contract value at current time t with maturity T, for $t \leq T$. By definition $Z(T,T) = 1$ for all T.*

Definition 3.2.2 (Day count convention). *Let α denote the interest accrual factor as the inverse of the compounding frequecy m times per annum.*

Suppose we have semi-annual compounding, implying that interest payments are made semi-annually, so that $m = 2$ and $\alpha = \frac{1}{2}$; for quarterly compounding and payments, $m = 4$ and $\alpha = \frac{1}{4}$, etc.

There are many different market conventions for calculating the actual accrual factor used when determining interest or coupon payments for a particular period. One can refer to Blyth (2014); Brigo & Mercurio (2006); Hull (2018) for detailed discussions on the various day count conventions. We mention the following three examples of the day count conventions commonly used in practice regarding fixed income instruments.

- `Actual/360`. With this convention a year is assumed to be 360 days long and the year fraction between two dates is the actual number of days between them divided

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by 360. Used on money market instruments in the United States. LIBOR is also quoted on an `actual/360` for all currencies except sterling, for which it is quoted on an `actual/365` basis.

- `Actual/Actual`. Here, the ratio is based on the actual number of days elapsed to the actual number of days in the period between interest payments. This day count convention is used for Treasury bonds in the United States; also, sterling and Euro-denominated bonds.

- `30/360`. With this convention, it assumes that there are 30 days in each month, and 360 days in each year. This is used for corporate and municipal bonds in the United States.

Table 3.1 illustrates the differences in the accrual factor for the day count convention mentioned above.

Table 3.1: Daycount conventions and accrual factors.

Day count convention	**α for 16 December 2011 to 16 March 2012**
`act/365`	$91/365 = 0.2493$
`act/act`	$15/365 + 76/366 = 0.2487$
`30/360`	$1/4 = 0.25$

Definition 3.2.3 (Zero coupon curve). *The zero coupon curve at time t is the graph of the function*

$$T \mapsto \begin{cases} L(t,T) & t < T \leq t+1 \text{ (years)}, \\ Y(t,T) & T > t+1 \text{ (years)}. \end{cases}$$

where $L(t,T)$ and $Y(t,T)$ are the simple-compounded spot interest rate and the annually compounded spot interest rate at time t for the maturity T respectively.

A zero coupon curve is also referred to as the *term structure of interest rates* or simply the *yield curve* at time t. It is a plot at time t made up of simple-compounded interest rates for all maturities T up to and including one year and of annually compounded rates for maturities T that are more than one year. Long-term interest rates tend to be higher than short-term interest rates. Thus, the yield curve is mostly upward sloping; it is downward sloping only when the market expects a steep decline in short-term rates.

3.2.2 Forward rates and LIBOR

Definition 3.2.4 (Forward contract). *A forward contract, or simply forward, is an agreement between two counterparties to trade a specific underlying asset, at a certain*

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future time T and at a certain price K. Here K is known as the delivery price, and the specified time T is the maturity.

In other words, at the current time $t \leq T$, one counterparty assumes a *long position* and agrees to buy the asset at T, for a certain specified price K. The other counterparty assumes a *short position* and agrees to sell the asset on the same date for the same price.

Let $V_K(t,T)$ be the value of a long forward contract at current time $t \leq T$ with delivery price K and maturity T, and $F(t,T)$ the forward price at current time $t \leq T$ with delivery price K such that $V_K(t,T) = 0$ (i.e. $V_{F(t,T)}(t,T) = 0$). For example, suppose a stock which pays no dividends has price 10, and interest rates are always 0. Then $F(t,T) = 10$ and $V_K(t,T) = 10 - K$, $\forall t \leq T$.

The London Interbank Offered Rate (LIBOR) is an interest rate average calculated by the Intercontinental Exchange (ICE) from estimates submitted by contributor banks in London and published at 11:00 A.M. by Thomson Reuters. It is an unsecured short-term borrowing rate between banks, and is the primary benchmark, along with the Euribor, for short-term interest rates around the world. The borrowing periods range from one day to one year. LIBOR is widely used as a reference rate for many financial instruments such as forward rate agreements (FRA), interest rate swaps (IRS), and swaptions, to name a few, which we will touch on in the following sections.

Maturities for LIBOR rates may vary, but are commonly 1, 3, 6, and 12 months. Thus, on any given day t, LIBOR rates for these periods $\alpha = 0.25, 0.5, 1, \ldots$ etc. are published. Let $L_t[t, t+\alpha]$ denote the LIBOR rate at time t for the period t to $t+\alpha$. The LIBOR rate $L_\tau[\tau, \tau+\alpha]$ for a future date $\tau > t$ is then a random variable. Thus, banks can deposit (or borrow) notional N at time t and receive (or pay back) $N(1+\alpha L_t[t, t+\alpha])$ at time $t+\alpha$. All interest is paid at the maturity of the deposit, and there is no interim compounding (simple interest).

Definition 3.2.5 (Forward rate agreement, FRA). *The forward rate aggreement is a forward contract to exchange two cashflows. Specifically, the buyer of the FRA with maturity T and delivery price or fixed rate K agrees at $t \leq T$ to pay αK and receive $\alpha L_T[T, T+\alpha]$ at time $T+\alpha$. Thus the payout of the FRA is $\alpha(L_T[T, T+\alpha] - K)$ at time $T+\alpha$.*

We further define $L_t[T, T+\alpha]$ as the *forward LIBOR rate*, and is the value of K such that the FRA has zero value at time $t \leq T$. Expressed as

$$L_t[T, T+\alpha] = \frac{Z(t,T) - Z(t, T+\alpha)}{\alpha Z(t, T+\alpha)}. \tag{3.2.1}$$

Proof. Consider a portfolio consisting of long one ZCB with maturity T and short $(1+\alpha K)$ ZCBs with maturity at $(T+\alpha)$. Then by definition $Z(T,T) = 1$ and $Z(T, T+\alpha) =$

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$-(1+\alpha K)$; place this 1 unit in a deposit for the period $[T, T+\alpha]$ with interest rate $L_T[T, T+\alpha]$ and accrual factor α.

Therefore, at time $(T+\alpha)$, the portfolio has value

$$(1+\alpha L_T[T, T+\alpha]) - (1+\alpha K) = \alpha(L_T[T, T+\alpha] - K),$$

which is the payout of a FRA. Let $V_K(t, T)$ be the value of the FRA at time t, then

$$V_K(t, T) = Z(t, T) - (1+\alpha K)Z(t, T+\alpha).$$

Following the definition of a forward LIBOR rate where $L_t[T, T+\alpha]$ is the value of K such that the FRA has zero value at $t \leq T$,

$$0 = Z(t, T) - (1+\alpha L_t[T, T+\alpha])Z(t, T+\alpha)$$

rearranging the terms, we have

$$L_t[T, T+\alpha] = \frac{Z(t, T) - Z(t, T+\alpha)}{\alpha Z(t, T+\alpha)} \qquad \blacksquare$$

3.2.3 Interest rate swaps

An *interest rate swap (IRS)* is a contractual agreement between two counterparties that agree to exchange streams of payment that are made in the same currency over time. The most typical IRS is the fixed-floating swap or the plain vanilla swap — referring to a swap where one stream is a fixed rate of interest, and the other is a floating rate of interest. The floating rate used in most IRS agreements is the LIBOR 3M (3-month LIBOR).

Suppose when one counterparty expects a decline in interest rates, one can enter into an IRS agreement with another counterparty (the Bank) to receive a fixed rate payment, say 5% per annum on a principal of \$100 million from the Bank, in exchange for a floating rate payment i.e. the 3-month LIBOR on the same principal. Effectively transforming a fixed rate liability into a floating rate liability, thus reducing the cost of capital. The counterparty (the Bank) paying the fixed rate to and receiving the floating rate from, is the fixed rate payer (the "payer"), the other party paying the floating rate is the floating rate payer (the "receiver").

As illustrated in Figure 3.1, let a swap has a start date T_0, maturity T_n, and payment dates T_i, $i = 1, \ldots, n$. Although in practice, the payment dates for the floating and fixed legs may be of different dates, the day count varies due to the underlying asset and for each period it is computed. For simplicity, we assume that the payment frequency for the floating and fixed legs are the same and a constant α for each period, i.e. $T_{i+1} = T_i + \alpha$,

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$i = 0, \dots, n-1$. We then have the floating leg of the swap that pays $\alpha L_{T_i}[T_i, T_i + \alpha]$ at time $T_i + \alpha$ — the LIBOR rate is set at time T_i for the period $[T_i, T_i + \alpha]$ and paid at $T_i + \alpha$; and αK paid at time $T_i + \alpha$ for the fixed leg of the swap.

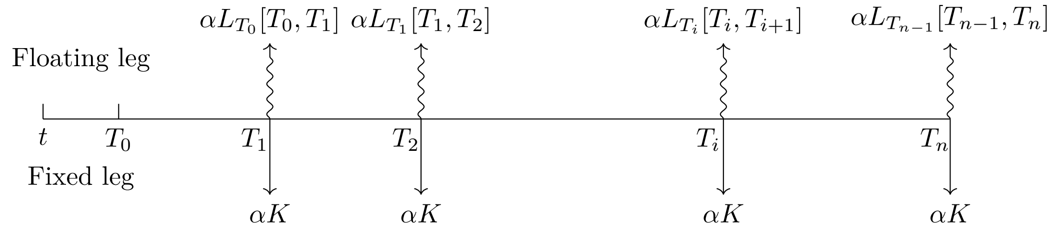

Figure 3.1: A swap

The value of the fixed leg can be expressed as

$$
\begin{aligned}
V^{\text{fix}}(t) &= K \sum_{i=0}^{n-1} \alpha Z(t, T_i + \alpha) \\
&\equiv K P_t[T_0, T_n],
\end{aligned}
$$

and the value of the floating leg is

$$
\begin{aligned}
V^{\text{float}}(t) &= \sum_{i=0}^{n-1} L_t[T_i, T_i + \alpha]\, \alpha Z(t, T_i + \alpha) \\
&= \sum_{i=0}^{n-1} \frac{Z(t, T_i) - Z(t, T_i + \alpha)}{\alpha Z(t, T_i + \alpha)}\, \alpha Z(t, T_i + \alpha) \\
&= \sum_{i=0}^{n-1} (Z(t, T_i) - Z(t, T_i + \alpha)) = Z(t, T_0) - Z(t, T_n)
\end{aligned}
$$

Definition 3.2.6 (Forward swap rate). *The forward swap rate at time t for a IRS from T_0 to T_n is defined to be the value $y_t[T_0, T_n]$ of the fixed rate K such that the IRS at t is 0.*

In other words, $y_t[T_0, T_n]$ is the rate in the fixed leg of the IRS that makes the IRS a fair contract at present time t. We have

$$
y_t[T_0, T_n] = \frac{Z(t, T_0) - Z(t, T_n)}{P_t[T_0, T_n]}.
$$

Therefore, the value of the swap $V^{\text{sw}}(t)$ at time $t \le T_0$ where one pay a fixed rate K and receive LIBOR is given by

$$
V^{\text{sw}}(t) = (y_t[T_0, T_n] - K)\, P_t[T_0, T_n]
$$

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3.2.4 Swaptions

Swap options or *swaptions* are options on swaps, giving the holder the right, but not the obligation, to enter into a swap. It can be viewed as a type of bond option. Institutions often use swaptions as a means to benefit from favourable interest rate movements while acquiring protection from or hedging against unfavourable interest rate movements. There are two main types of swaptions, a payer swaption and a receiver swaption.

A payer swaption, also known as a put swaption, gives the holder the right to pay fixed rate K (and receive floating, say LIBOR) in a swap. Whereas a receiver swaption (*a.k.a* call swaption) gives the holder the right to receive fixed (and pay floating) in a swap.

The payer swaption will only be exercised at a future agreed time T, if the market swap rate $y_T[T_0, T_n]$ is greater than or equal to the strike swap rate K; as it is clearly preferable to receive the market swap rate which is higher while paying a lower fixed rate that was struck at the start of the agreement.

We only consider European swaptions in this thesis where swaptions are only exercisable at maturity. The swaption maturity usually coincides with the first reset date of the underlying swap; and the market standard is that swaptions are cash settled so that counterparties avoid credit exposure to one another. It is also market practice to value swaptions with Black's formula.

3.2.4.1 Valuation of European swaptions — the Black-76 formula

Given that at the time $t \leq T_0$, the value of the swaption $V^{\mathrm{sw}}(t)$ where one pay a fixed rate K and receive LIBOR is

$$V^{\mathrm{sw}}(t) = (y_t[T_0, T_n] - K)\, P_t[T_0, T_n],$$

which can be seen as a call on a swap rate multiplied by the sum of ZCBs, $P_T[T, T_n]$.

Let T (i.e. $T_0 = T$ in the equation above) be the exercise date of the European payer's swaption, K the strike swap rate, where the swaption will only be exercised at T if $y_t[T, T_n] > K$. If we denote the price of the payer swaption at time $t \leq T$ by $\Psi_K(T, T, T_n)$, then its payout at exercise date T is given by

$$\begin{aligned}\Psi_K(T, T, T_n) &= (y_T[T, T_n] - K)^+\, P_T[T, T_n] \\ &= \max\{y_T[T, T_n] - K, 0\}\, P_T[T, T_n]\end{aligned}$$

Under risk-neutral measures and $P_t[T, T_n]$ as the numeraire, the value of the swaption

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$\Psi_K(T,T,T_n)$ can be expressed as

$$\frac{\Psi_K(t,T,T_n)}{P_t[t,T_n]} = \mathsf{E}^* \left\{ \frac{\Psi_K(T,T,T_n)}{P_T[T,T_n]} \right\} = \mathsf{E}^* \left\{ (y_T[T,T_n] - K)^+ \right\}$$

where E^* is the expectation under the risk-neutral measures with respect to $P_t[T,T_n]$. It is also assumed that the underlying forward swap rate $y_T[T,T_n]$ is lognormally distributed with parameters $y_T[T,T_n]$ and volatility σ, that is

$$y_t[T,T_n] \sim \text{lognormal}\left(\log y_t[t,T_n] - \frac{1}{2}\sigma^2(T-t), \sigma^2(T-t) \right)$$

Then, the Black-76 formula for swaptions can be written as

$$\Psi_K(t,T,T_n) = P_t[t,T_n] \left(y_t[T,T_n]\, \Phi(d_1) - K\, \Phi(d_2) \right), \tag{3.2.2}$$

where the cumulative normal distriution function $\Phi(\cdot)$ is given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, \mathrm{d}y$$

and

$$d_1 = \frac{\log\left(\frac{y_t[T,T_n]}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

3.2.5 Hull-White short rate model

An instantaneous short rate r_t is the rate that applies to an infinitesimally short period of time at time t; which is also a simple way to describe the term structure of interest rates and the valuation of interest rate linked options. The *no-arbitrage model* is a model designed to be exactly consistent with today's term structure of interest rates, and where the initial term structure is an input rather than an output. This will become evident when we contruct the interest rate tree in 3.3. We consider one of the no-arbitrage models — the Hull-White (one-factor) model Hull & White (1990, 1993c), as the basis model for the term structure of interest rates in this study. For this model is versatile enough to be extended to represent other models, such as Ho-Lee, and Black-Karasinski as its special cases, which is beyond the scope of this study, the capability of fitting an arbitrary initial term structure (as an input) together with the ability to accommodate negative interest rates, it is its analytical tractability that is most attractive — as the model can be calibrated to market data on interest option prices (3.4).

We assume the existence of a risk-neutral measure, implying that the arbitrage-free price of a contingent claim at time t with final payoff function H_T at maturity T is given

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by

$$H_t = \mathsf{E}\left\{ e^{-\int_t^T r_\tau \, \mathrm{d}\tau} H_T \,\middle|\, \mathcal{F}_t \right\}$$

where $\mathsf{E}\{\cdot \,|\, \mathcal{F}_t\}$ is the conditional expectation with respect to t under that measure. It is evident that the price of the ZCB, $Z(t,T)$ (and by definition $H_T \equiv 1$) can then be expressed as

$$Z(t,T) = \mathsf{E}\left\{ e^{-\int_t^T r_\tau \, \mathrm{d}\tau} \,\middle|\, \mathcal{F}_t \right\}.$$

In the general Hull-White (HW) model (Hull & White (1990, 1993c)), the short rate r_t follows a diffusion process

$$\mathrm{d}r_t = (\vartheta(t) - a\, r_t)\ \mathrm{d}t + b\, \mathrm{d}W_t \tag{3.2.3}$$

where W_t is a standard Wiener process, $a, b \in \mathbb{R}^+$, are the two volatility paramenters that are chosen to fit the current market prices of a set of actively traded interest rate options. $r_0 \in \mathbb{R}$ and $\vartheta(t)$ is the term structure parameter selected so as to fit the initial term structure currently observed in the market. Then rearranging $Z(t,T)$ in the following form (see Brigo & Mercurio (2006, (3.39)))

$$Z(t,T) = A(t,T)\, e^{B(t,T)\, r_t}$$

where

$$\begin{aligned} A(t,T) &= \frac{F^M(0,T)}{F^M(0,t)} \exp\left\{ B(t,T) f^M(0,t) - \frac{b^2}{4a}\left(1 - e^{2at}\right) B(t,T)^2 \right\} \\ B(t,T) &= \frac{1}{a}\left(1 - e^{-a(T-t)}\right), \end{aligned}$$

with $F^M(0,T)$ the market discount factor at current time for the maturity T, and $f^M(0,T)$ the market instantaneous forward rate at time 0 for the maturity T denoted as,

$$f^M(0,T) = -\frac{\partial \log F^M(0,T)}{\partial T}.$$

We have r_t conditional on $\mathcal{F}_s$, and is normally distributed with mean and variance given respectively by

$$\begin{aligned} \mathsf{E}\left\{ r_t \,|\, \mathcal{F}_s \right\} &= r_s\, e^{-a(t-s)} + \alpha(t) - \alpha(s) e^{-a(t-s)} \\ \operatorname{var}\left\{ r_t \,|\, \mathcal{F}_s \right\} &= \frac{b^2}{2a}\left(1 - e^{-2a(t-s)}\right) \end{aligned}$$

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where

$$\alpha(t) = f^M(0,t) + \frac{b^2}{2a^2}\left(1 - e^{-at}\right)^2.$$

In other words, at current time $t = 0$, $A(0,T)$ and $B(0,T)$ are defined by b, the current term structure of interest rates, and the current term structure of spot or forward interest rate volatilities.

Let $\text{ZBP}(t,T,S,K)$ be the price of a European put option at time $t \le T < S$, with strike K and maturity T, written on a pure discount bond (such as a coupon bond) maturing at time S, or simply the put option on the ZCB; then according to the arbitrage-free pricing principle

$$\text{ZBP}(t,T,S,K) = \mathsf{E}\left\{ e^{-\int_t^T r_\tau \, \mathrm{d}\tau} (K - Z(T,S))^+ \,\middle|\, \mathcal{F}_t \right\}$$

and the above expression can be rewritten as in (Brigo & Mercurio (2006, (3.41)))

$$\text{ZBP}(t,T,S,K) = KZ(t,T)\Phi\left(-h + \sigma_p\right) - Z(T,S)\Phi(-h)$$

where

$$\sigma_p = b\sqrt{\frac{1 - e^{-2a(T-t)}}{2a}} Z(T,S), \quad h = \frac{1}{\sigma_p}\log\frac{Z(t,S)}{Z(t,T)K} + \frac{\sigma_p}{2}.$$

Drawing from parts mentioned earlier in this section, and applying the decomposition technique in Jamshidian (1989), where the European put (payer) swaption can be decomposed into a portfolio of ZBPs. First, let $c_i = K\alpha$, $i = 1, 2, \ldots, n-1$, $c_n = 1 + K\alpha$ such that r^* is the solution that satisfies

$$\sum_{i=1}^{n} c_i \, A(T,T_i) \, e^{-B(T,T_i)r^*} = 1$$

and let $K_i = A(T,T_i)e^{-B(T,T_i)r^*}$, $i = 1, 2, \ldots, n$. Hence, the price of the European put (payer) swaption $\Psi_K(t,T,T_n)$ can be written as

$$\Psi_K(t,T,T_n) = \sum_{i=1}^{n} c_i \, \text{ZBP}(t,T,T_i,K_i). \tag{3.2.4}$$

3.3 Pricing of bonds — interest rate trees

3.3.1 Bond pricing

Most coupon bearing bonds pay periodic coupons to their holders, and its principal (i.e. *par value* or face value, usually set at 100) is paid at the bonds' maturity. The

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theoretical price of such bonds can be calculated as the present value of all the cashflows that will be received by the owner of these bonds. However, market interest rates are a function of time and stochastic in nature, thus the price of the bond at each time $t < T$ before its maturity varies.

We can compute the price of the bond at time t by determining an appropriate interest rate as the discounting factor for the expected cashflows over this period. Ideally, for the ease of computation, is for these discounting factors to be a constant; however, under real market conditions these discounting factors and ultimately the corresponding interest rates are random variables. Despite this, one can still construct an interest rate tree that represents the approximate evolution of that interest rate process.

A typical binomial tree is shown on the left of Figure 3.2, the possible outcomes of the bond price also increase with time. Under the no-arbitrage principle, in each time step, it has the same probability of moving up by a certain percentage amount as it has with moving down a certain percentage amount. As the final payoff value of the bond is known, one can follow the procedure on the right of Figure 3.2. Working backwards in time by discounting the expected cashflows by their respective interest rates, we can obtain the value of the bond at each node, and finally its initial price at $t = 0$.

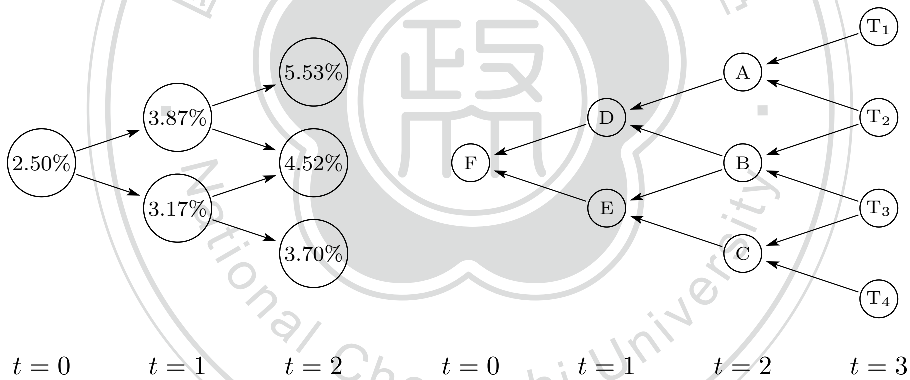

Figure 3.2: Typical binomial tree (left); corresponding states and order of discounting (right). T_1, T_2, T_3, T_4 represents the final payoff of the bond price; A, B, and C represents the outcome of the bond in the second year, with its respective corresponding interest rates $5.53\%, 4.52\%, 3.70\%$; D and E as the outcome of the bond after one year, with corresponding interest rates 3.87% and 3.17% respectively; F as the bond price at issuance, with initial rate at 2.50%.

Example. Consider a 3-year par yield bond that pays a coupon of 4.25% per annum, callable annually. Given the interest rate tree in Figure 3.2, compute the bond value at issuance.

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Solution.

$$\text{State } \mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3, \mathrm{T}_4:\ 100 + 4.25 = 104.25$$

$$\text{State A: } \frac{1}{2}\left(\frac{104.25}{1+5.53\%} + \frac{104.25}{1+5.53\%}\right) = 99.003$$

$$\text{State B: } \frac{1}{2}\left(\frac{104.25}{1+4.52\%} + \frac{104.25}{1+4.52\%}\right) = 99.742$$

$$\text{State C: } \frac{1}{2}\left(\frac{104.25}{1+3.70\%} + \frac{104.25}{1+3.70\%}\right) = 100.53 > 100;\ \text{callable, price} = 100$$

$$\text{State D: } \frac{1}{2}\left(\frac{99.003+4.25}{1+3.87\%} + \frac{99.742+4.25}{1+3.87\%}\right) = 99.761$$

$$\text{State E: } \frac{1}{2}\left(\frac{99.742+4.25}{1+3.17\%} + \frac{100+4.25}{1+3.17\%}\right) = 100.92 > 100;\ \text{callable, price} = 100$$

$$\text{State F (bond price at issuance): } \frac{1}{2}\left(\frac{99.761+4.25}{1+2.5\%} + \frac{100+4.25}{1+2.5\%}\right) = 101.59$$

■

3.3.2 Trinomial tree

A series of research was done by Hull and White (Hull & White (1990, 1993a,b,c,d, 1994a,b, 1996, 2001)) regarding the construction of an efficient numerical procedure, such as the interest rate tree, to determine option prices and the calibration of the parameters of these models. In the following section, we find the works in Brigo & Mercurio (2006, 3.3.3, 3.5, Appendix F) provides a succinct description on the construction of a trinomial interest rate tree through approximating the dynamics of a general diffusion process.

Consider a diffusion process X_t that evolves according to

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t$$

where μ and σ are smooth scalar real functions and W_t the scalar standard Brownian motion. The approximation of the diffusion process X of building the trinomial tree can be broken down into two stages and four steps — our first step is to select the spacing of the tree nodes according to time; and the second step is deciding on the spacing of the nodes with respect to the interest rate. Followed by choosing the branching process for each of the $x_{i,j}$'s through the nodes; and finally, shifting the tree by the displacement value at each point in time.

First Stage

Let $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$ be a finite set of times, and $\Delta t_i = t_{i+1} - t_i$. At each time t_i, we have a finite number of equispaced states on X, with constant vertical step Δx_i; set $x_{i,j} = j\Delta x_i$. We define $\{(i,j)\,|\,i = 0, 1, \ldots, n;\ j = j_i^-, j_i^- + 1, \ldots, j_i^+ - 1, j_i^+\}$ as the nodes of the tree, where j_i^-, j_i^+ are integers dependent of time i. We observe that the tree is symmetrical and that $j_i^- < 0$, $j_i^+ > 0$, are the mirror images of each other.

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Suppose at time t_{i+1}, the value $x_{i,j}$ on the j-th node moves to $x_{i+1,k+1}$, $x_{i+1,k}$, $x_{i+1,k-1}$ with probabilities p_{u}, p_{m}, p_{d} respectively. They are defined as the probabilities of the highest, middle, and lowest branches emanating from a node. Let the mean $M_{i,j}$ and variance $V_{i,j}$ conditional on $X(t_i) = x_{i,j}$ be denoted as

$$\begin{aligned}\mathsf{E}\{\, X(t_{i+1}) \,|\, X(t_i) = x_{i,j}\} &= M_{i,j} \\ \mathrm{var}\{\, X(t_{i+1}) \,|\, X(t_i) = x_{i,j}\} &= V_{i,j}\end{aligned}$$

and satisfies the following equations

$$\begin{aligned}p_{\mathrm{m}} + p_{\mathrm{d}} + p_{\mathrm{u}} &= 1 \\ p_{\mathrm{m}}\, x_{i+1,k} + p_{\mathrm{d}}\, x_{i+1,k-1} + p_{\mathrm{u}}\, x_{i+1,k+1} &= M_{i,j} \\ p_{\mathrm{m}}\, x_{i+1,k}^2 + p_{\mathrm{d}}\, x_{i+1,k-1}^2 + p_{\mathrm{u}}\, x_{i+1,k+1}^2 &= V_{i,j} + M_{i,j}^2\end{aligned}$$

Note that

$$x_{i+1,k+1} = x_{i+1,k} + \Delta x_{i+1}, \quad x_{i+1,k-1} = x_{i+1,k} - \Delta x_{i+1}$$

Setting

$$\eta_{i,j,k} = M_{i,j} - x_{i+1,k}$$

then the solution to these equations is

$$\begin{aligned}p_{\mathrm{u}} &= \frac{V_{i,j}}{2(\Delta x_{i+1})^2} + \frac{\eta_{i,j,k}^2}{2(\Delta x_{i+1})^2} + \frac{\eta_{i,j,k}}{2\Delta x_{i+1}} \\ p_{\mathrm{m}} &= 1 - \frac{V_{i,j}}{(\Delta x_{i+1})^2} - \frac{\eta_{i,j,k}^2}{(\Delta x_{i+1})^2} \\ p_{\mathrm{d}} &= \frac{V_{i,j}}{2(\Delta x_{i+1})^2} + \frac{\eta_{i,j,k}^2}{2(\Delta x_{i+1})^2} - \frac{\eta_{i,j,k}}{2\Delta x_{i+1}}\end{aligned}$$

In order to ensure that the probabilities p_{u}, p_{m}, p_{d} are all positive, some constraints are necessary. We assume

1. $V_{i,j}$ is independent of j, thus $V_{i,j} = V_i$;
2. $\Delta x_{i+1} = \sqrt{3V_i}$,
3. $k = \left\lfloor \frac{M_{i,j}}{\Delta x_{i+1}} \right\rceil$ such that $x_{i+1,k}$ is as close to $M_{i,j}$ as possible, i.e. the k-th node at time t_{i+1} is also the central node.

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Then the probabilities become

$$
\begin{aligned}
p_{\mathrm{u}} &= \frac{1}{6} + \frac{\eta_{i,j,k}^2}{6V_i} + \frac{\eta_{i,j,k}}{2\sqrt{3V_i}} \\
p_{\mathrm{m}} &= \frac{2}{3} - \frac{\eta_{i,j,k}^2}{3V_i} \\
p_{\mathrm{d}} &= \frac{1}{6} + \frac{\eta_{i,j,k}^2}{6V_i} - \frac{\eta_{i,j,k}}{2\sqrt{3V_i}}
\end{aligned}
$$

Second Stage

The second stage of the construction procedure consists of displacing the tree nodes to obtain the corresponding tree for the current term structure (see Hull & White (2001), Hull (2018, 32.5)). Let α_i denote the uniform displacement at time t_i, such that the shift at each node $(i,\cdot)$ is the same and the price of the discounted bonds obtained at these nodes are consistent with the initial term structure observed in the market. We have

$$
\alpha_0 = -\frac{\log F^M(0,t_1)}{t_1}
$$

Consider an underlying asset where it pays 1 unit currency if it reaches the node (i,j) and 0 otherwise; we define the present value of such an underlying asset as $Q_{i,j}$, then

$$
Q_{i+1,j} = \sum_h Q_{i,h}\, q(h,j)\, e^{-(\alpha_i + h\Delta x_i)\Delta t_i}, \quad j = j_{i+1}^-, \ldots, j_{i+1}^+
$$

where $q(h,j)$ is the probability of change from node (i,h) to $(i+1,j)$; with the summation taken over all values of h for which this is non-zero. Solving

$$
Z(0,t_{i+1}) = \sum_{j=j_i^-}^{j_i^+} Q_{i,j}\, e^{-(\alpha_i + j\Delta x_i)\Delta t_i},
$$

we get the solution to the above equation as

$$
\alpha_i = \frac{1}{\Delta t_i} \log \frac{\sum_{j=j_i^-}^{j_i^+} Q_{i,j}\, e^{-j\Delta x_i \Delta t_i}}{Z(0,t_{i+1})}.
$$

3.4 Numerical results

In the following section, we make use of the open source software QuantLib and its Python (Balaraman & Ballabio (2017); Ballabio (2017)) binding, together with publicly available data to construct a cost efficient pricing system that can be easily plugged-in by practitioners for the fitting of the interest rate term structure (yield curve), the parameter

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estimation of the general Hull-White model, and ultimately the mark-to-market valuation of the callable zero coupon bonds that make up the majority of the life insurer's assets in conformation with the latest IFRS-9 requirement.

3.4.1 Fitting the initial term structure of interest rates

In order to build a bond valuation system, one would start with the fitting of the term structure of the interest rates involved (Ametrano & Bianchetti (2013), Balaraman & Ballabio (2017, Chapters 6, 7)). Here, we use the publicly available data such as the US Treasury Constant Maturity Rate (CMY) and the USD LIBOR / ICE Swap Rate (FRED) to fit the term structure. Historical data on these rates are obtained via the Federal Reserve Bank's website[1], and their corresponding tickers are listed in Appendix A. By choosing these data that are published by the governing body not only ensures its neutrality but also as a cost efficient source to build a versatile internal pricing system.

As we start with the steps in constructing a trinomial tree, we first need to fit the yield curve. A few dates are selected so to fit a few yield curves to illustrate the variability (see Figure 3.3). We also list, for example, the actual rates as observed in the market on 25 March 2019 that is used to fit the yield curve for that particular day (see Table 3.2 and Table 3.3). The normal yield curve slopes upwards; the yield of long-dated maturity exceeds those of short-dated maturity simply due to the time value of money. The "inverted" yield curve as seen in (f)(g)(h) of Figure 3.3 shows that the short-term rates exceed the long-term rates. The formation of the inverted yield curve is commonly attributed to the increasing concerns of an impending recession and the excessive demand of long-term Treasury bonds to preserve capital in the falling market. Before the advent of inverted yield curves, the upward slope tends to flatten as the economic cycle slows down; this can be seen in (b)(c)(d)(e) of Figure 3.3.

3.4.2 Parameter estimation of the Hull-White model

Here we discuss how one would calibrate the volatility parameters in the Hull-White model so that enables us to price the various callable zero coupon bonds that are currently on the life insurers' books. As in Hull & White (2001) and later in Brigo & Mercurio (2006), both has discussed how one can calibrate the initial yield curve from the market quotes for actively traded options such as the European swaptions, caps, and floors; to date, they are the most commonly used source for calibration purposes. Implicitly, by using the market quotes of these actively traded options and their implied volatility implies that the risk premia had already been priced in, therefore we can safely use them without the need to determine or make assumptions on the cost of risk premia when modelling the term structure of interest under risk-neutral measure $\mathbb{Q}$ assumptions. Thus

[1] https://fred.stlouisfed.org/tags/series

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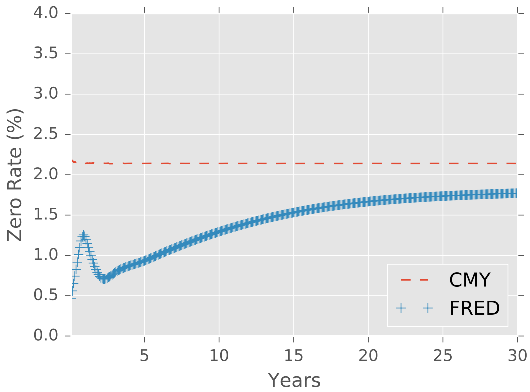
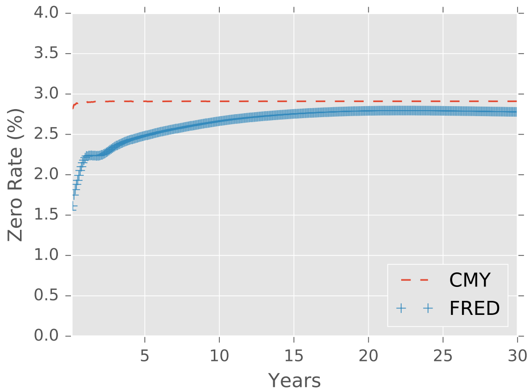
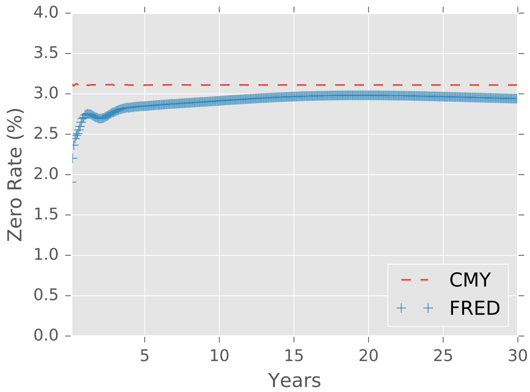


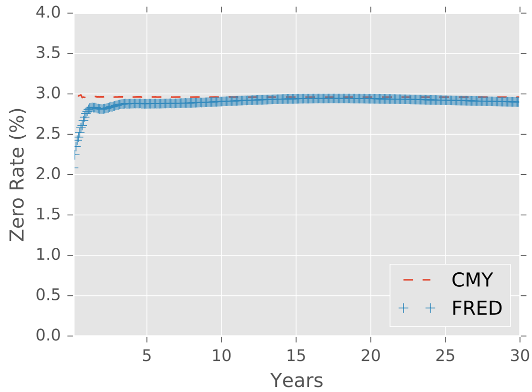
(a) 2016/07/05



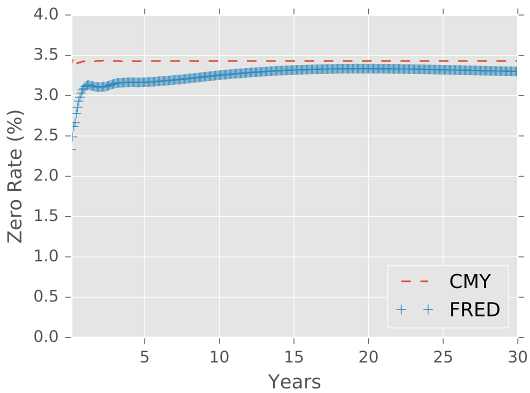
(b) 2018/01/19



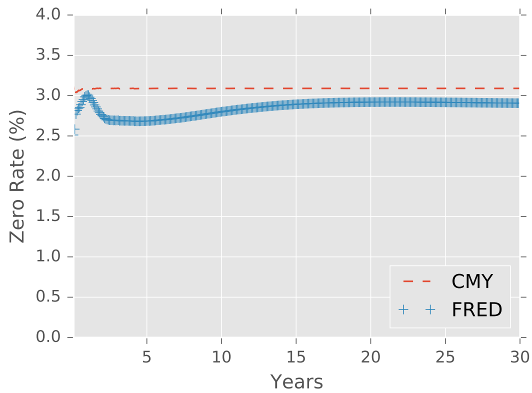
(c) 2018/04/19



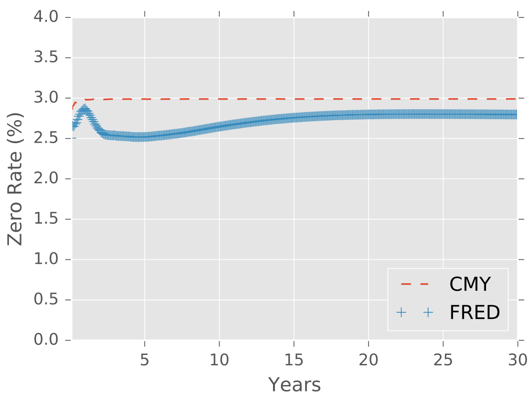
(d) 2018/07/19



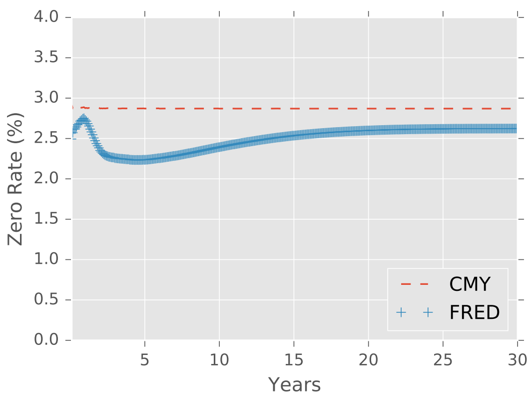
(e) 2018/11/08



(f) 2019/01/18



(g) 2019/02/19



(h) 2019/03/25

Figure 3.3: Yield curve for the various dates.

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Table 3.2: CMY tenor and its corresponding rates on 2019/03/25.

Interest rate tenor	Rate(%)
CMY 1M	2.47
CMY 3M	2.46
CMY 6M	2.49
CMY 1Y	2.41
CMY 2Y	2.26
CMY 3Y	2.19
CMY 5Y	2.21
CMY 7Y	2.32
CMY 10Y	2.43
CMY 20Y	2.68
CMY 30Y	2.87

Table 3.3: FRED tenor and its corresponding rates on 2019/03/25.

Interest rate tenor	Rate(%)
LIBOR 1W	2.41238
LIBOR 1M	2.48975
LIBOR 3M	2.60875
LIBOR 6M	2.673
LIBOR 12M	2.74575
ICE 2Y	2.367
ICE 3Y	2.276
ICE 5Y	2.253
ICE 6Y	2.273
ICE 7Y	2.3
ICE 10Y	2.403
ICE 15Y	2.531
ICE 20Y	2.587
ICE 30Y	2.612

the parameters computed will be a closer fit and a reflection of actual market conditions, which is what we are trying to achieve in terms of mark-to-market valuations.

Thus under an interest rate term structure and given a series of market prices on the European swaptions and its implied volatility σ_i, one can obtain the analytical price of European swaptions $\Psi^{\text{black}}(\sigma_i)$ under the Black-76 model (3.2.4.1) by working backwards through equation (3.2.2). In Hull & White (2001), the authors state that by using these market volatilities in the Black-76 model, the model produced mid-market priced options. Suppose these conditions can also be applied to the Hull-White model to solve the unknown volatility parameters (a^*, b^*) and derive a closed form solution $\Psi_i^{\text{hw}}(a, b)$ through equation (3.2.4). Let (a^*, b^*) be the set of parameters that minimizes the sum of the differences squared between the two swaption prices $\Psi^{\text{black}}(\sigma_i)$ and $\Psi_i^{\text{hw}}(a, b)$,

$$(a^*, b^*) = \underset{(a,b)}{\operatorname{argmin}} \sum_i \left(\Psi^{\text{black}}(\sigma_i) - \Psi_i^{\text{hw}}(a, b)\right)^2$$

There are alternative ways in calibrating the model parameters. One can choose the objective function to be the sum of the differences squared of the implied volatilities or weights can be assigned to each term of the objective function in order of importance, etc. However, each objective function will produce their own unique set of model parameters. Here, we choose the objective function to be the sum of the differences squared, where the differences can be seen as the spread between the option prices derived from the two models.

Since 11 April 2016, the Chicago Mercantile Exchange (CME) started acting as a clearing house for European swaptions, becoming the first of its kind to offer such services.

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At the same time, it also publishes the volatility data on these swaptions on a daily basis[2]. To the best of our knowledge, this is the only source where these data are available on the public domain.

We use a standard European swaption with the 3-month US Dollar LIBOR (3M USD LIBOR) as the underlying asset for the parameter estimation of the Hull-White model. We were able to use the at-the-money (ATM) lognormal implied volatility data from the volatility-price matrix that is published daily by the CME to construct the term structure of interest rates. This daily ATM matrix quotes five maturities (1-, 3-, 6-month, and 1-, 2-year) and seven tenors (1-, 2-, 5-, 10-, 15-, 20-, 30-year), giving us a total of 35 combinations. The maturities quoted represent the life of the option, whereas the tenors are the life of the swap; one assumes the swap to start when the option expires, thus the total life span of the swaption is the option life plus the swap life. By entering CME's implied volatility data into equation (3.2.2), one can obtain the analytical price of the European swaptions $\Psi^{\text{black}}(\sigma_i)$. These $5 \times 7 = 35$ swaption prices are listed in the third column of Tables 3.4 and 3.5. Similarly, with rates from FRED and CMY, and equation (3.2.4), we can derive $\Psi_i^{\text{hw}}(a, b)$, as displayed in the fourth column of these aforementioned tables. The set of parameters (a^*, b^*), which is dependent on the daily yield curve, varies daily.

We enter the data obtained from the CME, FRED, and CMY into the Hull-White model to compute the implied volatilities. An example of the comparative results of our daily estimations are tabulated in Tables 3.4 and 3.5.

3.4.3 Pricing of a callable zero coupon bond

We have outlined the process and laid down the definitions and theorems essential to the pricing of a zero coupon bond. A typical abbreviated term sheet or the "product specifications" of these USD-denominated callable zero coupon bonds listed on the TPEx are set out in Table 3.6. The examples illustrated herein are in accordance with TPEx specifications.

For the avoidance of doubt, in the following sections where the use of interest rate term structures are concerned, we refer to the most recent date i.e. 25 March 2019 as our point of referral. By applying the 25 March 2019 FRED term structure to the interest rate trees, we can obtain the initial value of a 30-year non-callable bond for that day to be 45.7395, and the yield-to-maturity (YTM) as $\left(\frac{100}{45.7395}\right)^{\frac{1}{30}} - 1 = 2.6416\%$.

By fixing certain variables that determine the initial value of the callable bond, such as the term structure of interest rates and the bond issuance period, we can then study the dynamics between the remaining variables — internal rate of return (IRR), non-callable period (NC), and redemption frequency (FREQ); the latter two determines

[2] ftp://ftp.cmegroup.com/irs/

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Table 3.4: Hull-White parameter estimation results under the FRED term structure for 25 March 2019.

Maturity	Tenor	CME IV	FRED IV	Error
1M	10Y	25.6468 %	23.2315 %	-2.4153 %
1Y	10Y	24.9363 %	23.2388 %	-1.6976 %
2Y	10Y	24.9987 %	22.4798 %	-2.5189 %
3M	10Y	24.7444 %	23.2248 %	-1.5196 %
6M	10Y	24.4420 %	23.2160 %	-1.2260 %
1M	15Y	23.5886 %	21.9450 %	-1.6435 %
1Y	15Y	22.7436 %	21.9306 %	-0.8130 %
2Y	15Y	22.9923 %	21.4890 %	-1.5033 %
3M	15Y	22.6666 %	21.9351 %	-0.7314 %
6M	15Y	22.3059 %	21.9266 %	-0.3793 %
1M	1Y	13.5499 %	20.5396 %	6.9897 %
1Y	1Y	27.1094 %	28.5188 %	1.4095 %
2Y	1Y	32.0742 %	26.8778 %	-5.1964 %
3M	1Y	15.9987 %	21.1397 %	5.1411 %
6M	1Y	20.3893 %	22.7718 %	2.3825 %
1M	20Y	22.5306 %	21.4266 %	-1.1040 %
1Y	20Y	21.7343 %	21.4546 %	-0.2797 %
2Y	20Y	21.9417 %	21.1766 %	-0.7651 %
3M	20Y	21.5389 %	21.4258 %	-0.1131 %
6M	20Y	21.2608 %	21.4304 %	0.1696 %
1M	2Y	23.4340 %	24.1010 %	0.6670 %
1Y	2Y	29.5842 %	27.6356 %	-1.9486 %
2Y	2Y	32.3447 %	26.2886 %	-6.0561 %
3M	2Y	24.3470 %	24.7168 %	0.3698 %
6M	2Y	26.0533 %	25.6451 %	-0.4082 %
1M	30Y	21.5371 %	21.2294 %	-0.3077 %
1Y	30Y	20.8012 %	21.2636 %	0.4624 %
2Y	30Y	21.1676 %	21.1010 %	-0.0666 %
3M	30Y	20.6135 %	21.2339 %	0.6204 %
6M	30Y	20.3160 %	21.2395 %	0.9235 %
1M	5Y	28.3160 %	24.9714 %	-3.3447 %
1Y	5Y	28.3852 %	25.8078 %	-2.5775 %
2Y	5Y	28.5634 %	24.7178 %	-3.8456 %
3M	5Y	27.5204 %	25.1006 %	-2.4197 %
6M	5Y	27.4131 %	25.3208 %	-2.0923 %
$a^* = 0.00009884$, $b^* = 0.00541078$			Average error: 1.8317%	

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Table 3.5: Hull-White parameter estimation results under the CMY term structure for 25 March 2019.

Maturity	Tenor	CME IV	CMY IV	Error
1M	10Y	25.6468 %	22.4542 %	-3.1926 %
1Y	10Y	24.9363 %	22.4776 %	-2.4587 %
2Y	10Y	24.9987 %	22.5041 %	-2.4946 %
3M	10Y	24.7444 %	22.4552 %	-2.2892 %
6M	10Y	24.4420 %	22.4637 %	-1.9784 %
1M	15Y	23.5886 %	22.3789 %	-1.2097 %
1Y	15Y	22.7436 %	22.3999 %	-0.3437 %
2Y	15Y	22.9923 %	22.4197 %	-0.5726 %
3M	15Y	22.6666 %	22.3825 %	-0.2841 %
6M	15Y	22.3059 %	22.3881 %	0.0822 %
1M	1Y	13.5499 %	22.5980 %	9.0481 %
1Y	1Y	27.1094 %	22.6269 %	-4.4824 %
2Y	1Y	32.0742 %	22.6576 %	-9.4166 %
3M	1Y	15.9987 %	22.6037 %	6.6050 %
6M	1Y	20.3893 %	22.6114 %	2.2221 %
1M	20Y	22.5306 %	22.3081 %	-0.2225 %
1Y	20Y	21.7343 %	22.3219 %	0.5876 %
2Y	20Y	21.9417 %	22.3367 %	0.3950 %
3M	20Y	21.5389 %	22.3128 %	0.7739 %
6M	20Y	21.2608 %	22.3144 %	1.0536 %
1M	2Y	23.4340 %	22.5814 %	-0.8526 %
1Y	2Y	29.5842 %	22.6097 %	-6.9745 %
2Y	2Y	32.3447 %	22.6404 %	-9.7042 %
3M	2Y	24.3470 %	22.5885 %	-1.7585 %
6M	2Y	26.0533 %	22.5942 %	-3.4591 %
1M	30Y	21.5371 %	22.1797 %	0.6426 %
1Y	30Y	20.8012 %	22.1767 %	1.3755 %
2Y	30Y	21.1676 %	22.1739 %	1.0062 %
3M	30Y	20.6135 %	22.1780 %	1.5645 %
6M	30Y	20.3160 %	22.1781 %	1.8621 %
1M	5Y	28.3160 %	22.5341 %	-5.7819 %
1Y	5Y	28.3852 %	22.5594 %	-5.8259 %
2Y	5Y	28.5634 %	22.5892 %	-5.9742 %
3M	5Y	27.5204 %	22.5369 %	-4.9834 %
6M	5Y	27.4131 %	22.5444 %	-4.8686 %
$a^* = 0.00150478$, $b^* = 0.00641566$			Average error: 3.0385%	

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Table 3.6: Terms of a Typical Callable Zero Coupon Bond

Denominated Currency	USD
Issue Price	100%
Tenor	Maturity Date specific
Coupon	Zero coupon
Accrual Yield	Quoted as a percentage per annum, calculated on the issue date
Day Count Fraction / Convention	30/360
Call Option (redemption rights)	Callable on each Redemption Date, at the call option of the issuer
Callable Structure	Non-call Period × Call Frequency
Early Redemption	Applicable; in the event of a Call Option exercised by the issuer
Partial Redemption	Not applicable, i.e. only full redemption is allowed
Final Redemption Date	Maturity Date
Status	Senior debt
Credit rating	Between S&P BBB- and AA+

Note: *For unsecured bonds, issuers require long-term credit rating between S&P BBB- and AA+, or their equivalent approved by the Securities and Futures Bureau (SFB). For secured bonds, guarantors are required to have long-term credit rating between S&P BBB- and AA+, or their equivalent approved by the SFB.* Source: Taipei Exchange.

the redemption dates, while IRR affects the callable bond's initial value through its redemption value at each redemption date. Let the initial value of the callable bond be $P(\mathrm{IRR}, \mathrm{NC}, \mathrm{FREQ})$, then by setting the interest rate term structure (25 March 2019) and the tenor of the callable bond to be 30 years, we can obtain the initial value of the callable bond under various combinations of IRR, NC, and FREQ (Figure 3.4). We set the IRR between 3.5% to 5.5%. The nine scenarios of NC, from a minimum of one year to eight years, and a maximum of ten years; where FREQ as no redemption, annual redemption or redemption exercised every five years. These reflect the current trading terms struck between the life insurance companies and the bond provider in accordance with regulation. Furthermore, the value of a callable bond can be decomposed in the form (Fabozzi & Mann (2010, p.170))

$$\text{Value of a callable bond} = \text{Value of an option-free bond - Value of a call option.}$$

In other words, the value of the initial embedded option is the value difference of the non-callable bond and the callable bond; the results are shown in Figure 3.5.

As a matter of interest, we compared the "theoretical price" of these zero coupon callable international bonds as published by TPEx with ours, an excerpt is listed in Table 3.7. The TPEx' system construct and its model assumptions are not publicly available, thus one cannot make a straight line comparison between the two systems; however, while comparing these theoretical prices, we did discover some anomalies:

- Both bonds F03905 and F03906 are issued by DBS Bank Ltd.. While the issue date, maturity date, and IRR are the same, it's redemption structure are 1×1 and 2×1 respectively. However, TPEx shows both with the same price of 112.196%. Given that under the same issue date, maturity date, and IRR, the embedded option of a

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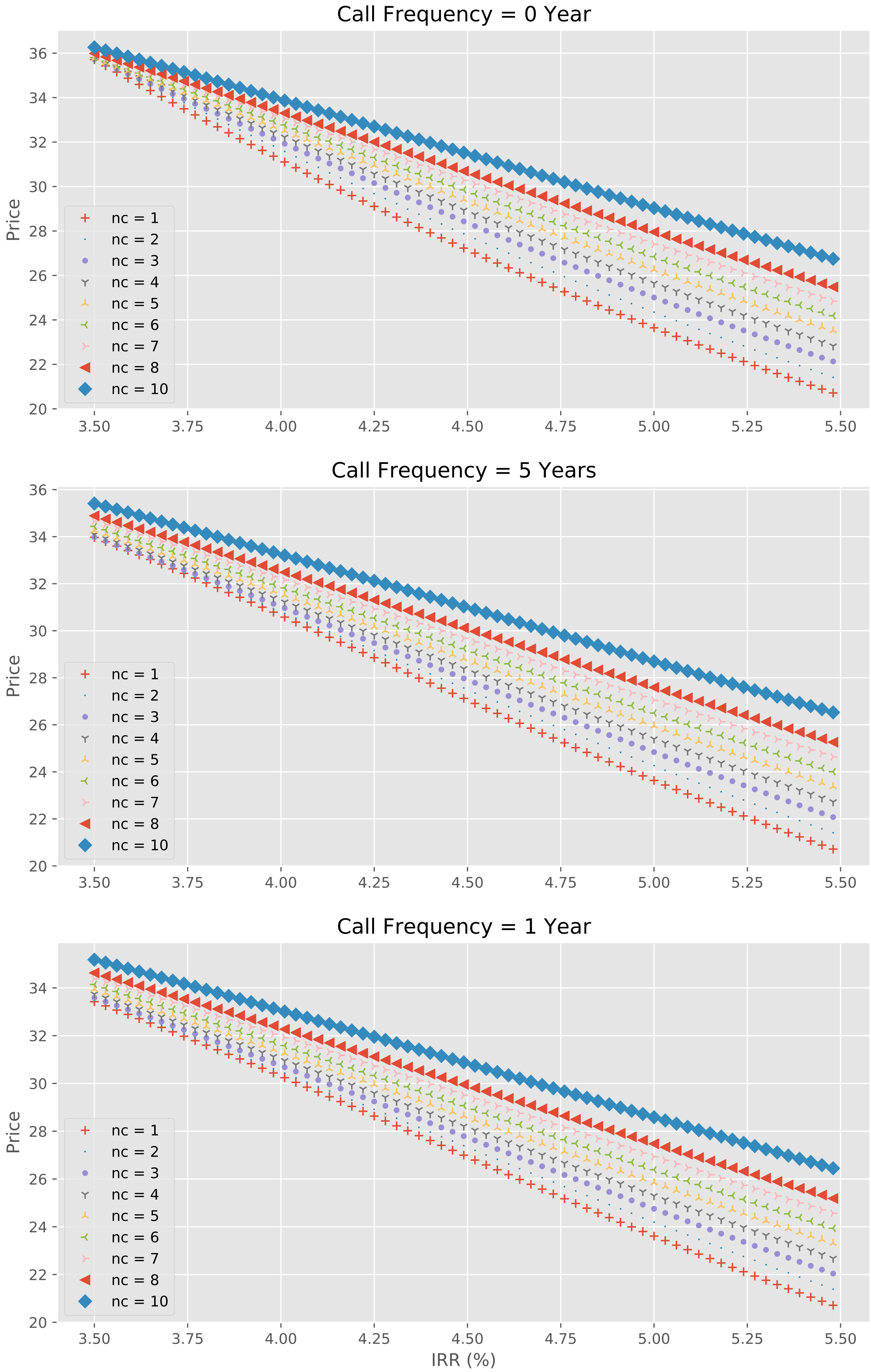

Figure 3.4: The dynamics of the initial value of the 30-year callable bond, non-callable period, call frequency, and IRR under the interest rate term structure for 25 March 2019.

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1×1 redemption structure would be more valuable than a 2×1, due to it having an extra opportunity to call. According to the price computed by our program, they are 112.178% and 112.867% respectively; it is evident that these prices should not be identical.

- F03915 and F03916 both have a tenor of 30 years, with IRR of 4.6% and 4.5% respectively; while their issue dates are 22 February 2019 and 11 March 2019, their price are listed by TPEx as 107.767% and 107.102% on valuation date (25 March 2019). Given that these two bonds are issued within four weeks of the valuation date, and the IRRs are below 5.0%, it is unreasonable to expect a price increase of over 7.0% at par.

3.5 Concluding remarks

Due to the sheer volume of the international bonds listed on the Taipei Exchange and the lack of a liquid secondary market, the need for pricing transparency and a reliable source of reference is of utmost importance. We provide the life insurers the means to evaluate the mark-to-market value of these callable bonds without having to rely on third parties to do so. We first assume the market's short rate follows the Hull-White process, then collate publicly available data and make use of open source software to construct a bespoke system that is able to independently evaluate the fixed income type asset the life insurer holds. We computed the theoretical value of the international bonds based on the single factor Hull-White short rate model, these results are yet to be compared with other interest rate models; future research can investigate a more sophisticated model to incorporate factors such as exchange rates, interest rate movements, and default risk.

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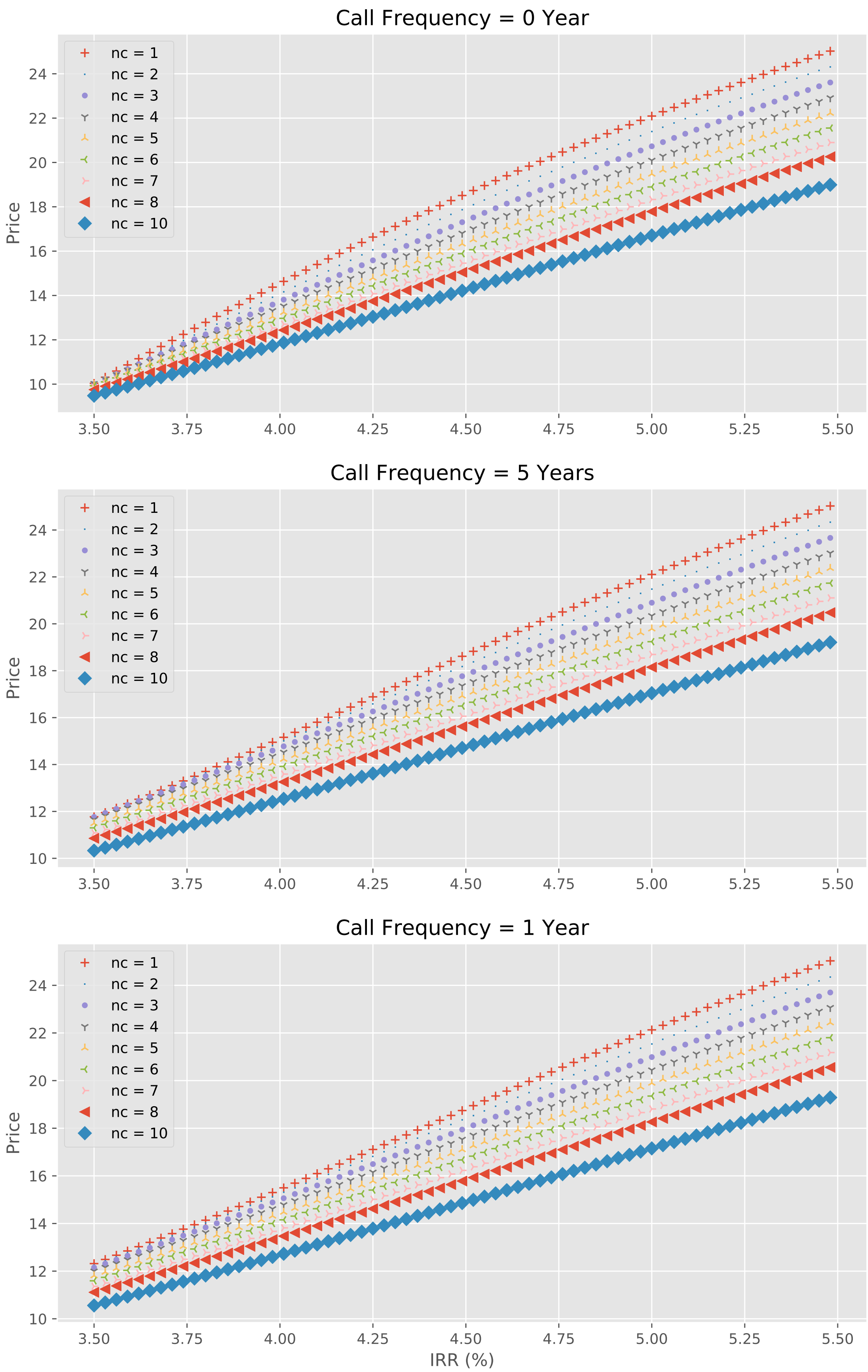

Figure 3.5: The dynamics of the initial embedded option value of the 30-year callable bond, non-callable period, call frequency, and IRR under the interest rate term structure for 25 March 2019.

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Table 3.7: Numerical illustration for selected callable bonds listed on the TPEx under the term structure for 25 March 2019.

Bond	Rating	Issue Date	Maturity	IRR	Spec	TPEx	FRED	CMY
F03905	AA-	2016/05/06	2046/05/06	4.000	2 x 1	112.196	112.178	109.543
F03906	AA-	2016/05/06	2046/05/06	4.000	1 x 1	112.196	112.867	109.878
F03907	AA-	2016/07/26	2046/07/26	3.610	2 x 1	101.539	111.109	107.454
F03908	AA-	2016/07/26	2046/07/26	3.630	1 x 1	102.128	111.384	107.619
F03909	AA-	2016/08/30	2046/08/30	3.470	1 x 1	97.060	110.769	106.776
F03910	AA-	2017/01/24	2047/01/24	3.900	3 x 1	107.810	108.143	106.194
F03911	AA-	2018/02/02	2048/02/02	4.020	5 x 1	106.140	103.918	102.548
F03912	AA-	2018/02/12	2048/02/12	4.060	1 x 1	105.821	104.813	103.579
F03913	AA-	2018/04/03	2048/04/03	4.350	5 x 1	109.551	103.192	102.284
F03914	AA-	2018/06/01	2048/06/01	4.500	5 x 1	109.782	102.605	101.917
F03915	AA-	2019/02/22	2049/02/22	4.600	5 x 1	107.797	100.282	100.193
F03916	AA-	2019/03/11	2049/03/11	4.500	5 x 1	107.102	100.125	100.085
F04001	A-	2014/10/03	2044/10/03	4.750	5 x 1	124.268	113.558	113.472
F04003	A-	2015/02/03	2045/02/03	4.150	5 x 1	116.445	114.592	111.903
F04004	A-	2015/02/26	2045/02/26	4.300	5 x 1	119.977	113.806	111.845
F04005	A-	2015/06/10	2045/06/10	4.600	5 x 1	120.435	111.750	110.935
F04008	A-	2016/02/03	2046/02/03	4.600	5 x 1	117.899	109.848	108.691
F04009	A-	2016/03/18	2046/03/18	4.630	5 x 1	117.583	109.428	108.297
F04012	A-	2016/08/03	2046/08/03	3.980	2 x 1	102.831	110.998	108.563
F04013	A-	2016/11/10	2046/11/10	4.110	5 x 1	105.301	108.201	105.933
F04014	A-	2017/02/09	2047/02/09	4.430	5 x 1	112.736	106.924	105.432
F04015	A-	2017/06/21	2047/06/21	4.350	5 x 1	109.325	105.800	104.362
F04016	A-	2017/09/05	2047/09/05	4.300	12 x 1	106.623	105.151	101.827
F04017	A-	2017/11/03	2047/11/03	4.250	5 x 5	104.223	104.627	103.454
F04018	A-	2018/02/06	2048/02/06	4.230	6 x 5	102.236	103.784	102.477
F04019	A-	2018/03/14	2048/03/14	4.530	6 x 5	109.340	103.307	102.323
F04020	A-	2018/06/20	2048/06/20	4.880	6 x 5	110.373	102.313	101.731
F04101	A+	2014/10/03	2044/10/03	4.750	10 x 5	133.145	114.862	108.852
F04102	A+	2014/10/21	2044/10/21	4.720	10 x 1	132.502	114.796	108.567
F04103	A+	2015/01/23	2045/01/23	4.350	5 x 1	120.950	113.989	112.243
F04104	A+	2015/06/30	2045/06/30	4.800	10 x 1	129.822	112.376	106.956
F04105	A+	2015/09/18	2045/09/18	4.800	5 x 2	121.144	110.289	110.045
F04106	A+	2016/02/03	2046/02/03	4.600	8 x 1	123.017	110.340	106.641
F04107	A+	2016/06/13	2046/06/13	4.250	8 x 1	118.189	109.473	105.555
F04108	A+	2017/01/24	2047/01/24	4.300	6 x 1	114.358	107.240	104.998
F04302	A-	2014/12/15	2044/12/15	4.830	1 x 1	123.923	121.943	120.680
F04303	A-	2014/12/15	2044/12/15	4.800	3 x 1	123.746	117.016	116.726
F04308	A-	2016/02/02	2046/02/02	4.700	1 x 1	117.089	115.482	114.129
F04309	A-	2016/05/13	2046/05/13	4.400	1 x 1	113.400	112.776	111.520
F04310	A-	2016/11/25	2046/11/25	4.450	3 x 1	111.677	107.621	107.511

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4 The Copula-GARCH Model: Application to Variable Annuity Guarantee Valuations on Multiple Assets

∗ This chapter is based on Hsuan & Chang (2018a).

4.1 Introduction

The premise of effective risk management is the ability to delineate the probabilistic relationship between multiple underlying assets and resources and to derive quantitative indicators which reflect the current status of the system to be controlled. Developments of modern computing technologies have enabled the transition from traditional simplistic models to full-fledged stochastic ones with real-world considerations; multivariate probability models that can faithfully characterize their elements are in ever greater need.

The copula is such a multivariate probability distribution for which the marginal probability distribution of each variable is uniform. Sklar's theorem (c.f. theorem 4.2.1) states that any multivariate joint distribution can be written in terms of the composite of univariate marginal distribution functions and a copula function which describes the dependence structure between the variables. The copula approach could be useful to high-dimensional statistical applications as one is allowed to estimate the distribution of random vectors by estimating marginals and copula separately. The converse of Sklar's theorem, which states that the composite of arbitrary univariate marginal distribution functions and a copula function is a valid multivariate joint distribution, is equally useful for applied researchers as one may exploit the provided freedom to build the more adaptive model. For time series applications the copula concept could be extended to reflect the model dynamics; the semiparametric copula-based multivariate dynamic (SCOMDY) model proposed in Chen & Fan (2006a,b), which encompasses the copula-GARCH model (e.g. Chan et al. (2009); Jondeau & Rockinger (2006); Patton (2006)) as a special case, is such a general formulation.

In this work we review the essence of the copula-GARCH model and the associated

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statistical tests recently obtained in Bai (2003); Genest & Rémillard (2004, 2008); Ghoudi & Rémillard (2014); Nasri & Rémillard (2019); Rémillard (2011, 2012, 2017) where they deserve to be better known for their mathematical correctness. As an illustration for the copula-GARCH techniques we are able to show rigorously that the co-movement of the monthly S&P500 and S&P600 indices is best described by a certain copula-GARCH model and subsequently apply this probability model for the evaluation of corresponding variable annuity product.

A variable annuity (VA) is a type of annuity contract that allows for the accumulation of capital on a tax-deferred basis. Variable annuities offer investors the opportunity to generate higher rates of returns by investing in equity subaccounts. VA comes with embedded guarantees, also known as rider benefits, which protect the policyholder's savings. Each type of guarantee can be categorized into subclasses with different payment terms and conditions; all these guarantees exhibit option features and can be priced using financial engineering techniques.

Most practitioners use simplistic univariate stochastic models for the evaluation of VA; however, the majority of VAs are linked to multiple assets. Significant drawbacks of the univariate approach are listed in Ng & Li (2013). In a nutshell, the joint probability distribution of random variables are usually quite different and complicated to determine from the probability of each random variable; simplistic reduction leads to inaccuracies of risk assessment. In Ng & Li (2013), a discrete-time multivariate framework for pricing and hedging of VA is proposed, and is demonstrated via the development of a bivariate model with two underlying asset processes with specified linear correlations. Two asset processes, namely the multivariate regime-switching lognormal (RSLN) process and the generalized autoregressive conditional heteroskedastic (GARCH) process are utilized. In Da Fonseca & Ziveyi (2017), the continuous-time setup is considered; the underlying asset process being the multivariate generalization of the Heston process, which exhibits the stochastic volatility. Here we follow the multivariate pricing paradigm as advocated in Da Fonseca & Ziveyi (2017); Ng & Li (2013) and adopt the copula-GARCH approach for the formulation of the underlying asset processes.

The remainder of this chapter is organized as follows: section 4.2 reviews the concept of copula relevant in this study; section 4.3 introduces the copula-GARCH model; section 4.4 introduces the associated statistical tests; section 4.5 gives a detailed demonstration of the rigorous data fitting process and then proceeds on the computation of the corresponding VA contract; finally section 4.6 concludes.

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4.2 Review of the copula concept

4.2.1 Notions of copula

An d-dimensional copula $C\left(u_1, u_2, \ldots, u_d\right)$ is a joint cumulative distribution function in the unit hypercube $[0,1]^d$ with uniform margins. As an example, let $d = 2$, a copula function $C(u_1, u_2)$ can be written as

$$C(u_1, u_2) \equiv \mathsf{P}\left(U_1 \leqslant u_1, U_2 \leqslant u_2\right),$$

where the random variables U_1, U_2 are uniformly distributed on $[0, 1]$.

Given a random variable X, the cumulative distribution function F_X of X is defined as

$$F_X(x) \equiv \mathsf{P}(X \leqslant x).$$

The quantile function (generalized inverse) function F_X^{-1} of F_X is defined as

$$F_X^{-1}(u) = \inf\{x | F_X(x) \geqslant u\}, \quad u \in (0, 1).$$

Lemma. 1. Given a $[0, 1]$-uniformly distributed random variable U and a cumulative distribution function F, the random variable $X \equiv F^{-1}(U)$ has the cumulative distribution function F.

2. The random variable $U \equiv F_X(X)$ is uniformly distributed on $[0, 1]$.

Proof. 1.

$$\begin{aligned} F_X(x) &= \mathsf{P}\left(X \leqslant x\right) \\ &= \mathsf{P}\left(F^{-1}(U) \leqslant x\right) \\ &= \mathsf{P}\left(U \leqslant F(x)\right) \\ &= F(x). \end{aligned}$$

The last equality holds for U is $[0, 1]$-uniformly distributed.

2.

$$\begin{aligned} F_U(u) &= \mathsf{P}\left(U \leqslant u\right) \\ &= \mathsf{P}\left(F_X(X) \leqslant u\right) \\ &= \mathsf{P}\left(X \leqslant F_X^{-1}(u)\right) \\ &= F_X\left(F_X^{-1}(u)\right) = u, \quad u \in [0, 1] \end{aligned}$$

which is the cumulative distribution function of a $[0, 1]$-uniformly distributed random

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variable.

Using the lemma, let $X_1, X_2, \ldots, X_d$ be random variables with cumulative distribution functions $F_{X_1},\ F_{X_2},\ \ldots,\ F_{X_d}$; the random variables $U_i \equiv F_{X_i}(X_i), i = 1, 2, \ldots, d$ are uniformly distributed on $[0,1]$. Furthermore, the function

$$\begin{aligned}
C_{(X_1,X_2,\ldots,X_d)}(u_1,u_2,\ldots,u_d) &\equiv \mathsf{P}\left(U_1 \leqslant u_1, U_2 \leqslant u_2, \ldots, U_d \leqslant u_d\right) \\
&= \mathsf{P}\left(F_{X_1}(X_1) \leqslant u_1, F_{X_2}(X_2) \leqslant u_2, \ldots, F_{X_d}(X_d) \leqslant u_d\right) \\
&= \mathsf{P}\left(X_1 \leqslant F_{X_1}^{-1}(u_1), X_2 \leqslant F_{X_2}^{-1}(u_2), \ldots, X_d \leqslant F_{X_d}^{-1}(u_d)\right)
\end{aligned}$$

is a copula. The joint cumulative distribution function of $(X_1, X_2, \ldots, X_d)$ can be recovered as

$$\begin{aligned}
&\mathsf{P}\left(X_1 \leqslant x_1, X_2 \leqslant x_2, \ldots, X_d \leqslant x_d\right) \\
&\quad = \mathsf{P}\left(F_{X_1}(X_1) \leqslant F_{X_1}(x_1), F_{X_2}(X_2) \leqslant F_{X_2}(x_2), \ldots, F_{X_d}(X_d) \leqslant F_{X_d}(x_d)\right) \\
&\quad = \mathsf{P}\left(U_1 \leqslant F_{X_1}(x_1), U_2 \leqslant F_{X_2}(x_2), \ldots, U_d \leqslant F_{X_d}(x_d)\right) \\
&\quad = C_{(X_1,X_2,\ldots,X_d)}\left(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_d}(x_d)\right)
\end{aligned}$$

The above discussion culminates in the

Theorem 4.2.1 (Sklar (1959); c.f. Durante & Sampi (2016); Joe (2014))**.** *For a d-variate distribution F with j-th univariate margin F_{X_j}, the copula associated with F is a distribution function $C : [0,1]^d \to [0,1]$ with $U(0,1)$ margins that satisfies*

$$F(x_1, x_2, \ldots, x_d) = C\left(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_d}(x_d)\right). \tag{4.2.1}$$

If F is a continuous d-variate distribution function with univariate margins $F_{X_1}, F_{X_2}, \cdots, F_{X_d}$ and quantile functions $F_{X_1}^{-1}, F_{X_2}^{-1}, \ldots, F_{X_d}^{-1}$, then

$$C(u_1, u_2, \ldots, u_d) = F\left(F_{X_1}^{-1}(u_1), F_{X_2}^{-1}(u_2), \ldots, F_{X_d}^{-1}(u_d)\right) \tag{4.2.2}$$

is the unique choice.

The copula function $C : [0,1]^d \to [0,1]$ has the following properties:

1. $C(u_1, u_2, \ldots, u_d) = 0$ if at least one $u_i = 0$.

2. $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$.

3. For each d-dimensional rectangle $\mathbb{T} \equiv [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d],\ a_i, b_i \in [0,1], a_i < b_i$,

$$0 \leqslant \sum_{(m_1,m_2,\ldots,m_d)\in\mathbb{T}} (-1)^{\#\{j:m_j=a_j\}} C(m_1, m_2, \ldots, m_d) \leqslant 1.$$

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For example, if $n = 2$, then

$$0 \leqslant (-1)^0 C(b_1, b_2) + (-1)^2 C(a_1, a_2) + (-1)^1 C(b_1, a_2) + (-1)^1 C(a_1, b_2) \leqslant 1.$$

The copula uniquely determined in $[0,1]^d$ for distributions F under absolutely continuous margins F_{X_i} has the density function (obtained by successively differentiate the cdf (4.2.1) with respect to its arguments)

$$f(x_1, x_2 \ldots, x_d) = c\left(F_{X_1}(x_1), F_{X_2}(x_2) \ldots, F_{X_d}(x_d)\right) \prod_{i=1}^{d} f_{X_i}(x_i) \tag{4.2.3}$$

where f_{X_i} are the marginal densities and c is the density function of the copula given by

$$c(u_1, u_2 \ldots, u_d) = \frac{f\left(F_{X_1}^{-1}(u_1), F_{X_2}^{-1}(u_2) \ldots, F_{X_d}^{-1}(u_d)\right)}{\prod_{i=1}^{d} f_{X_i}\left(F_{X_i}^{-1}(u_i)\right)}.$$

Another theorem important for the inference of copula is

Theorem 4.2.2 (Rank-Invariance)**.** *(c.f. Durante & Sampi (2016, Theorem 2.4.1)) Suppose the random variables $X_1, X_2, \ldots, X_d$ have continuous marginals and copula C_X and consider d continuous and strictly increasing mappings $\phi_i : \operatorname{ran} X_i \to \mathbb{R}$, $i = 1, 2, \ldots, d$. Then the dependence of the random variables $Y_i = \phi_i(X_i)$, $i = 1, 2, \ldots, d$ is also given by the copula C_X.*

Proof. Note that

$$F_{Y_i}(t) = \mathsf{P}(Y_i \leqslant t) = \mathsf{P}(\phi_i(X_i) \leqslant t) = \mathsf{P}(X_i \leqslant \phi_i^{-1}(t)) = F_{X_i}(\phi_i^{-1}(t)),$$

so a priori

$$F_{Y_i}^{-1}(t) = \phi_i\left(F_{X_i}^{-1}(t)\right). \tag{4.2.4}$$

Now

$$\begin{aligned}
C_Y(u_1, u_2, \ldots, u_d) &= F_Y\left(F_{Y_1}^{-1}(u_1), F_{Y_2}^{-1}(u_2), \ldots, F_{Y_d}^{-1}(u_d)\right) \quad \text{by (4.2.2)} \\
&= \mathsf{P}\left(Y_1 \leqslant F_{Y_1}^{-1}(u_1), Y_2 \leqslant F_{Y_2}^{-1}(u_2), \ldots, Y_d \leqslant F_{Y_d}^{-1}(u_d)\right) \\
&= \mathsf{P}\left(\phi_1(X_1) \leqslant F_{Y_1}^{-1}(u_1), \phi_2(X_2) \leqslant F_{Y_2}^{-1}(u_2), \ldots, \phi_d(X_d) \leqslant F_{Y_d}^{-1}(u_d)\right) \\
&= \mathsf{P}\left(X_1 \leqslant \phi_1^{-1}(F_{Y_1}^{-1}(u_1)), X_2 \leqslant \phi_2^{-1}(F_{Y_2}^{-1}(u_2)), \ldots, X_d \leqslant \phi_d^{-1}(F_{Y_d}^{-1}(u_d))\right) \\
&= \mathsf{P}\left(X_1 \leqslant F_{X_1}^{-1}(u_1), X_2 \leqslant F_{X_2}^{-1}(u_2), \ldots, X_d \leqslant F_{X_d}^{-1}(u_d)\right) \quad \text{by (4.2.4)} \\
&= C_X(u_1, u_2, \ldots, u_d) \quad \text{by (4.2.2)}.
\end{aligned}$$

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4.2.2 Inference of copula

Consider the independent observations $\{(x_{1t}, x_{2t}, \ldots x_{dt});\ t = 1, 2, \ldots, T\}$; in the full-parametric case assume that we have parametric models $F_{X_1}(\cdot; \alpha_1)$, $F_{X_2}(\cdot; \alpha_2), \ldots, F_{X_d}(\cdot; \alpha_d)$ for the marginal cumulative distribution functions and the parametric copula $C(\cdot; \theta)$ for the copula density. We wish to estimate the parameters $\widehat{\alpha_1}, \widehat{\alpha_2}, \ldots, \widehat{\alpha_d}, \widehat{\theta}$ such that the observations fit the parametric model best; one of the most venerable way of doing this is the maximum likelihood method: define the likelihood function

$$\mathcal{L}(\alpha_1, \alpha_2, \ldots, \alpha_d, \theta) = \prod_{t=1}^{T} \left\{ c\left(F_{X_1}(x_{1t}; \alpha_1), F_{X_2}(x_{2t}; \alpha_2) \ldots, F_{X_d}(x_{dt}; \alpha_d);\ \theta\right) \prod_{i=1}^{d} f_{X_i}(x_{it};\ \alpha_i) \right\}$$

then $\widehat{\alpha_1}, \widehat{\alpha_2}, \ldots, \widehat{\alpha_d}, \widehat{\theta}$ are determined by

$$(\widehat{\alpha_1}, \widehat{\alpha_2}, \ldots, \widehat{\alpha_d}, \widehat{\theta}) = \underset{\alpha_1, \alpha_2, \ldots, \alpha_d, \theta}{\operatorname{argmax}} \mathcal{L}(\alpha_1, \alpha_2, \ldots, \alpha_d, \theta).$$

This $\widehat{\alpha_1}, \widehat{\alpha_2}, \ldots, \widehat{\alpha_d}, \widehat{\theta}$ also maximize the logarithm of $\mathcal{L}(\alpha_1, \alpha_2, \ldots, \alpha_d, \theta)$,

$$\begin{aligned}\log \mathcal{L}(\alpha_1, \alpha_2, \ldots, \alpha_d, \theta) = \sum_{t=1}^{T} \log c\left(F_{X_1}(x_{1t}; \alpha_1), F_{X_2}(x_{2t}; \alpha_2), \ldots, F_{X_d}(x_{dt}; \alpha_d);\ \theta\right) \\ + \sum_{t=1}^{T} \sum_{i=1}^{d} \log f_{X_i}(x_{it};\ \alpha_i),\end{aligned} \tag{4.2.5}$$

which is more computationally convenient.

A variation of the above maximum likelihood method is that, instead of simultaneously determining $\widehat{\alpha_1}, \widehat{\alpha_2}, \ldots, \widehat{\alpha_d}, \widehat{\theta}$ by maximizing the log likelihood function of (4.2.5), one obtain the optimal parameters $\widetilde{\alpha_1}, \widetilde{\alpha_2}, \ldots, \widetilde{\alpha_d}, \widetilde{\theta}$ successively by

$$\widetilde{\alpha_i} = \underset{\alpha_i}{\operatorname{argmax}} \sum_{t=1}^{T} \log f_{X_i}(x_{it}, \alpha_i), \quad i = 1, 2, \ldots, d$$

and

$$\widetilde{\theta} = \underset{\theta}{\operatorname{argmax}} \sum_{t=1}^{T} \log c\left(F_{X_1}(x_{1t}; \widetilde{\alpha_1}), F_{X_2}(x_{2t}; \widetilde{\alpha_2}), \ldots, F_{X_d}(x_{dt}; \widetilde{\alpha_d});\ \theta\right).$$

Joe (2005) establishes the asymptotic efficiency of this optimization scheme.

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4.2.3 Parametric copula families

We have put forward some salient features of the parametric copula families pertinent to this study; for an extensive overview of parametric copula families see Joe (2014, Chapter 4).

4.2.3.1 Gaussian copula

Using Sklar's theorem (4.2.2), given a correlation matrix R, the Gaussian copula can be written as

$$C(u_1, u_2, \ldots, u_d) = \Phi_R\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots \Phi^{-1}(u_d)\right)$$

where $\Phi_R : \mathbb{R}^d \to \mathbb{R}$ is the joint cumulative distribution function of a multivariate Gaussian distribution with mean 0 and covariance R, and Φ^{-1} is the inverse of the cumulative distribution function of a standard Gaussian distribution. In other words,

$$C(u_1, u_2, \ldots, u_d) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \cdots \int_{-\infty}^{\Phi^{-1}(u_d)} \frac{\exp\left(-\frac{1}{2}x' R^{-1} x\right)}{\sqrt{(2\pi)^d |R|}} \, \mathrm{d}x$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}\tau^2\right) \, \mathrm{d}\tau.$$

The Gaussian copula is of the elliptical family, which means the trace of distribution is elliptically contoured and concentrated.

4.2.3.2 Student t copula

A random vector $x \in \mathbb{R}^d$ is said to be a multivariate t distribution with ν degrees of freedom, mean vector μ and positive-definite dispersion matrix Σ if the probabilistic density function $f(x)$ is

$$f(x) = \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^d |\Sigma|}} \left(1 + \frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{\nu}\right)^{-\frac{\nu+d}{2}}$$

Again using (4.2.2), the Student t copula can be written as

$$C(u_1, u_2, \ldots, u_d) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_d)} \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^d |P|}} \left(1 + \frac{x' P^{-1} x}{\nu}\right)^{-\frac{\nu+d}{2}} \, \mathrm{d}x$$

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where the covariance matrix $P = \frac{\nu}{\nu-2}\Sigma$ and

$$t_\nu(x) = \int_{-\infty}^{x} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}} \left(1 + \frac{\tau^2}{\nu}\right)^{-\frac{\nu+1}{2}} \, \mathrm{d}\tau.$$

For $d = 2$, P can be written as

$$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

with constant ρ. The Student t copula is also of the elliptical family, but with more tail points than the Gaussian copula and better suited for extreme events formulation.

4.2.3.3 Archimedean copula

A copula is called Archimedean if it has the representation

$$C(u_1, u_2, \ldots, u_d; \theta) = \psi^{-1}(\psi(u_1; \theta) + \psi(u_2; \theta) + \cdots + \psi(u_d; \theta); \theta)$$

where $\psi : [0,1] \times \Theta \to [0, \infty)$ is a continuous, strictly increasing and convex function with $\psi(1; \theta) = 0$, and ψ^{-1} is the pseudo inverse function of ψ defined as

$$\psi^{-1}(t; \theta) = \begin{cases} \psi^{(-1)}(t; \theta) & 0 \leqslant t \leqslant \psi(0, \theta) \\ 0 & \psi(0, \theta) \leqslant t \leqslant \infty \end{cases}$$

and $\psi^{(-1)}$ is the ordinary inverse function of ψ. Archimedean copulas are popular for their simple, closed-form representations. Table 4.1 is a summary of common bivariate Archimedean copulas.

name	$\psi(t;\theta)$	$\psi^{-1}(t;\theta)$	$C(u_1,u_2)$	range of θ
Ali-Mikhail-Haq	$\log \frac{1-\theta(1-t)}{t}$	$\frac{1-\theta}{e^t-\theta}$	$\frac{u_1 u_2}{1-\theta(1-u_1)(1-u_2)}$	$\theta \in [-1, 1)$
Clayton	$\frac{1}{\theta}\left(t^{-\theta} - 1\right)$	$(1+\theta t)^{-\frac{1}{\theta}}$	$\left(\max(u_1^{-\frac{1}{\theta}} + u_2^{-\frac{1}{\theta}} - 1, 0)\right)^{-\frac{1}{\theta}}$	$\theta \in [-1, \infty) \setminus \{0\}$
Frank	$-\log \frac{e^{-\theta t}-1}{e^{-\theta}-1}$	$-\frac{1}{\theta}\log\left(1 - e^{-t} + e^{-(\theta+t)}\right)$	$-\frac{1}{\theta}\log\left(1 + \frac{\left(e^{-\theta u_1}\right)\left(e^{-\theta u_2}-1\right)}{e^{-\theta}-1}\right)$	$\theta \in \mathbb{R} \setminus \{0\}$
Gumbel	$(-\log t)^\theta$	$e^{-t^{\frac{1}{\theta}}}$	$e^{-\left((-\log u_1)^\theta + (-\log u_2)^\theta\right)^{\frac{1}{\theta}}}$	$\theta \in [1, \infty)$

Table 4.1: Common Archimedean Copulas

4.2.4 Conditional copula and beyond

The application we have in mind is the contingent claim pricing problem

$$(\text{claim price}) = e^{-rT}\, \mathsf{E}^Q \left\{\Phi(S_T) | \mathcal{F}_0\right\}$$

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where r is the risk-free interest rate, T the time horizon, $S_T \equiv (S_{1,T}, S_{2,T}, \dots, S_{d,T})$ the value of d assets S_i at time T, Φ the payoff function, Q the martingale measure, and $\mathcal{F}_0$ the filtration at initial time 0. We will specify the distribution of S_T as a copula C_T, and naturally the term $C_t(\cdot|\mathcal{F}_0)$ also needs to be specified. In Patton (2006), the author formalizes the idea as follows. Let $X = (X_1, X_2, \dots, X_d)$ be a d-dimensional random vector from $(\Omega, \mathcal{A}_0, \mathsf{P})$ to $\mathbb{R}^d$. For a sub-algebra $\mathcal{A} \subseteq \mathcal{A}_0$, the conditional copula C^* associated with $(X, \mathcal{A})$ is defined as, for $x \equiv (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$,

$$\mathsf{P}\left(X \leqslant x|\mathcal{A}\right) = C^*\left(\mathsf{P}\left(X_1 \leqslant x_1|\mathcal{A}\right), \mathsf{P}\left(X_2 \leqslant x_2|\mathcal{A}\right), \dots, \mathsf{P}\left(X_d \leqslant x_d|\mathcal{A}\right)|\mathcal{A}\right)$$

Sklar's theorem holds under this extension: $C^*(\cdot|\mathcal{A})$ is an uniquely defined copula for $\mathcal{A} \subseteq \mathcal{A}_0$. However, in this formulation the information set $\mathcal{A}$ should be the same for all margins, which is inconvenient for practitioners; it is desirable to estimate each margin using its own past information and introduce the dependency later.

Consider the d-dimensional process $\{X_m \equiv (X_{m,1}, X_{m,2}, \dots, X_{m,d})\}$, $m \in \mathbb{N}$; we are interested in the copula function $C^*(\cdot|\mathcal{A}_m)$, where $\mathcal{A}_m = \sigma(X_{m-1}, X_{m-2}, \dots, X_1)$ is the filtration generated by past $m-1$ vectors. Similarly, define $\mathcal{A}_{m,i} = \sigma(X_{m-1,i}, X_{m-2,i}, \dots, X_{1,i})$ as the filtration generated by past $m-1$ i-th component of X.. Formally, we would like to have

$$\mathsf{P}\left(X_m \leqslant x|\mathcal{A}_m\right) = C^*\left(\mathsf{P}\left(X_{m,1} \leqslant x_1|\mathcal{A}_{m,1}\right), \mathsf{P}\left(X_{m,2} \leqslant x_2|\mathcal{A}_{m,2}\right), \dots, \mathsf{P}\left(X_{m,d} \leqslant x_d|\mathcal{A}_{m,d}\right)\right)$$

Should C^* be a copula, by property 2 we have

$$\mathsf{P}\left(X_{m,1} \leqslant x_1|\mathcal{A}_m\right) = C^*\left(\mathsf{P}\left(X_{m,1} \leqslant x_1|\mathcal{A}_{m,1}\right), 1, 1, \dots, 1\right) = \mathsf{P}\left(X_{m,1} \leqslant x_1|\mathcal{A}_{m,1}\right)$$

and the same relation holds for all margins. The implication is strong — each variable depends on its own history only, which is not always the case in applications. Fermanian & Wegkamp (2012) introduce notions to mitigate this problem, but inference and estimation problems persist. In order to proceed, we have to content ourselves with this simplifying assumption as in (almost all) other papers, e.g. Chiou & Tsay (2008); Patton (2006); Rosenberg (1998, 2003); van den Goorbergh et al. (2005); Zhang & Guégan (2008).

4.3 The Copula-GARCH model

The Semiparametric Copula-Based Multivariate Dynamic (SCOMDY) model introduced in Chan et al. (2009); Chen & Fan (2006a,b) is a class of multivariate time series model which is characterized by parametric multivariate conditional mean and variance, an infinite-dimensional marginal distribution of the individual standardized innovation, and the parametric copula of all standardized innovations. The d-dimensional time series $\boldsymbol{x}_t$

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which satisfies the SCOMDY model can be represented as

$$\boldsymbol{x}_t = \boldsymbol{\mu}_t + \boldsymbol{\sigma}_t\, \boldsymbol{\varepsilon}_t \tag{4.3.1}$$

where $\boldsymbol{\mu}_t = \mathsf{E}\{\boldsymbol{x}_t \,|\, \mathcal{F}_{t-1}\}$, $\boldsymbol{\sigma}_t = \operatorname{diag}\left(\sqrt{\mathsf{E}\{(\boldsymbol{x}_t - \boldsymbol{\mu}_t)^2 \,|\, \mathcal{F}_{t-1}\}}\right)$ and $\mathcal{F}_{t-1}$ denotes the σ-algebra (information set) up to time $t-1$. The standardized innovations $\boldsymbol{\varepsilon}_t \equiv (\varepsilon_{1t}, \varepsilon_{2t}, \ldots, \varepsilon_{dt})$ are independent of $\mathcal{F}_{t-1}$, i.i.d. distributed with $\mathsf{E}\{\varepsilon_{it}\} = 0$ and $\mathsf{E}\{\varepsilon_{it}^2\} = 1$ for each i, and the distribution function of $\boldsymbol{\varepsilon}_t$ is of the form $c(F_1(\varepsilon_{1t}), F_2(\varepsilon_{2t}), \ldots, F_d(\varepsilon_{dt}))$, where each F_i is the marginal distribution of ε_i and c is a parametric copula.

The Copula-GARCH model is a special case of SCOMDY where each component of the multivariate time series is of GARCH(p, q); an univariate time series x_t is said to be of GARCH(p, q) if

$$x_t = \mu + \sigma_t\, \varepsilon_t, \quad \sigma_t^2 = \omega + \sum_{j=1}^{q} \alpha_j\, \varepsilon_{t-j}^2 + \sum_{j=1}^{p} \beta_j\, \sigma_{t-j}^2$$

where μ, ω are constants, σ_t^2 denotes the conditional variance, and ε_t the i.i.d. random numbers with mean 0 and variance 1. For $p, q = 1$ we simply write $\alpha \equiv \alpha_1$ and $\beta \equiv \beta_1$. Two of the most common choices of the distribution of the innovation ε_t are the Gaussian distribution and the generalized error distribution. A random variable X is of the generalized error distribution with parameter ν, denoted as $X \sim \operatorname{ged}(\nu)$, if the probability density function $f(x)$ of X is

$$f(x) = \frac{1}{\zeta\, 2^{1+\frac{1}{\nu}}\, \Gamma\left(1 + \frac{1}{\nu}\right)} e^{-\frac{1}{2}\left(\frac{|x|}{\zeta}\right)^{\nu}}, \quad x \in \mathbb{R}, \quad \zeta = 2^{-\frac{1}{\nu}} \frac{\Gamma\left(\frac{1}{\nu}\right)}{\Gamma\left(\frac{3}{\nu}\right)}.$$

In this case $\mathsf{E}\, X = 0$, $\mathsf{E}\, X^2 = 1$ and hence $\operatorname{var} X = 1$. Note that the special case $\nu = 2$ corresponds to the Gaussian distribution.

4.4 Statistical tests

Here we briefly introduce all statistical tests used in our research. Theoretical underpinnings of these tests are results of empirical processes (c.f. Billingsley (1999); Dudley (1999); Gaenssler & Stute (1987); Kosorok (2008); Shorack & Wellner (1986); van der Vaart & Wellner (1996)), bootstrap sampling, and properties of copula; the full elucidation and validation of these tests is beyond the scope of this chapter and are detailed in the original papers of Bai (2003); Genest & Rémillard (2004, 2008); Ghoudi & Rémillard (2014); Nasri & Rémillard (2018); Rémillard (2011, 2012, 2017).

The necessity of introducing these statistical tests is amplified by the prevalent practice of the followings:

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Independence test using autocorrelation: the Ljung-Box test. Given an univariate stationary time series $\{x_1, x_2, \ldots, x_n\}$, to test the serial independency within the sequence people often resort to test the autocorrelation; one of the most used tests is the Ljung-Box statistic

$$Q_{\mathrm{LB}} = n(n+2)\sum_{j=1}^{h} \frac{\hat{\rho}(j)^2}{n-j}$$

where n is the sample size and $\hat{\rho}(j)$ is the sample autocorrelation. Under the null hypothesis of serial independence the distribution of Q_{LB} is asymptotically $\chi^2(h)$. However it can be shown that the following time series

$$x_1 \sim U(0,1),\ x_{i+1} = 1 - |2x_i - 1|,\ i \geqslant 1.$$

has strong serial dependence but zero autocorrelations (c.f. Rémillard (2013, pp.64)).

Normality test using empirical skewness/kurtosis: the Jarque-Berra test. The Jarque-Bera test is a goodness-of-fit test of whether sample data have the skewness and kurtosis matching a normal distribution. The test statistic JB is defined as

$$\mathrm{JB} = n\left(\frac{s^2}{6} + \frac{(c-3)^2}{24}\right)$$

where n is the sample size and s, c are the sample skewness and curtosis, respectively. Under the null hypothesis of normality the distribution of JB is asymptotically $\chi^2(2)$. However, for small samples the chi-squared approximation is overly sensitive, often rejecting the null hypothesis when it is true. Furthermore, the distribution of p-values departs from a uniform distribution and becomes a right-skewed uni-modal distribution, especially for small p-values. This leads to a large Type I error rate (c.f. Rémillard (2013, pp.66)).

Goodness-of-fit test of copula using AIC exclusively. The Akaike information criterion (AIC) is an estimator of the relative quality of statistical models for a given set of data; it orginates from information theory considerations. Let k be the number of estimated parameters in the model and $\hat{L}$ be the maximum value of the likelihood function of the model. Then the AIC value of the model is

$$\mathrm{AIC} = 2k - \log \hat{L}$$

which is relatively simple to compute. AIC does not provide a test of a model in the sense of testing a null hypothesis. It tells nothing about the absolute quality of a model, only the quality relative to other models. Thus, if all the candidate models fit poorly, AIC will not give any warning. However, many applied papers use AIC as the only means for

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copula goodness-of-fit testing.

To decide whether a typical multivariate time series belongs to a certain copula-GARCH model we follow the steps

1. test of structural change to see if the probability distribution is time-invariant;
2. test of independence;
3. test if each component of the multivariate time series satisfy a certain GARCH model;
4. test the copula goodness-of-fit to decide what kind of the copula family does the innovation of the multivariate time series belong to.

Hereafter we introduce the statistical tests in this order.

4.4.1 Structural change tests

4.4.1.1 Structural change test for univariate time series

Given the observations $e_1, e_2, \ldots, e_n$, we wish to detect the inherent structural changes. The null hypothesis is that all the observations have the same distribution, while the alternative hypothesis is that there exists a $\tau < n$ such that $e_1, e_2, \ldots, e_\tau$ are of the same distribution, say D_1, and $e_{\tau+1}$ is of a distribution differ from D_1. Set

$$\mathcal{T}_n \equiv \mathcal{T}_n(e_1, e_2, \ldots, e_n) = \frac{1}{\sqrt{n}} \max_{1 \leqslant k \leqslant n} \max_{1 \leqslant i \leqslant n} \left| \sum_{j=1}^{k} \mathbb{1}\left(e_j \leqslant e_i\right) - kT_n(e_i) \right|$$

where

$$T_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\left(e_i \leqslant x\right),$$

it is shown in Rémillard (2011, 2012) that under the null hypothesis $\mathcal{T}_n$ converges in distribution to a parameter-free distribution; for $k = 1, 2, \ldots, N$, generate n independent $(0,1)$ uniformly distributed random variable $U_1^{(k)}, U_2^{(k)}, \ldots, U_n^{(k)}$, compute

$$\mathcal{T}_n^{(k)} = \mathcal{T}_n\left(U_1^{(k)}, U_2^{(k)}, \ldots, U_n^{(k)}\right)$$

Then the associated p-value is estimated by

$$\frac{1}{N} \sum_{k=1}^{N} \mathbb{1}\left(\mathcal{T}_n^{(k)} > \mathcal{T}_n\right).$$

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4.4.1.2 Structural change test for copula

Set

$$\mathcal{K}_n(e_1, e_2, \ldots, e_n) = \frac{1}{\sqrt{n}} \max_{1 \leqslant k \leqslant n} \max_{1 \leqslant i \leqslant n} \left| \sum_{j=1}^{k} \mathbb{1}\left(e_j \leqslant e_i\right) - kK_n(e_i) \right|$$

where

$$K_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\left(e_i \leqslant x\right)$$

for $k = 1, 2, \ldots, N$, generate n independent standard Gaussian random variable $\xi_1^{(k)}$, $\xi_2^{(k)}$, $\ldots, \xi_n^{(k)}$.

$$\mathcal{K}_n^{(k)}(e_1, e_2, \ldots, e_n) = \frac{1}{\sqrt{n}} \max_{1 \leqslant k \leqslant n} \max_{1 \leqslant i \leqslant n} \left| \sum_{j=1}^{k} \xi_j^{(k)} \left\{\mathbb{1}\left(e_j \leqslant e_i\right) - K_n(e_i)\right\} \right|$$

The p-value is estimated by

$$\frac{1}{N} \sum_{k=1}^{N} \mathbb{1}\left(\mathcal{K}_n^{(k)} > \mathcal{K}_n\right).$$

4.4.2 Independence test

Following Genest & Rémillard (2004), let $X_1, X_2, \ldots, X_n$ be n identically distributed d-dimensional random vectors, each with continuous margins $F_1, F_2, \ldots, F_d$. If $d \geqslant 2$, then for $A \in \mathcal{A}_d = \{B \subset \{1, 2, \ldots, d\} \mid |B| > 1\}$ and any $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, let

$$\begin{aligned}
\mathcal{G}_{n,A}(x) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \prod_{j \in A} \left\{\mathbb{1}\left(X_{ij} \leqslant x_j\right) - F_{jn}(x_j)\right\} \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \prod_{j \in A} \left\{\mathbb{1}\left(U_{ij} \leqslant F_j(x_j)\right) - D_{jn}(F_j(x_j))\right\}
\end{aligned}$$

where $U_{ij} = F_j(X_{ij})$, $|A|$ is the cardinality of A, and

$$F_{jn}(y) = D_{jn}(F_j(y)) = \frac{1}{n} \mathbb{1}\left(X_{ij} \leqslant y\right) = \frac{1}{n} \mathbb{1}\left(U_{ij} \leqslant F_j(y)\right).$$

Under the null hypothesis of independence, the Cramér-von Mises statistics

$$V_{n,A} = \frac{6^{|A|}}{n} \sum_{i=1}^{n} \mathcal{G}_{n,A}(X_i)^2$$

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converge jointly in law to V_A. If $|A| = k$, then V_A has the same law of

$$\frac{6^k}{\pi^{2k}} \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \frac{Z^2_{i_1\, i_2\, \ldots\, i_k}}{(i_1\, i_2\, \ldots\, i_k)^2}$$

where each $Z_{i_1\, i_2\, \ldots\, i_k}$ is i.i.d standard Gaussian. For each $k = 1, 2, \ldots, N$, generate n independent random vectors

$$U_i^{(k)} = (U_{i1}^{(k)}, U_{i2}^{(k)}, \ldots, U_{id}^{(k)}) \sim C_\perp, \quad i = 1, 2, \ldots, n$$

Compute the associated $V_{n,A}^{(k)}$ for $A \in \mathcal{A}_d$; the associated p-value $p_{n,A}$ is estimated by

$$p_{n,A} = \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}\left(V_{n,A}^{(k)} > V_{n,A}\right), \quad A \in \mathcal{A}_d.$$

Set

$$\mathcal{F}_n = -2 \sum_{A \subset \mathcal{A}_d} \log p_{n,A}$$

Under the null hypothesis of independence, the p-values $p_{n,A}$ converge to independent uniform variables, so $\mathcal{F}_n \rightsquigarrow \mathcal{F}$, where $\mathcal{F}$ is a chi-square distribution with $2^{d+1} - 2d - 2$ degrees of freedom.

4.4.2.1 Serial independence test

Similarly as above, let $X_1, X_2, \ldots, X_n$ be identically distributed observations. Let $d \geqslant 2$, then for $A \in \mathcal{A}_d = \{B \subset \{1, 2, \ldots, d\} | |B| > 1, 1 \in B\}$ and any $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, set

$$\mathcal{G}_{n,A}(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \prod_{j \in A} \{\mathbb{1}\left(X_{i+j-1} \leqslant x_j\right) - H_n(x_j)\}$$

where $|A|$ is the cardinality of A and

$$H_n(y) = \frac{1}{n} \mathbb{1}\left(X_i \leqslant y\right).$$

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We also set $X_{n+i} = X_i$ for $i \in \{1, 2, \dots, d\}$. Under the null hypothesis of independence, the Cramér-von Mises statistics

$$V_{n,A} = \frac{6^{|A|}}{n} \sum_{i=1}^{n} \mathcal{G}_{n,A}(X_i, X_{i+1}, \dots, X_{i+d-1})^2$$

converge jointly in law to V_A. If $|A| = k$, then V_A has the same law of

$$\frac{6^k}{\pi^{2k}} \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \frac{Z^2_{i_1\, i_2 \dots i_k}}{(i_1\, i_2\, \dots\, i_k)^2}$$

where each $Z_{i_1\, i_2 \dots i_k}$ is i.i.d. standard Gaussian. For each $k = 1, 2, \dots, N$, generate n independent $(0,1)$-uniform random variables $U_1^{(k)}, U_2^{(k)}, \dots, U_n^{(k)}$. Compute the associated $V_{n,A}^{(k)}$ for $A \in \mathcal{A}_d$; the associated p-value $p_{n,A}$ is estimated by

$$p_{n,A} = \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}\left(V_{n,A}^{(k)} > V_{n,A}\right), \quad A \in \mathcal{A}_d.$$

Set

$$\mathcal{F}_{n,d} = -2 \sum_{A \subset \mathcal{A}_d} \log p_{n,A}$$

Under the null hypothesis of independence, the p-values $p_{n,A}$ converge to independent uniform variables, so $\mathcal{F}_{n,d} \rightsquigarrow \mathcal{F}_d$, where $\mathcal{F}_d$ is a chi-square distribution with $2^d - 2$ degrees of freedom.

4.4.3 Specification tests of GARCH models

Specification tests for the GARCH model consists of two parts: testing for the independence and the goodness-of-fit of the innovation distribution; the former is treated in the previous section, and in this section we focus on the latter.

4.4.3.1 Bai's test

One of the first rigorous goodness-of-fit tests of the Gaussian innovation is given in Bai (2003). Given the observations $e_1, e_2, \dots, e_n$, set $u_i = \mathcal{N}(e_i), i = 1, 2, \dots, n$ and $v_i, i = 1, 2, \dots, n$ be the ordered statistics of $u_i, i = 1, 2, \dots, n$ and $v_0 = 0, v_{n+1} = 1$. Let

$$g(s) = \left(1, -\mathcal{N}^{-1}(s), 1 - \left(\mathcal{N}^{-1}(s)\right)^2\right)^\top$$

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and

$$C(s) = \int_s^1 g(\tau)\, g(\tau)^\top \,\mathrm{d}\tau.$$

Set $V_n(s) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \{\mathbb{1}(v_i \leqslant s) - s\}$, the Khmaladze transform W_n of V_n is

$$W_n(s) = V_n(s) - \int_0^s \left\{ g(t)^\top C^{-1}(t) \int_t^1 g(\tau)\,\mathrm{d}V_n(\tau) \right\} \mathrm{d}t.$$

Under the null hypothesis that $e_i, i = 1, 2, \ldots, n$ are Gaussian, W_n converges to a Brownian motion. The Kolmogorov-Smirnov statistic KS and the Cramér-von Mises statistic CvM are defined respectively by

$$\mathrm{KS} = \max_{i=1,2,\ldots,n} |W_n(v_i)|, \quad \mathrm{CvM} = \frac{1}{n} \sum_{i=1}^n W_n(v_i)^2 (v_{i+1} - v_i)\,.$$

4.4.3.2 Parametric bootstrap for generic innovation distribution

Following Genest & Rémillard (2008); Rémillard (2011), to estimate the p-value of a generic test statistic S, perform the following steps:

- Use $e_1, e_2, \ldots, e_n$ to estimate the GARCH model with the given innovation distribution; let the estimation be θ and the computed test statistic be S.
- For each $i = 1, 2, \ldots, N$ simulate the GARCH series with parameter θ; estimate the new GARCH parameters using the simulated series and compute the test statistics S_i.
- The approximated p-value is

$$\frac{1}{N} \sum_{i=1}^N \mathbb{1}(S_i > S)$$

4.4.4 Goodness-of-Fit test of copula

Following Kojadinovic & Yan (2010); Rémillard (2017), the parametric bootstrap consists of the steps:

- Compute C_n from the pseudo-observations $U_1, U_2, \ldots, U_n$.
- Compute the test statistics S_n,

$$S_n = \sum_{i=1}^n \left(C_n(U_i) - C_{\theta_n}(U_i)\right)^2.$$

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- For a large N, repeat the following steps for $k = 1, 2, \ldots, N$:
  - Generate a random sample $X_1^{(k)}, X_2^{(k)}, \ldots, X_n^{(k)}$ from copula C_{θ_n} and compute the associated pseudo-observations $U_1^{(k)}, U_2^{(k)}, \ldots, U_n^{(k)}$.
  - Let

$$C_n^{(k)}(u) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\left(U_i^{(k)} \leqslant u\right), \quad u \in [0,1]^d$$

    and compute an estimate $\theta_n^{(k)}$ of θ from $U_1^{(k)}, U_2^{(k)}, \ldots, U_n^{(k)}$.
  - Compute $S_n^{(k)}$ by

$$S_n^{(k)} = \sum_{i=1}^{n} \left(C_n(U_i) - C_{\theta_n}(U_i)\right)^2 .$$

- An approximate p-value for the test is given by

$$\frac{1}{N} \sum_{i=1}^{N} \mathbb{1}\left(S_n^{(k)} > S_n\right).$$

4.5 Valuation

Algorithm (The Copula-GARCH Model Calibration and Simulation). Given the data matrix $\{X_i\}_{i=1}^N$ with rows $X_i = (x_{i1}, x_{i2}, \ldots, x_{id})$, fit the copula-GARCH model and generate l paths, each has m spots.

1. Test the data columnwise to see if the GARCH requirements are satisfied. Derive the corresponding GARCH parameters and the residuals series; assemble the column vectors of residuals into a matrix Y.
2. Test if Y satisfies the static copula conditions. Perform goodness-of-fit (GoF) test of the copulas to select a copula that best fits the data; fit the selected copula with $\text{pobs}(Y)$, where $\text{pobs}(\cdot)$ is the pseudo-observation function.
3. Generate m random variates $\{U_i \equiv (u_{i1}, u_{i2}, \ldots, u_{id})\}_{i=1}^m$ from the fitted copula.
4. Compute the standardized residuals $\{Z_i \equiv (z_{i1}, z_{i2}, \ldots, z_{id})\}_{i=1}^m$, where $z_{ij} = q_j(u_{ij})$ and q_j is the quantile function of generalized error distribution $\text{ged}(\nu_j)$.
5. Simulate the GARCH model using the dependent standardized residuals Z_i as innovations.

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4.5.1 The data

Monthly data from the S&P 500 Index (S&P500) and the S&P SmallCap 600 Index (S&P600) for the period October 1994 (S&P600 index inception date) to December 2017 are used. The S&P500 is a representation of large-cap U.S. stocks, whereas the S&P600 represents the small-cap range.

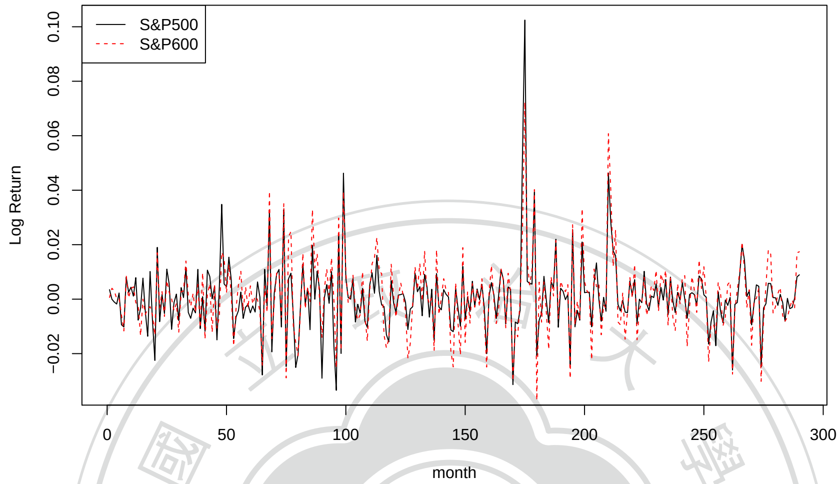

Figure 4.1: Monthly Log Return of S&P500 and S&P600

4.5.2 Statistical tests and model calibration

We have set the number of bootstrap replications to 2,000 in the following sections[1].

4.5.2.1 GARCH specification tests

Here we use the R package `rugarch` (Ghalanos (2018)) for GARCH model calibration and the R package `copula` (Hofert et al. (2017)) for bootstrap tests. First we perform the structural change test to the residual series to see if this basic assumption of GARCH holds; the relatively large p-values show that one cannot reject the null hypothesis of no structural changes at the 5% confidence level (Table 4.2).

We assume that each of the log return series satisfies GARCH(1,1) model with Gaussian innovations and estimate the GARCH coefficients and the residuals (Table 4.3). Then we perform the goodness-of-fit test against the residuals. Using Bai's algorithm in 4.4.3.1 for the computation of CvM and KS statistics and the corresponding 95% confidence band, Figures 4.2 and 4.3 show that the innovations of both series do not lie

[1]The bootstrap method is a stochastic sampling procedure first proposed in 1979 by Bradley Efron to estimate the probability distribution. Due to its time-consuming nature, Prof. Efron suggests that 100 — 1000 repetitions are sufficient for the purpose.

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Table 4.2: Structural Changes Test of Residuals

Underlying	KS statistic	p-value of KS (%)
S&P 500	0.5718	50.4
S&P 600	0.5568	56.5

within the 95% confidence band. Furthermore the result in Table 4.4 indicates that, with the tiny p-values, the null hypothesis that the innovations obey the Gaussian law should be rejected at the 5% level.

Table 4.3: Estimated GARCH coefficient: Gaussian Innovations

Underlying	μ	ω	α	β
S&P 500	4.71e-5	3.25e-5	0.23	0.57
S&P 600	6.22e-4	1.37e-5	0.0911	0.842

Table 4.4: Result: Goodness-of-Fit Test of Gaussian Innovation

Underlying	CvM statistic	KS statistic	p-value of CvM (%)	p-value of KS (%)
S&P 500	54.2635	20.8522	$\ll 1$	$\ll 1$
S&P 600	36.2823	19.4951	$\ll 1$	$\ll 1$

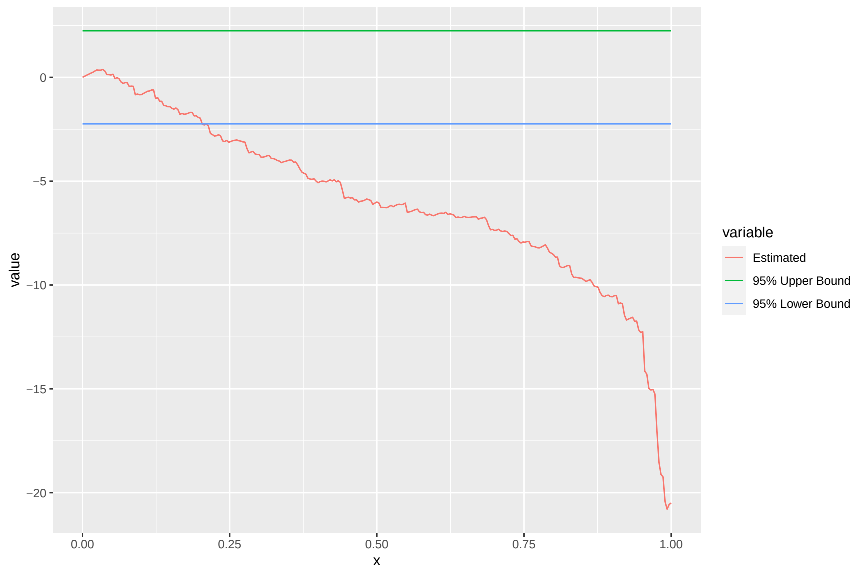

Figure 4.2: Goodness-of-Fit Test of GARCH with Gaussian Innovation: S&P500 Monthly Series

Now we assume that each of the log return series satisfies GARCH(1,1) model with GED innovations and estimate the GARCH coefficients and the residuals (Table 4.5). Again we perform the goodness-of-fit test against the residuals. Using the parametric

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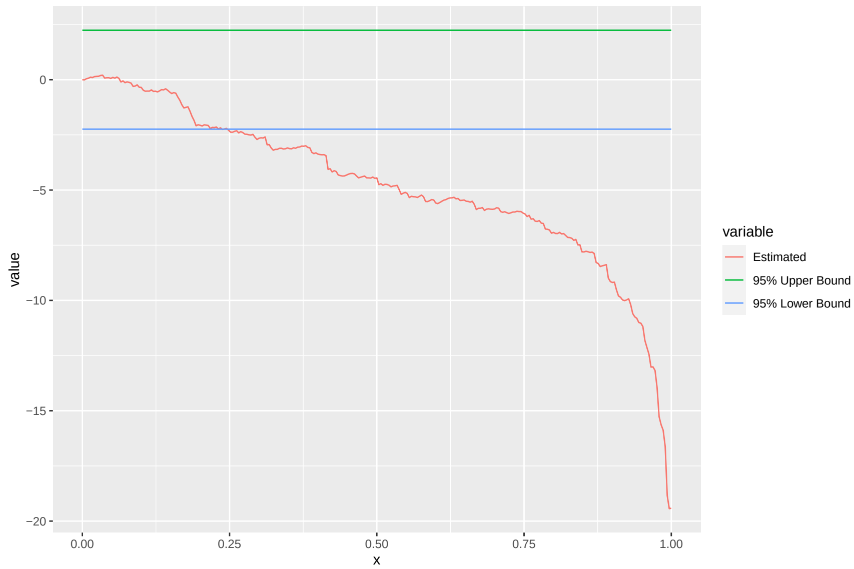

Figure 4.3: Goodness-of-Fit Test of GARCH with Gaussian Innovation: S&P600 Monthly Series

bootstrap procedure in 4.4.3.2 for the computation of CvM and KS statistics and the corresponding 95% confidence band, Figures 4.4 and 4.5 show that the innovations of both series do lie within the 95% confidence band. Furthermore the result in Table 4.6 indicates that, with the relatively large p-values, the null hypothesis that the innovations obey the GED law cannot be rejected at the 5% confidence level.

Table 4.5: Estimated GARCH coefficients: GED Innovations

Underlying	μ	ω	α	β	ν
S&P 500	3.28e-4	3.29e-5	0.215	0.566	0.929
S&P 600	5.62e-4	3.58e-5	0.130	0.680	1.120

Table 4.6: Result: Goodness-of-Fit Test of GED Innovations

Underlying	CvM statistic	KS statistic	p-value of CvM (%)	p-value of KS (%)
S&P 500	0.0246	0.5419	87.70	68.60
S&P 600	0.0338	0.5203	65.55	62.80

4.5.2.2 Copula related

All computations in this section are done using the R package `copula` (Hofert et al. (2017)). We test if the collection of residuals form a stationary— i.e. without structural changes— copula; the null hypothesis of being stationary cannot be rejected at the 5% confidence level (Table 4.7). This is vital for our subsequent development.

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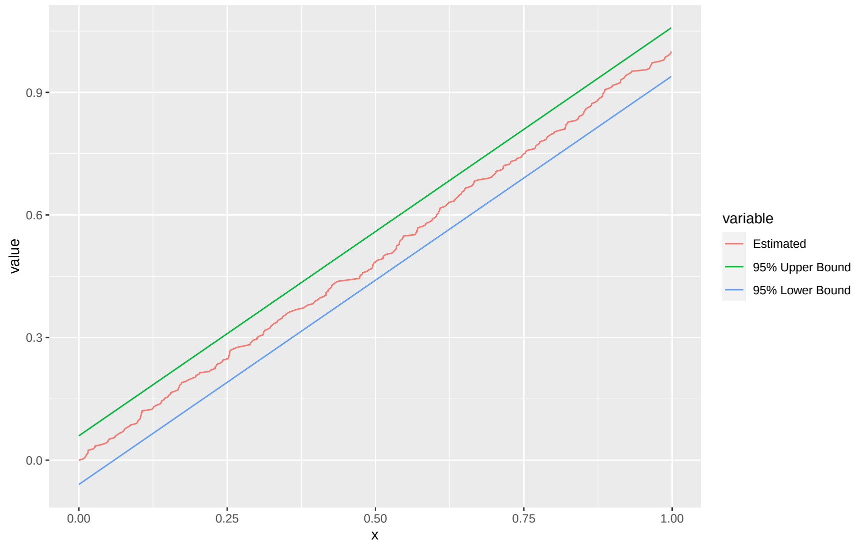

Figure 4.4: Goodness-of-Fit Test of GARCH with GED Innovation: S&P500 Monthly Series

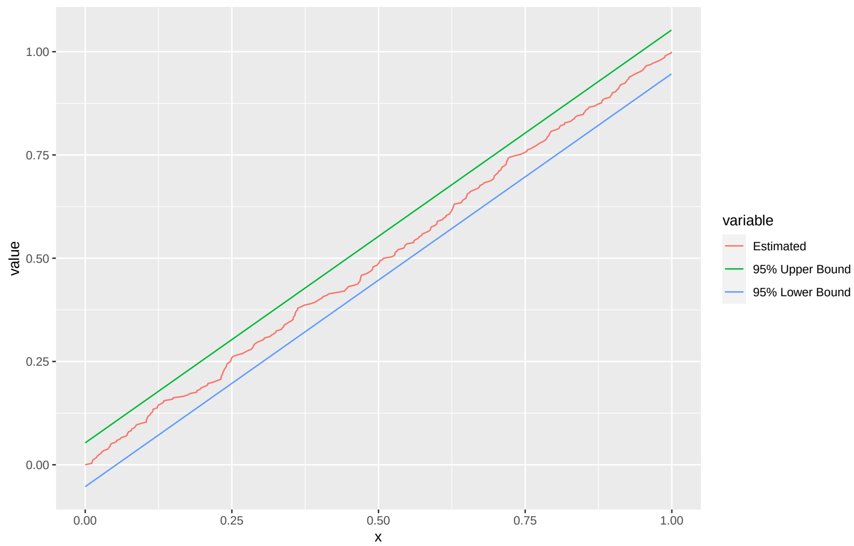

Figure 4.5: Goodness-of-Fit Test of GARCH with GED Innovation: S&P600 Monthly Series

Table 4.7: Test of Structural Changes: Copula

Threshold KS statistic	Empirical p-value (%)
0.5704	100

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Next we perform the goodness-of-fit test of copula to select the best fit among the common parametric copula families. The copula with the highest p-value would be the model best suited to our data, in this case, the Student t copula. (Table 4.8).

Table 4.8: Result: Goodness-of-Fit Test of Copulas

Copula	p-value (%)	parameters
Clayton	0.025	
Frank	0.375	
Gumbel	4.320	
Gaussian	79.20	
Student t	82.40	$\rho = 0.879$, $\nu = 6.244$

4.5.3 Variable annuities contract evaluation

Following the setup in Da Fonseca & Ziveyi (2017); Ng & Li (2013), we first evaluate guaranteed minimum maturity benefit (GMMB) riders written on a fund with multiple underlying assets. Suppose at time 0 the policyholder makes a single lump sum payment P and invested in n different subaccount funds. The portfolio of the policyholder consists w_i units of the i-th subaccount fund with its corresponding subaccount value $S_{i,t}$ at time t; note that $P = \sum_{i=1}^{n} w_i\, S_{i,0}$, where the sum of the weighted subaccount values equals the payment P. Here we assume that the w_i's are constant over time and the subaccount funds are non-dividend-paying. At time t the value of the portfolio F_t is

$$F_t = \sum_{i=1}^{n} w_i\, S_{i,T}(1-m)^t$$

where m is the monthly management charge. Then the payoff function of the GMMB is

$$\max\left(Pg - F_T, 0\right)$$

where g is the guarantee level in percentage. The value of the GMMB is

$${}_Tp_x \cdot e^{-rT}\, \mathsf{E}^Q\{\max\left(Pg - F_T, 0\right)|\mathcal{F}_0\}$$

where ${}_Tp_x$ denotes the probability that the policyholder is x years old at policy inception and still alive and had not withdrawn at $t = T$, r denotes the risk-free interest rate, and $\mathsf{E}^Q\{\cdot|\mathcal{F}_0\}$ denotes the conditional expectation under the risk-neutral martingale measure Q with respect to the initial filtration $\mathcal{F}_0$.

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Similarly, we evaluate the guaranteed minimum death benefit (GMDB) rider

$$\sum_{t=1}^{T} {}_{t|}q_x \cdot e^{-rt}\,\mathsf{E}^Q\{\max(Pg - F_t, 0)\,|\mathcal{F}_0\}$$

where ${}_{t|}q_x$ denotes the probability that the policyholder is x years old at policy inception and dies during t and $t+1$ and had not withdrawn at t; note that ${}_{t|}q_x = {}_tp_x - {}_{t+1}p_x$.

We set the following parameters for comparison purposes:

- age at policy inception (x): 45, 50, 60 years old;
- premium (P): \$10 000 paid at policy inception;
- portfolio: 2 non-dividend-paying subaccount funds;
- withdrawal: \$0 (none, for simplicity);
- guarantee level (g): 60%, 80%, 100%, 120%;
- annual management charge (AMC): the management charge is quoted as an annual percentage charge and deducted monthly from the fund. An AMC of 3% is equivalent to the monthly management charge $m = \frac{0.03}{12} = 0.0025$;
- maturity (T): 5, 10, 20 years;
- mortality: assuming that $\mu_x = A + Bc^x$ (Makeham's Law) with $A = 5.4\times10^{-4}$, $B = 9.5929\times10^{-6}$, $c = 1.1085$.

Note that (c.f. (2.19) in Dickson et al. (2013))

$${}_tp_x = \exp\left\{-\int_0^t \mu_{x+\tau}\,\mathrm{d}\tau\right\} = \exp\left\{-\int_0^t A + Bc^{x+\tau}\,\mathrm{d}\tau\right\} = e^{-At-\frac{Bc^x(c^t-1)}{\log c}}$$

The values are computed using Monte Carlo simulations; each value is computed via 10^6 simulations. Table 4.9 tabulates the results of the guarantee values expressed as a percentage of initial premium for both the GMMB and GMDB riders under the copula-GARCH model.

4.6 Conclusion

We reviewed the copula concept and its parametric families, adopted the multivariate copula-GARCH model for the valuation of the various levels of guarantees embedded in variable annuities where there are multiple underlying assets in the portfolio or fund. To reduce the error of model misspecification, rigorous statistical inference procedures

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based on the empirical processes theory are implemented throughout; specifically, the GARCH specification test and the copula goodness-of-fit test. Although copulas are used by practitioners as a standard tool for portfolio risk management due to its ability to model the dependency of individual risks within the insurer's portfolio faithfully, however, a vast majority of existing literature often making model assumptions without justification or conducting insufficient statistical tests to verify the adequacy of such selection. We provide a theoretically and mathematically sound methodology of which we base our model selection upon. As a result, we were able to show through demonstration, that by implementing this methodology, one can derive a more precise assessment of the costs of these embedded guarantees in variable annuities.

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Table 4.9: Simulated Guarantee Values Expressed As Percentage of Initial Premium: Copula-GARCH Model

Maturity (yrs)	Starting Age	Guarantee Level	GMMB (%)	GMDB (%)
5	45	0.60	0.0000	0.0000
5	45	0.80	0.0000	0.0000
5	45	1.00	9.6444	0.0625
5	45	1.20	27.5488	0.2464
5	50	0.60	0.0000	0.0000
5	50	0.80	0.0000	0.0000
5	50	1.00	9.5559	0.0927
5	50	1.20	27.4372	0.3663
5	60	0.60	0.0000	0.0000
5	60	0.80	0.0000	0.0000
5	60	1.00	9.3794	0.2269
5	60	1.20	26.8793	0.8927
10	45	0.60	0.0000	0.0000
10	45	0.80	0.1985	0.0007
10	45	1.00	16.1619	0.2667
10	45	1.20	32.1962	0.6964
10	50	0.60	0.0000	0.0000
10	50	0.80	0.2228	0.0012
10	50	1.00	15.9686	0.4065
10	50	1.20	31.7953	1.0567
10	60	0.60	0.0000	0.0000
10	60	0.80	0.1877	0.0026
10	60	1.00	15.2762	1.0130
10	60	1.20	30.2171	2.6071
20	45	0.60	0.0000	0.0000
20	45	0.80	10.1725	0.4171
20	45	1.00	22.5676	1.4938
20	45	1.20	35.0072	2.7326
20	50	0.60	0.0000	0.0000
20	50	0.80	9.7553	0.6553
20	50	1.00	21.6061	2.3233
20	50	1.20	33.5523	4.2451
20	60	0.60	0.0000	0.0000
20	60	0.80	8.0011	1.5143
20	60	1.00	17.7338	5.4932
20	60	1.20	27.4773	10.0656

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5 Conclusion and Future Work

In the opening chapter we examine the fair pricing of interest rate sensitive life insurance policies that are commonly sold in Taiwan. It is well known that the asset's log-return distribution is non-Gaussian; inadvertent uses of Gaussian models could lead to underestimation of losses and hugely mispriced derivative products. With the reference portfolio following Heston's stochastic volatility process, the payoff function of these policies consists of a series of forward-start options. Although the option to surrender are standard features of these policies, policyholders incur heavy penalties should they exercise such option. Given certain policyholder behaviour, we study the impact of the minimum guaranteed interest rate, and the annually declared bonus rate on the issuing company's solvency. Parameters in the models are calibrated from Taiwanese data; the liability reserve, the expected surplus / deficit of the bonus stabilization reserve, and standard risk measures such as VaR and ES are computed through Monte Carlo simulations.

Given the sheer volume of the international bonds listed on the Taipei Exchange that are held by the life insurers in Taiwan, coupled with the lack of a liquid secondary market, the need for pricing transparency and a reliable source of reference is of utmost importance. We provide the life insurers the means to evaluate the mark-to-market value of these callable bonds without having to rely on third parties to do so. We are able to collate publicly available data and make use of open source software to construct a bespoke system that can independently price the international bonds.

In the last chapter, the copula concept with its multivariate time-series model generalization, namely the copula-GARCH model, and robust statistical inference procedures based on the theory of empirical processes are investigated in depth. A vast majority of existing literature on applications of copula often makes assumptions without justification or conducts inadequate statistical tests for verifications. Here we demonstrate what we believed to be the preferred way of using copula for financial and risk management applications by the detailed valuation of guarantees embedded in variable annuities with multiple underlying assets.

Throughout these studies we have insisted on model calibration using open-sourced

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subroutines and actual market data to ensure maximum transparency, reproducibility and applicability. In the ISL chapter, our model is constructed to the specifications of the most popular life insurance policy in Taiwan and its parameters estimated using Taiwanese market data. It can be easily adopted by other markets with similar products and our results are of interest to practitioners and the regulatory authorities. The international bonds evaluation presented here is based on the single factor Hull-White short rate model with publicly available datasets and open source library QuantLib. The development of the copula-GARCH model makes heavy use of the R programming language and associated high-quality packages.

The current studies could be expanded in various ways. The values of long-maturity insurance products are especially susceptible to the comovement of asset price and interest rate; it is much desirable to have a stochastic model that can faithfully characterize the market. The Heston stochastic volatility asset model with uncorrelated CIR stochastic interest rate model setup could be replaced by other correlated stochastic volatility and interest rate combinations such as the Schöbel-Zhu-Hull-White (SZHW) model (van Haastrecht et al. (2009)). An additional benefit of adopting the SZHW model is the possible incorporation of the surrender option mechanism in pricing and hedging of ISL policies. The Hull-White short-term interest rate model with trinomial tree are used for callable bond pricing, but an even more sensible choice would be the LIBOR market model with explicit considerations of exchange rates and credit risks. Aside from the tree methods, the embedded option pricing algorithm could adopt the least-square Monte Carlo (LSM) method (Longstaff & Schwartz (2001)) and its extensions. From the standpoint of the issuer of international bonds, concerns regarding hegding proves to be more important than pricing itself. The pursuit of these avenues of research should be rewarding and is left for future work.

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Appendix A
Interest Rate Tickers

Table A.1: US Treasury Constant Maturity Rate

Treasury Constant Maturity Rate	Code
30-Year	DGS30
20-Year	DGS20
10-Year	DGS10
7-Year	DGS7
5-Year	DGS5
3-Year	DGS3
2-Year	DGS2
1-Year	DGS1
6-Month	DGS6MO
3-Month	DGS3MO
1-Month	DGS1MO

Table A.2: USD LIBOR Rates

LIBOR Rates (USD)	Code
Overnight	USDONTD156N
1-Week	USD1WKD156N
1-Month	USD1MTD156N
2-Month	USD2MTD156N
3-Month	USD3MTD156N
6-Month	USD6MTD156N
12-Month	USD12MD156N

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Table A.3: ICE Swap Rates (USD), 11:00 A.M. (London Time)

ICE Swap Rates	Code
1 Year	ICERATES1100USD1Y
2 Year	ICERATES1100USD2Y
3 Year	ICERATES1100USD3Y
4 Year	ICERATES1100USD4Y
5 Year	ICERATES1100USD5Y
6 Year	ICERATES1100USD6Y
7 Year	ICERATES1100USD7Y
8 Year	ICERATES1100USD8Y
9 Year	ICERATES1100USD9Y
10 Year	ICERATES1100USD10Y
15 Year	ICERATES1100USD15Y
20 Year	ICERATES1100USD20Y
30 Year	ICERATES1100USD30Y

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B

International Bond Valuation: Theory and Implementation — Code

```python
# -*- coding: utf-8 -*-

import ftplib, os, tempfile, shutil, codecs
from datetime import datetime
from collections import namedtuple

os.chdir(os.path.dirname(__file__))
td = tempfile.mkdtemp(dir=os.getcwd())
cat = os.path.join

from QuantLib import *
import pandas as pd
from pandas_datareader.data import DataReader as dr
import matplotlib.pyplot as plt
plt.style.use('ggplot')
import numpy as np
import requests
from bs4 import BeautifulSoup
from tabulate import tabulate

irr_nc_price_pkl = 'irr_nc_price.pkl'
irr_nc_oas_pkl = 'irr_nc_oas.pkl'
plot_styles = {
  '.':        'point marker',
  ',':        'pixel marker',
```


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```
    'o':        'circle marker',
    'v':        'triangle_down marker',
    '^':        'triangle_up marker',
    '<':        'triangle_left marker',
    '>':        'triangle_right marker',
    '1':        'tri_down marker',
    '2':        'tri_up marker',
    '3':        'tri_left marker',
    '4':        'tri_right marker',
    's':        'square marker',
    'p':        'pentagon marker',
    '*':        'star marker',
    'h':        'hexagon1 marker',
    'H':        'hexagon2 marker',
    '+':        'plus marker',
    'x':        'x marker',
    'D':        'diamond marker',
    'd':        'thin_diamond marker',
    '|':        'vline marker',
    '_':        'hline marker',
}
all_styles = sorted(plot_styles.keys())

def format_real(v, digits=3):
    _ = '%%.%df' % digits
    return _ % v

def get_cmy(dt):
    syms = ['DGS30', 'DGS20', 'DGS10', 'DGS7', 'DGS5', 'DGS3', 'DGS2', 'DGS1', 'DGS6MO', 'DGS3MO', 'DGS1MO']
    yc = dr(syms, 'fred', start=dt, end=dt)

    names = dict(zip(syms, ['30Y', '20Y', '10Y', '7Y', '5Y',  '3Y', '2Y',  '1Y', '6M',  '3M',  '1M']))
    yc = yc.rename(columns=names)
    yc = yc[['1M', '3M', '6M', '1Y', '2Y', '3Y', '5Y', '7Y', '10Y', '20Y', '30Y']].tail(1)

```


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```python
    t = pd.Timestamp(yc.index.values[0])
    calc_date = Date(t.day, t.month, t.year)

    Settings.instance().evaluationDate = calc_date
    calendar = UnitedStates()
    business_convention = Unadjusted
    day_count = Thirty360()
    end_of_month = True
    settlement_days = 0
    face_amount = 100.
    coupon_frequency = Period(Semiannual)

    helpers = []
    for r, m in zip(yc.values.tolist()[0], [Period(i, Months) for i in [1, 3, 6, 12, 24, 36, 60, 84, 120, 240, 360]]):
        termination_date = calc_date + m
        schedule = Schedule(calc_date, termination_date, coupon_frequency, calendar, business_convention, business_convention, DateGeneration.Backward, end_of_month)
        bond_helper = FixedRateBondHelper(QuoteHandle(SimpleQuote(face_amount)), settlement_days, face_amount, schedule, [r / 100.], day_count, business_convention)
    helpers.append(bond_helper)

    yts = PiecewiseLogCubicDiscount(0, TARGET(), helpers, Actual365Fixed())
    yts.enableExtrapolation()
    return calc_date, RelinkableYieldTermStructureHandle(yts), yts

def get_fred(dt):
    syms = ['USDONTD156N', 'USD1WKD156N', 'USD1MTD156N', 'USD2MTD156N', 'USD3MTD156N', 'USD6MTD156N', 'USD12MD156N', 'ICERATES1100USD1Y', 'ICERATES1100USD2Y', 'ICERATES1100USD3Y', 'ICERATES1100USD4Y', 'ICERATES1100USD5Y', 'ICERATES1100USD6Y', 'ICERATES1100USD7Y',  'ICERATES1100USD8Y', 'ICERATES1100USD9Y', 'ICERATES1100USD10Y', 'ICERATES1100USD15Y', 'ICERATES1100USD20Y', 'ICERATES1100USD30Y']
```


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```python

yc = dr(syms, 'fred', start=dt, end=dt)

names = dict(zip(syms, ['on', '1W', '1M', '2M', '3M', '6M', '12M', '1Y', '2Y', '3Y', '4Y', '5Y', '6Y', '7Y', '8Y', '9Y', '10Y', '15Y', '20Y', '30Y']))
yc = yc.rename(columns=names)
yc = yc[['1W', '1M', '3M', '6M', '12M', '2Y', '3Y', '5Y', '6Y', '7Y', '10Y', '15Y', '20Y', '30Y']].tail(1)

t = pd.Timestamp(yc.index.values[0])
calc_date = Date(t.day, t.month, t.year)
Settings.instance().evaluationDate = calc_date
settlement_days = 0
calendar = UnitedStates()
business_convention = Unadjusted
day_count = Thirty360()

end_of_month = True
l = yc.values.tolist()[0]

#helpers = [OISRateHelper(2, tenor, QuoteHandle(SimpleQuote(rate / 100.)), Eonia()) for rate, tenor in zip(l[:5], [Period(1, Weeks),] + [Period(i, Months) for i in [1, 3, 6, 12]])]

helpers = [DepositRateHelper(QuoteHandle(SimpleQuote(rate / 100.)), m, settlement_days, calendar, business_convention, end_of_month, day_count) for rate, m in zip(l[:5], [Period(1, Weeks),] + [Period(i, Months) for i in [1, 3, 6, 12]])]

helpers += [SwapRateHelper(QuoteHandle(SimpleQuote(rate / 100.)), tenor, TARGET(), Semiannual, Unadjusted, Thirty360(), USDLibor(Period(3, Months))) for rate, tenor in [(i, Period(j, Years)) for i, j in zip(l[5:], [2, 3, 5, 6, 7, 10, 15, 20, 30])]]

yts = PiecewiseLogCubicDiscount(0, TARGET(), helpers, Actual360())
yts.enableExtrapolation()
```


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```
    return calc_date, RelinkableYieldTermStructureHandle(yts), yts

def get_tpex_spec():
    ## download xls from tpex
    #r = requests.get('http://www.tpex.org.tw/web/bond/tradeinfo' +
    #         '/internationalbond/TheoreticalValue.php?l=zh-tw')
    #soup = BeautifulSoup(r.content, 'lxml')
    #all_xls = soup.find_all('a', 'btn btn-xls')
    #r = requests.get('http://www.tpex.org.tw' + all_xls[0]['href'],
    ↪   stream=True)
    #xls = cat(td, 'tpex.xls')
    #with open(xls, 'wb') as fd:
    #    for chunk in r.iter_content(chunk_size=1024):
    #        fd.write(chunk)

    xls = 'BDdos209.20190325-C.xls'

    # bond_code, short_name, issuer, issue_date, maturity_date,
    ↪   credit_rating, irr, optional_redemption, bond_price,
    ↪   accrued_interest, clean_price
    df = pd.read_excel(xls, 'BDdos209')

    def str_to_date(s):
        return Date(*[int(i) for i in s.split('/')][::-1])

    its, its_raw = [], []
    zcb = namedtuple('zcb', 'bond_code short_name issuer issue_date
    ↪   maturity_date irr nc freq maturity bond_price')
    for i in range(len(df.index)):
        iat = df.iat
        head = iat[i, 0]
        if i == 0:
            # determine calc_date by inspecting xls row 2
            calc_date = str_to_date(head[-10:])
        try:
            if not head.startswith('F'):
                continue
        except:
```


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```
            continue

        issue_date = str_to_date(iat[i, 3])
        issue_date_raw = iat[i, 3]
        maturity_date = str_to_date(iat[i, 4])
        maturity_date_raw = iat[i, 4]
        if issue_date.month() != maturity_date.month() or issue_date.dayOfMonth() != maturity_date.dayOfMonth():
            continue
        irr = .01 * float(iat[i, 6])
        irr_raw = format_real(float(iat[i, 6]))
        nc, freq = [int(float(s.strip())) for s in str(iat[i, 7]).split('x')]
        schedule = '%s x %s' % (str(nc), str(freq))
        maturity = maturity_date.year() - issue_date.year()
        bond_price = float(iat[i, 8])
        bond_code = iat[i, 0]
        short_name = iat[i, 1]
        issuer = iat[i, 2]
        rating = iat[i, 5]
        its.append(zcb(bond_code=bond_code, short_name=short_name, issuer=issuer, issue_date=issue_date, maturity_date=maturity_date, irr=irr, nc=nc, freq=freq, maturity=maturity, bond_price=bond_price))
        its_raw.append((bond_code, short_name, issuer, rating, issue_date_raw, maturity_date_raw, irr_raw, schedule, maturity, format_real(bond_price)))
    return {'calc_date': calc_date, 'zcbs': its, 'zcbs_raw': its_raw}

def get_cme_vols():
    ftp = ftplib.FTP('ftp.cmegroup.com')
    ftp.login()
    ftp.cwd('irs')
    l, ll = [], []
    ftp.retrlines('NLST', l.append)
    for fn in l:
        if fn.find('CME_ATM') == 0:
            ll.append(fn)
```


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```python
    fn = sorted(ll)[-1]
    fn_tmp = cat(td, fn)
    ftp.retrbinary('RETR ' + fn, open(fn_tmp, 'wb').write)
    ftp.quit()
    #fn_tmp = 'CME_ATM_VolCube_20190325.csv'
    vols = []
    for nn, l in enumerate(open(fn_tmp, 'r')):
        if nn == 0:
            continue
        date, currency, expiry, tenor, moneyness, strike, normal_vol, lnormal_vol, option_price, annuity = l.strip().split(',')
        strike = float(strike)
        normal_vol = float(normal_vol)
        lnormal_vol = float(lnormal_vol)
        option_price = float(option_price)
        annuity = float(annuity)
        vols.append((Period(expiry), Period(tenor), lnormal_vol))
    return vols

def calibrate(model, helpers, l, name):
    def format_vol(v, digits = 2):
        _ = '%%.%df %%%%' % digits
        return _ % (v * 100)

    _ = '%12s |%12s |%12s |%12s |%12s'
    header = _ % ('maturity', 'length', 'volatility', 'implied', 'error')
    rule = '-' * len(header)
    dblrule = '=' * len(header)
    print(dblrule)
    print(name)
    print(rule)

    model.calibrate(helpers, Simplex(l), EndCriteria(5000, 250, 1e-7, 1e-7, 1e-7))

    print('Parameters: %s' % model.params())
    print(rule)
```


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```
    print(header)
    print(rule)

    table = []
    total_err = 0.
    for swaption, helper in zip(vols, helpers):
        maturity, length, vol = swaption
        NPV = helper.modelValue()
        implied = helper.impliedVolatility(NPV, 1.e-8, 10000, 1.e-6, .99)
        error = implied - vol
        total_err += abs(error)
        print(_ % (maturity, length, format_vol(vol, 4), format_vol(implied, 4), format_vol(error, 4)))
        table.append((maturity, length, format_vol(vol, 4), format_vol(implied, 4), format_vol(error, 4)))
    avg_err = total_err / len(helpers)

    #codecs.open('hw_.txt', 'w', 'utf-8').write(tabulate(table, headers=[u'到期日', u'交換期間', u'CME IV', u'FRED IV', u'誤差',], tablefmt='latex'))

    print(rule)
    _ = '%%%ds' % len(header)
    print(_ % ('Average error: ' + format_vol(avg_err, 4)))
    print(dblrule)

    return model.params()

def calibrated_hw(vols, ts):
    index = USDLibor(Period(3, Months), ts)
    helpers = [SwaptionHelper(maturity, length, QuoteHandle(SimpleQuote(vol)), index, index.tenor(), index.dayCounter(), index.dayCounter(), ts) for maturity, length, vol in vols]
    model = HullWhite(ts)

    for h in helpers:
```


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```python
        h.setPricingEngine(JamshidianSwaptionEngine(model))

    a, b = calibrate(model, helpers, .01, 'Hull-White (Semi-Analytic)')
    print('\nmean_revert: %s, volatility: %s' % (a, b))
    return HullWhite(ts, a, b)

def cbond(model, calc_date, issue_date, maturity_date, irr, nc, freq, maturity):
    settlement_days = 2
    face_amount = 100.
    redemption = 100.
    grids = 1800

    Settings.instance().evaluationDate = calc_date

    calendar = NullCalendar()
    call_schedule = CallabilitySchedule()
    if nc:
        call_date = calendar.advance(issue_date, nc, Years)
        mm = ((maturity - nc - 1) // freq) if freq else 0
        for i in range(mm + 1):
            c_price = CallabilityPrice(face_amount / (1 + irr)**(maturity - nc - i * freq), CallabilityPrice.Clean)
            call_schedule.append(Callability(c_price, Callability.Call, call_date))
            call_date = calendar.advance(call_date, freq, Years)

    b = CallableFixedRateBond(settlement_days, face_amount, Schedule(issue_date, maturity_date, Period(Once), UnitedStates(UnitedStates.GovernmentBond), Unadjusted, Unadjusted, DateGeneration.Backward, False), [0.], Thirty360(Thirty360.BondBasis), Following, redemption, issue_date, call_schedule)

    b.setPricingEngine(TreeCallableFixedRateBondEngine(model, grids))
    return b

```


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```python
def oas2price(oas, ts, b):
    return b.cleanPriceOAS(oas, ts, Thirty360(), Compounded, Annual)

def price2oas(price, ts, b):
    return b.OAS(price, ts, Thirty360(), Compounded, Annual, Date(),
    ↪ 1.e-8, 10000, .002)

def plot_spot_rate(dt):
    def get_spot_rates(curve, day_count=Thirty360(),
    ↪ calendar=UnitedStates(), months=361):
        spots, tenors = [], []
        ref_date = curve.referenceDate()
        calc_date = ref_date
        compounding = Compounded
        freq = Semiannual
        for month in range(months):
            yrs = month / 12.
            d = calendar.advance(ref_date, Period(month, Months))
            zero_rate = curve.zeroRate(yrs, compounding, freq)
            tenors.append(yrs)
            eq_rate = zero_rate.equivalentRate(day_count, compounding,
            ↪ freq, calc_date, d).rate()
            spots.append(100. * eq_rate)
        return pd.DataFrame(zip(tenors, spots), columns=['maturities',
        ↪ 'curve'], index=[''] * len(tenors))

    fig = plt.figure(figsize=(5, 3.5))
    sr_cmy = get_spot_rates(curve_cmy)
    sr_fred = get_spot_rates(curve_fred)
    plt.plot(sr_cmy['maturities'], sr_cmy['curve'], '--', label='CMY')
    plt.plot(sr_fred['maturities'], sr_fred['curve'], '+', label='FRED')
    plt.xlabel('Years')
    plt.ylabel('Zero Rate (%)')
    plt.xlim(.1, 30)
    plt.ylim([0., 4.])
    plt.legend(loc=4)
    fig.savefig('ytm_%s_%s_%s.pdf' % tuple([str(s).zfill(2) for s in
    ↪ dt]), bbox_inches='tight')
```


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```python

#for _ in [(2016, 7, 5), (2018, 11, 8), (2018, 1, 19), (2018, 4, 19), (2018, 7, 19), (2019, 1, 18), (2019, 2, 19), (2019, 3, 25)]:
#     dt = datetime(*_)
#     calc_date, ts_cmy, curve_cmy = get_cmy(dt)
#     calc_date, ts_fred, curve_fred = get_fred(dt)
#     plot_spot_rate(_)

dt = datetime(2019, 3, 25)
calc_date, ts_cmy, curve_cmy = get_cmy(dt)
calc_date, ts_fred, curve_fred = get_fred(dt)

vols = get_cme_vols()
tpex = get_tpex_spec()
shutil.rmtree(td)

model_cmy = calibrated_hw(vols, ts_cmy)
model_fred = calibrated_hw(vols, ts_fred)

def create_table():
    table = [[zcb[i] for i in [0, 3, 4, 5, 6, 7, -1]] for zcb in tpex['zcbs_raw'][100:143]]
    calc_date = tpex['calc_date']
    for ii, zcb in enumerate(tpex['zcbs'][100:143]):
        pars = zcb[3:9]
        p = cbond(model_fred, calc_date, *pars).cleanPrice()
        p0 = cbond(model_fred, zcb[3], *pars).cleanPrice()
        table[ii].append(p / p0 * 100.)
        pp = cbond(model_cmy, calc_date, *pars).cleanPrice()
        pp0 = cbond(model_cmy, zcb[3], *pars).cleanPrice()
        table[ii].append(pp / pp0 * 100.)

    codecs.open('table.txt', 'w', 'utf-8').write(tabulate(table, headers=[u'債券', u'信評', u'發行日', u'到期日', u'利率', u'贖回', u'TPEx', u'FRED', u'CMY'], tablefmt='latex'))

def test_price():
    x, y = tpex['zcbs'][100][3:5]
```


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```python
    irr, nc, freq = .055, 5, 1
    b = cbond(model_fred, x, x, y, irr, 0, 0, 30)
    price = cbond(model_fred, x, x, y, irr, nc, freq, 30).cleanPrice()
    print(price)
    oas = price2oas(price, ts_fred, b)
    print(oas)
    print(oas2price(oas, ts_fred, b))

def compute_irr_nc_price():
    irr_nc_price = namedtuple('irr_nc_price', 'irr nc freq price')
    _ = []
    x, y = tpex['zcbs'][100][3:5]
    for nc in [1, 2, 3, 4, 5, 6, 7, 8, 10]:
        for freq in [0, 1, 5]:
            for irr in np.linspace(.035, .055, 201):
                _.append(irr_nc_price(irr=irr, nc=nc, freq=freq, price=cbond(model_fred, x, x, y, irr, nc, freq, 30).cleanPrice()))
    df = pd.DataFrame(_, columns=_[0]._fields)
    df.to_pickle(irr_nc_price_pkl)

def compute_irr_nc_oas():
    irr_nc_oas = namedtuple('irr_nc_oas', 'irr nc freq oas')
    _ = []
    x, y = tpex['zcbs'][100][3:5]
    b = cbond(model_fred, x, x, y, .01, 0, 0, 30)
    df_ = pd.read_pickle(irr_nc_price_pkl)
    for nc in [1, 2, 3, 4, 5, 6, 7, 8, 10]:
        for freq in [0, 1, 5]:
            for irr in np.linspace(.035, .055, 201):
                price = df_.loc[(df_['nc'] == nc) & (df_['freq'] == freq) & (df_['irr'] == irr)]['price'].values.tolist()[0]
                _.append(irr_nc_oas(irr=irr, nc=nc, freq=freq, oas=price2oas(price, ts_fred, b)))
    df = pd.DataFrame(_, columns=_[0]._fields)
    df.to_pickle(irr_nc_oas_pkl)
```


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```python
def plot_irr_nc_price():
    df = pd.read_pickle(irr_nc_price_pkl)
    #df['price'] = df['price'].map(lambda x: 1.e4 / x)
    #df['price'] = df['price'].map(lambda x: 45.7395 - x)
    df['irr'] = df['irr'].map(lambda x: 100. * x)

    fig = plt.figure(figsize=(10, 16))
    ax1 = fig.add_subplot(3, 1, 1)
    ax2 = fig.add_subplot(3, 1, 2)
    ax3 = fig.add_subplot(3, 1, 3)

    def __(df_):
        return df_[::3]

    for ax, freq in [(ax1, 0), (ax2, 5), (ax3, 1)]:
        for key, grp in df.loc[df['freq'] == freq].groupby(['nc']):
            ax.plot(__(grp['irr']), __(grp['price']), linestyle='',
             ↪  marker=all_styles[key], label='nc = ' + str(key),)
        ax.set_title('Call Frequency = %s' % (str(freq) + ' Year' + ('s'
         ↪  if freq > 1 else ''),))
        if freq == 1:
            ax.set_xlabel('IRR (%)')
        else:
            ax.set_xlabel('')
        ax.set_ylabel(u'Price')
        ax.legend(loc=3)

    fig.savefig('irr_nc_price.pdf', bbox_inches='tight')

def plot_irr_nc_oas():
    df = pd.read_pickle(irr_nc_oas_pkl)
    df['oas'] = df['oas'].map(lambda x: 1.e4 * x)
    df['irr'] = df['irr'].map(lambda x: 100. * x)

    fig = plt.figure(figsize=(10, 16))
    ax1 = fig.add_subplot(3, 1, 1)
    ax2 = fig.add_subplot(3, 1, 2)
    ax3 = fig.add_subplot(3, 1, 3)
```


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```python

for ax, freq in [(ax1, 0), (ax2, 5), (ax3, 1)]:
    for key, grp in df.loc[df['freq'] == freq].groupby(['nc']):
        ax.plot(grp['irr'], grp['oas'], linestyle='', marker=all_styles[key], label='nc = ' + str(key),)
    ax.set_title('Call Frequency = %s' % (str(freq) + ' Year' + ('s' if freq > 1 else ''),))
    if freq == 1:
        ax.set_xlabel('IRR (%)')
    else:
        ax.set_xlabel('')
    ax.set_ylabel(u'OAS (bps)')
    ax.legend()

fig.savefig('irr_nc_oas.pdf', bbox_inches='tight')
```


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Bibliography

Ametrano, F. M., & Bianchetti, M. (2013). Everything you always wanted to know about multiple interest rate curve bootstrapping but were afraid to ask. Social Science Research Network (SSRN) Working Paper Series. `https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2219548`.

Andersen, L. (2008). Simple and efficient simulation of the Heston stochastic volatility model. *Journal of Computational Finance*, *11*(3), 1–42.

Andersen, L., & Piterbarg, V. (2010a). *Interest Rate Modeling. Volume I: Foundations and Vanilla Models*. London: Atlantic Financial Press.

Andersen, L., & Piterbarg, V. (2010b). *Interest Rate Modeling. Volume II: Term Structure Models*. London: Atlantic Financial Press.

Andersen, L., & Piterbarg, V. (2010c). *Interest Rate Modeling. Volume III: Products and Risk Management*. London: Atlantic Financial Press.

Bacinello, A. R. (2001). Fair pricing of life insurance participating policies with a minimum interest rate guaranteed. *ASTIN Bulletin*, *31*, 275–297.

Bacinello, A. R., & Ortu, F. (1996). Fixed income linked life insurance policies with minimum guarantees: Pricing models and numerical results. *European Journal of Operatoinal Research*, *91*, 235–249.

Bai, J. (2003). Testing parametric conditional distributions of dynamic models. *The Review of Economics and Statistics*, *85*(3), 531–549.

Bakshi, G., & Madan, P. (2000). Spanning and derivative-security valuation. *Journal of Financial Economics*, *55*, 205–238.

Balaraman, G., & Ballabio, L. (2017). *QuantLib Python Cookbook*. Leanpub.

DOI:10.6814/NCCU202101641

Ballabio, L. (2017). *Implementing QuantLib*. Leanpub.

Barbarin, J., & Devolder, P. (2005). Risk measure and fair valuation of an investment guarantee in life insurance. *Insurance: Mathematics and Economics*, *37*(2), 297–323.

Bauer, D., Kiesel, R., Kling, A., & Ruß, J. (2006). Risk-neutral valuation of participating life insurance contracts. *Insurance: Mathematics and Economics*, *39*(2), 171–183.

Bernard, C., Courtois, O. L., & Quittard-Pinon, F. (2006). Development and pricing of a new participating contract. *North American Actuarial Journal*, *10*, 179–195.

Billingsley, P. (1999). *Convergence of Probability Measures*. New York: John Wiley & Sons, Second ed.

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, *81*(3), 637–654.

Blyth, S. (2014). *An Introduction to Quantitative Finance*. Oxford, UK: Oxford University Press.

Brennan, M. J., & Schwartz, E. S. (1976). The pricing of equity-linked life insurance policies with an asset value guarantee. *Journal of Financial Economics*, *3*, 195–213.

Brigo, D., & Mercurio, F. (2006). *Interest Rate Models – Theory and Practice: With Smile, Inflation and Credit*. Berlin: Springer-Verlag, Second ed.

Briys, E., & de Varenne, F. (1994). Life insurance in a contingent claim framework: pricing and regulatory implications. *The Geneva Papers on Risk and Insurance Theory*, *19*(1), 53–72.

Briys, E., & de Varenne, F. (1997). On the risk of insurance liabilities: Debunking some common pitfalls. *Journal of Risk and Insurance*, *64*(4), 673–694.

Chan, N.-H., Chen, J., Chen, X., Fan, Y., & Peng, L. (2009). Statistical inference for multivariate residual copula of GARCH models. *Statistica Sinica*, *19*, 53–70.

Chang, S. C., & Wu, J. W. (2016). Risk assessment of international bond investment in Taiwan life insurance industry. *Taiwan Insurance Review*, *32*(4), 1–33.

Chen, X., & Fan, Y. (2006a). Estimation and model selection of semiparametric copula-based multivariate dynamic models under copula misspecification. *Journal of Econometrics*, *135*, 125–154.

Chen, X., & Fan, Y. (2006b). Estimation of copula-based semiparametric time series models. *Journal of Econometrics*, *130*, 307–335.

DOI:10.6814/NCCU202101641

Chiou, S.-C., & Tsay, R.-S. (2008). A copula-based approach to option pricing and risk assessment. *Journal of Data Science*, *6*, 273–301.

Corb, H. (2012). *Interest Rate Swaps and Other Derivatives.* New York: Columbia University Press.

Cox, J., Ingersoll, J., & Ross, S. (1985). A theory of the term structure of interest rates. *Econometrica*, *53*, 385–407.

Da Fonseca, J., & Ziveyi, J. (2017). Valuing variable annuity guarantees on multiple assets. *Scandinavian Actuarial Journal*, *2017*(3), 209–230.

Dai, T. S. (2017). The theoretical prices of USD zero-coupon callable international bonds are now available online. Taipei Exchange.
URL https://nweb.tpex.org.tw/TPEX150/en/en_news2.html

Dickson, D., Hardy, M., & Waters, H. (2013). *Actuarial Mathematics for Life Contingent Risks.* Cambridge: Cambridge University Press, Second ed.

Dudley, R. (1999). *Uniform Central Limit Theorem.* Cambridge: Cambridge University Press.

Durante, F., & Sampi, C. (2016). *Principles of Copula Theory.* Boca Raton, F.L.: Chapman & Hall/CRC.

Fabozzi, F., & Mann, S. (2010). *Introduction to Fixed Income Analytics: Relative Value Analysis, Risk Measures, and Valuation.* Hoboken, N.J.: John Wiley & Sons, Second ed.

Feller, W. (1951). Two singular diffusion problems. *Annals of Mathematics*, *54*(1), 173–182.

Fermanian, J.-D., & Wegkamp, M. H. (2012). Time-dependent copulas. *J. Multivariate Anal.*, *110*, 19–29.

Gaenssler, P., & Stute, W. (1987). *Seminar on Empirical Processes.* Basel: Birkhaüser.

Gatzert, N., & Kling, A. (2007). Analysis of participating life insurance contracts: a unification approach. *Journal of Risk and Insurance*, *74*(3), 547–570.

Gatzert, N., & Schmeiser, H. (2013). New life insurance financial products. In G. Dionne (Ed.) *Handbook of Insurance*, (pp. 1061–1095). Springer-Verlag, Second ed.

Genest, C., & Rémillard, B. (2004). Tests of independence and or randomness based on the empirical copula process. *Test*, *13*, 335–369.

DOI:10.6814/NCCU202101641

Genest, C., & Rémillard, B. (2008). Validity of the parametric bootstrap for goodness-of-fit testing in semiparametric models. *Ann. Inst. H. Poincarè Sect. B.*, *44*, 1096–1127.

Ghalanos, A. (2018). `rugarch`*: Univariate GARCH models.* R package version 1.4-0. URL `https://cran.r-project.org/web/packages/rugarch/index.html`

Ghoudi, K., & Rémillard, B. (2014). Comparison of specification tests for GARCH models. *Computational Statistics and Data Analysis*, *76*, 291–300.

Gilli, M., & Schumann, E. (2010). Calibrating option pricing models with heuristics. In A. Brabazon, M. O'Neill, & D. Maringer (Eds.) *Natural Computing in Computational Finance: Volume 4*, (pp. 9–37). New York: Springer-Verlag. `http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1566975`.

Graf, S., Kling, A., & Ruß, J. (2011). Risk analysis and valuation of life insurance contracts: Combining actuarial and financial approaches. *Insurance: Mathematics and Economics*, *49*, 115–125.

Graf, S., Kling, A., & Ruß, J. (2012). Financial planning and risk-return profiles. *European Actuarial Journal*, *2*(1), 77–104.

Grosen, A., & Jørgensen, P. L. (1997). Valuation of early exercisable interest rate guarantees. *Journal of Risk and Insurance*, *64*(3), 481–503.

Grosen, A., & Jørgensen, P. L. (2000). Fair valuation of life insurance liabilities: The impact of interest rate guarantees, surrender options, and bonus policies. *Insurance: Mathematics and Economics*, *26*, 37–57.

Grosen, A., & Jørgensen, P. L. (2002). Life insurance liabilities at market value: An analysis of insolvency risk, bonus policy, and regulatory intervention rules in a barrier option framework. *Journal of Risk and Insurance*, *69*, 63–91.

Hayre, L. (Ed.) (2001). *Salomon Smith Barney Guide to Mortgage-Backed and Asset-Backed Securities.* New York: John Wiley & Sons.

Heston, S. (1993). A closed-form solutions for options with stochastic volatility. *The Review of Financial Studies*, *6*, 327–343.

Hofert, M., Kojadinovic, I., Mächler, M., & Yan, J. (2017). `copula`*: Multivariate Dependence with Copulas.* R package version 0.999-18. URL `http://cran.r-project.org/package=copula`

Homer, S., Leibowitz, M. L., Bova, A., & Kogelman, S. (2013). *Inside the Yield Book: The Classic That Created the Science of Bond Analysis.* Hoboken, N.J.: John Wiley & Sons, Third ed.

DOI:10.6814/NCCU202101641

Hsuan, W., & Chang, S. C. (2018a). The Copula-GARCH model: Application to variable annuity guarantee valuations on multiple assets. *Journal of Risk Management*, *20*(2), 131–164.

Hsuan, W., & Chang, S. C. (2018b). Risk assessment of the life insurer in Taiwan: An examination of the interest sensitive life insurance policies. *Journal of Risk Management*, *20*(1), 5–29.

Hsuan, W., & Chang, S. C. (2019). International bonds valuation system: Theory and practice. *Taiwan Insurance Review*, *35*(3), 1–34.

Hull, J. C. (2018). *Options, Futures, and Other Derivatives*. Boston: Pearson, Tenth ed.

Hull, J. C., & White, A. (1990). Pricing interest rate derivative securities. *The Review of Financial Studies*, *3*, 573–592.

Hull, J. C., & White, A. (1993a). Bond option pricing based on a model for the evolution of bond prices. *Advances in Futures and Options Research*, *6*, 1–13.

Hull, J. C., & White, A. (1993b). Efficient procedures for valuing European and American path-dependent options. *Journal of Derivatives*, *1*, 21–31.

Hull, J. C., & White, A. (1993c). One-factor interest rate models and the valuation of interest rate derivative securities. *Journal of Financial and Quantitative Analysis*, *28*(2), 235–254.

Hull, J. C., & White, A. (1993d). The pricing of options on interest-rate caps and floors using the Hull-White model. *The Journal of Financial Engineering*, *2*, 287–296.

Hull, J. C., & White, A. (1994a). Numerical procedures for implementing term structure models I: Single-factor models. *Journal of Derivatives*, *2*, 7–16.

Hull, J. C., & White, A. (1994b). Numerical procedures for implementing term structure models II: Two-factor models. *Journal of Derivatives*, *2*, 37–47.

Hull, J. C., & White, A. (1996). Using Hull-White interest rate trees. *Journal of Derivatives*, *3*(3), 26–36.

Hull, J. C., & White, A. (2001). The general Hull-White model and supercalibration. *Financial Analysts Journal*, *57*(6), 34–43.

Iacus, S. (2008). *Simulation and Inference for Stochastic Differential Equations With R Examples*. New York: Springer-Verlag.

Jamshidian, J. (1989). An exact bond option formula. *Journal of Finance*, *44*(1), 205–209.

DOI:10.6814/NCCU202101641

Jensen, B., Jørgensen, P. L., & Grosen, A. (2001). A finite difference approach to the valuation of path dependent life insurance liabilities. *The Geneva Papers on Risk and Insurance Theory*, *26*, 57–84.

Jha, S. (2011). *Interest Rate Markets: A Practical Approach to Fixed Income*. Hoboken, N.J.: John Wiley & Sons.

Joe, H. (2005). Asymptotic efficiency of the two-stage estimation method for copula-based models. *J. Multivariate Anal.*, *94*, 401–419.

Joe, H. (2014). *Dependence Modeling with Copulas*. Boca Raton, F.L.: Chapman & Hall/ CRC.

Jondeau, E., Poon, S. H., & Rockinger, M. (2007). *Financial Modeling Under Non-Gaussian Distributions*. Springer-Verlag.

Jondeau, E., & Rockinger, M. (2006). The copula-GARCH model of conditional dependencies: An international stock market application. *Journal of International Money and Finance*, *25*(5), 827–853.

Kladıvko, K. (2007). Maximum likelihood estimation of the Cox-Ingersoll-Ross process: the MATLAB implementation. In *Technical Computing Prague*. working paper.

Kling, A., Richter, A., & Ruß, J. (2007). The interaction of guarantees, surplus distribution, and asset allocation in with-profit life insurance policies. *Insurance: Mathematics and Economics*, *40*, 164–178.

Kojadinovic, I., & Yan, J. (2010). Modelling multivariate distributions with continuous margins using the `copula` r package. *J. Statistical Software*, *34*(9).

Kosorok, M. (2008). *Introduction to Empirical Processes and Semiparametric Inference*. New York: Springer-Verlag.

Longstaff, F., & Schwartz, E. (2001). Valuing American options by simulation: A simple least squares approach. *The Review of Financial Studies*, *14*(1), 113–147.

Miltersen, K. R., & Persson, S. A. (2003). Guaranteed investment contracts: distributed and undistributed excess return. *Scandinavian Actuarial Journal*, *4*, 257–279.

Nasri, B., & Rémillard, B. (2018). Copula-based dynamic models for multivariate time series. Social Science Research Network (SSRN) Working Paper Series.
URL `https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3181605`

Nasri, B. R., & Rémillard, B. N. (2019). Copula-based dynamic models for multivariate time series. *J. Multivariate Anal.*, *172*, 107–121.

DOI:10.6814/NCCU202101641

Ng, C.-Y., & Li, S.-H. (2013). Pricing and hedging variable annuity guarantees with multiasset stochastic investment models. *North American Actuarial Journal*, *17*(1), 41–62.

Patton, A. (2006). Modelling asymmetric exchange rate dependence. *International Economic Review*, *47*, 527–556.

Price, K., Storn, R. M., & Lampinen, J. A. (2005). *Differential Evolution: A Practical Approach to Global Optimization*. New York: Springer-Verlag.

Rémillard, B. (2011). Validity of the parametric bootstrap for goodness-of-fit testing in dynamic models. Social Science Research Network (SSRN) Working Paper Series. URL https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1966476

Rémillard, B. (2012). Non-parametric change point problems using multipliers. Social Science Research Network (SSRN) Working Paper Series. URL https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2043632

Rémillard, B. (2013). *Statistical Methods for Financial Engineering*. Boca Raton, F.L.: Chapman & Hall/CRC.

Rémillard, B. (2017). Goodness-of-fit tests for copulas of multivariate time series. *Econometrics*, *5*(1), 1–23.

Rosenberg, J. (1998). Pricing multivariate contingent claims using estimated risk-neutral density functions. *Journal of International Money and Finance*, *17*, 229–247.

Rosenberg, J. (2003). Nonparametric pricing of multivariate contingent claims. *Journal of Derivatives*, *10*, 9–26.

Rouah, F. (2013). *The Heston Model and its Extensions in MATLAB and C#*. Hoboken, N.J.: John Wiley & Sons.

Shorack, G., & Wellner, J. A. (1986). *Empirical Processes with Applications to Statistics*. New York: John Wiley & Sons.

Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publications de L'Institut de Statistique de L'Université de Paris*, *8*, 229–231.

Storn, R., & Price, K. (1997). Differential evolution – a simple and efficient heuristic for global optimization over continuous spaces. *Journal of Global Optimization*, *11*, 341–359.

Sundaresan, S. (2009). *Fixed Income Markets and Their Derivatives*. Burlington, MA: Academic Press, Third ed.

DOI:10.6814/NCCU202101641

Tanskanen, A. J., & Lukkarinen, J. (2003). Fair valuation of path-dependent participating life insurance contracts. *Insurance: Mathematics and Economics*, *33*, 595–609.

Tuckman, B., & Serrat, A. (2012). *Fixed Income Securities: Tools for Today's Markets*. Hoboken, N.J.: John Wiley & Sons, Third ed.

van den Goorbergh, R., Genest, C., & Werker, B. (2005). Bivariate option pricing using dynamic copula models. *Insurance: Mathematics and Economics*, *37*, 101–114.

van der Vaart, A. W., & Wellner, J. A. (1996). *Weak Convergence and Empirical Processes: With Applications to Statistics*. New York: Springer-Verlag.

van Haastrecht, A., Lord, R., Pelsser, A., & Schrager, R. (2009). Pricing long-dated insurance contracts with stochastic interest rates and stochastic volatility. *Insurance: Mathematics and Economics*, *45*, 436–448.

Veronesi, P. (2010). *Fixed Income Securities: Valuation, Risk, and Risk Management*. Hoboken, N.J.: John Wiley & Sons.

Wilmott, P. (2002). Cliquet options and volatility models. *Wilmott Magazine*, *December*, 78–83.

Zhang, J., & Guégan, D. (2008). Pricing bivariate option under GARCH processes with time-varying copula. *Insurance: Mathematics and Economics*, *42*, 1095–1103.

DOI:10.6814/NCCU202101641