國立政治大學應用數學系

碩士學位論文

漢諾圖上的哈密頓路徑
Hamiltonian Walks on the Hanoi Graph

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中文摘要

本文給出了 n 階 2 維漢諾圖（又稱漢諾塔圖、河內圖）上哈密頓路徑的數量，其漸進表現是 $h(n) \sim \frac{25\times16^n}{624}$。這類漢諾圖上的哈密頓路徑總數量與起點在最上面的顶點的哈密頓路徑數量的對數的比值漸進至 2。同時，當這類漢諾圖上三個方向的平行邊分別被 x, y, z 這三個數加權後，我們也推導出了它們的哈密頓路徑的加權和，其漸進表現為 $h'(n) \sim \frac{25w}{16\times27\times13}16^n(xyz)^{3^{n-1}}$，其中 $w = \dfrac{(x+y+z)^2}{xyz}$。

關鍵字：漢諾圖、哈密頓路徑、漸進表現

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Abstract

We've derived the number of Hamiltonian walks on the two-dimensional Hanoi graph at stage n, whose asymptotic behaviour is given by $h(n) \sim \frac{25\times 16^n}{624}$. And the asymptotic behaviour the logarithmic ratio of the number of Hamiltonian walks on these Hanoi graphs with that one end at the topmost vertex is given by 2. When the parallel edges in the three directions on these Hanoi graphs are weighted by three numbers, x, y, z, the weighted sum of their Hamiltonian paths is also derived by us, and the asymptotic behaviour of it is given by $h'(n) \sim \frac{25w}{16\times 27\times 13}16^n(xyz)^{3^{n-1}}$, in which $w = \dfrac{(x+y+z)^2}{xyz}$.

Key word: Hanoi graph, Hamiltonian walk, asymptotic behaviour

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Chapter 1

Introduction

1.1 Hanoi graphs

Hanoi graph, a kind of simple fractal lattice, are undirected graphs whose vertices represent the possible states of the Tower of Hanoi puzzle [7], its edges represent the allowable moves between pairs of states. The puzzle consists of a set of disks of different sizes, which are placed on a set of fixed towers in increasing order of size. The Hanoi graph for the puzzle with n disks on the k towers is denoted by H_k^n [5] [6]. Each state of the puzzle is determined by choosing a tower for each disk, so the graph has k^n vertices [6].

In the Tower of Hanoi puzzle denoted by H_k^n, each state of the puzzle is determined by choosing a tower for each disk. So this graph has k^n vertices [6].

In the movement of this puzzle, the smallest disk on each tower could be moved to an unoccupied tower or a tower with a larger smallest disk. If there are i unoccupied towers, the number of allowable moves is

$$\binom{k-i}{2} + i(k-i) = \binom{k}{2} - \binom{i}{2}, \quad i = 0, 1, \cdots, k-1,$$

which there are $\binom{k}{2} \sum_{j=o}^{i} \binom{i}{j} (i-j)^n (-1)^j$ vertices on behalf of it. Whenever, in H_k^n, there are $\binom{k}{2} \sum_{j=o}^{i} \binom{i}{j} (i-j)^n (-1)^j$ vertices have degree $\binom{k}{2} - \binom{i}{2}$.

In other hand, there are $\binom{k}{2} k^{n-i} (k-2)^{i-1}$ edges which is on behalf of moving the i^{th}

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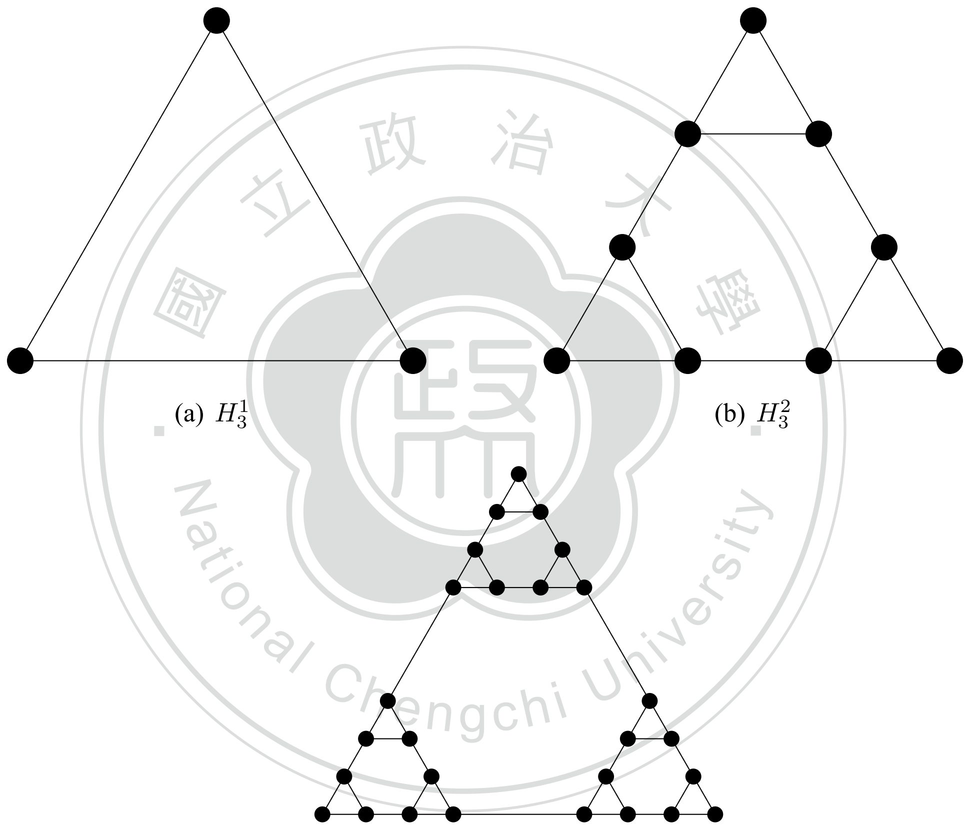
smallest disk. So,

$$e\left(H_k^n\right) = \sum_{i=1}^{n} \binom{k}{2} k^{n-i}(k-2)^{i-1} = \frac{1}{2}\binom{k}{2}\left(k^n - (k-2)^n\right).$$

Therefore, we have a combinatorial identity which is

$$\frac{1}{2}\sum_{i=0}^{k-1}\left(\binom{k}{2} - \binom{i}{2}\right)\binom{k}{2}\sum_{j=o}^{i}\binom{i}{j}(i-j)^n(-1)^j = \frac{1}{2}\binom{k}{2}\left(k^n - (k-2)^n\right).$$



(a) H_3^1 (b) H_3^2

(c) H_3^3

Figure 1.1: Hanoi graphs with 3 towers

1.2 Hamiltonian walk

A Hamiltonian walk is defined to be a walk on a graph that visits each vertex once and only once [3], is also a special spanning tree. In general, problems about Hamiltonian walks is an important issue in graph theory. It's difficult to determine whether such a walk is in general

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graphs. People have made some processing in some especially graphs, such as the Sierpinski gaskets [2] [3] and fractal lattices [1] [4] [8].

In this paper, we will only discuss the Hamiltonian walks in H_3^n. In chapter 2, we will show the number of Hamiltonian walks given by 3 for n=1 and

$$h(n) = \frac{25}{624}16^n + \frac{1}{3}4^n + \frac{3}{13}3^n + \frac{1}{3} \tag{1.2.1}$$

for $n \geq 2$.

The asymptotic behaviour of it is

$$h(n) \sim \frac{25 \times 16^n}{624}, \tag{1.2.2}$$

while $' \sim '$ is defined as

$$f(n) \sim g(n) \Leftrightarrow \lim_{x \to +\infty} \frac{f(n)}{g(n)} = 1. \tag{1.2.3}$$

No more than that, we've also obtained that the ratio of logarithms for the number of Hamiltonian walks on the Hanoi graph with one end at the topmost vertex is given by 2.

In chapter 3, we will derive the weighted sum of Hamiltonian walks with edges in three directions is weighted by x, y, z, and got the asymptotic behaviour as

$$h'(n) \sim \frac{25w}{16 \times 27 \times 13} 16^n (xyz)^{3^{n-1}}. \tag{1.2.4}$$

in which $w = \dfrac{(x+y+z)^2}{xyz}$.

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Chapter 2

Hamiltonian paths in weightless Hanoi graph

In this chapter, we will count the number of Hamiltonian paths in Hanoi graphs without being weighted.

2.1 Preliminaries

In this section, we will do some relevant definitions at first.

To count the number of Hamiltonian walks on H_3^n, let us define some quantities as follows.

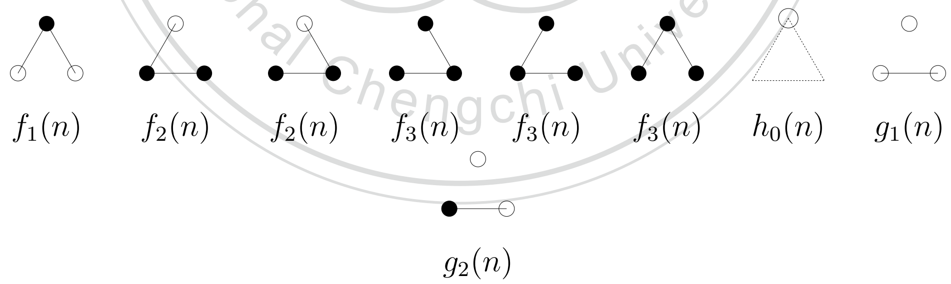

Figure 2.1: Illustration for the quantities $f_1(n)$,$f_2(n)$,$f_3(n)$,$g_1(n)$,$g_2(n)$ and $h_0(n)$. $f_2(n)$ is the numbers of both the second graph and the third one.

Definition 2.1.1. *Consider the Hanoi graph H_3^n for a puzzle with n disks on 3 towers.*

(i) Define $h(n)$ as the number of Hamiltonian walks on H_3^n.

(ii) Define $h_0(n)$ as the number of Hamiltonian walks on H_3^n with one end at the topmost vertex.

(iii) Define $f_1(n)$ as the number of Hamiltonian walks on H_3^n with one end at the rightmost

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vertex, another one at the leftmost vertex.

(iv) Define $f_2(n)$ as the number of Hamiltonian walks on H_3^n with one end at the topmost vertex, another one not at rightmost vertex nor left one.

(v) Define $f_3(n)$ as the number of Hamiltonian walks on H_3^n with no end at any of the outmost vertices.

In figure 2.1, An open circle denoting an outmost vertex corresponds to an end of a walk; a solid one corresponds it to be a middle point; if a vertex is not denoted by any of them, that means any of these would be done. In addition, two outmost vertices connected by a solid line belong to a same path without another outmost vertex; if an outmost vertex is not connected with any others, then it's not connected to other outmost vertices; if two outmost vertices connected by a solid line, that means they are connected but the another outmost vertex may in the middle of them.

According to the definitions of these functions, we can find

$$h_0(n) = 2f_1(n) + f_2(n), \tag{2.1.1}$$

and

$$h(n) = 3f_1(n) + 3f_2(n) + f_3(n). \tag{2.1.2}$$

But it's not enough to get the value of $f_1(n)$, $f_2(n)$ and $f_3(n)$. To calculate them, we need following definitions.

Definition 2.1.2. *Consider the Hanoi graph H_3^n for a puzzle with n disks on 3 towers.*

(i) Define $g_1(n)$ as the number of spanning subgraphs with two walks such that the topmost vertex be the end of one walk and the other two outmost vertices be ends of another walk on H_3^n.

(ii) Define $g_2(n)$ as the number of spanning subgraphs with two walks such that the topmost vertex be the end of one walk and the other two outmost vertices belong to another walk, and only the right one be an end.

Then, we can easily get the initial values as

$$f_1(1) = g_1(1) = 1, \quad g_2(1) = f_2(1) = f_3(1) = 0 \quad and \quad g_1(2) = 5.$$

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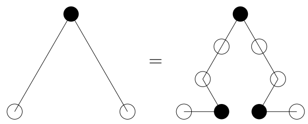

Figure 2.2: Illustration for the expression of $f_1(n+1)$

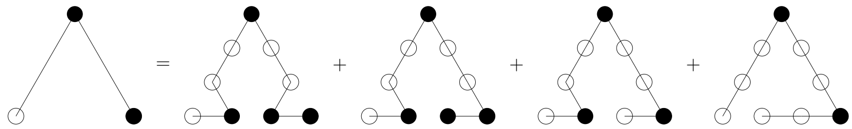

Figure 2.3: Illustration for the expression of $f_2(n+1)$

2.2 Building recursions to count the number of Hamiltonian walks

In this section, we will enumerate the number of Hamiltonian walks $h(n)$ and show the relationships between these number by building some recursions.

Consider the quantity $f_1(n+1)$, according to the figure 2.2, we have

$$f_1(n+1) = f_1^3(n) = f_1^{3^n}(1) = 1 \tag{2.2.1}$$

and get

$$h_0(n) = f_2(n) + 2 \tag{2.2.2}$$

by referencing (2.1.1).

Consider the quantity $f_2(n+1)$, which has two equivalence cases. According to the figure (2.3), we have

$$\begin{aligned} f_2(n+1) &= 2(f_1^2(n)f_2(n) + f_1^3(n) + f_1^2(n)g_1(n)) \\ &= 2(f_2(n) + 1 + g_1(n)) = 2(h_0(n) - 1 + g_1(n)) \\ \Rightarrow \quad h_0(n+1) - 2 &= 2(h_0(n) - 1 + g_1(n)) \\ \Rightarrow \quad h_0(n+1) &= 2(h_0(n) + g_1(n)). \end{aligned} \tag{2.2.3}$$

Noticed that the first graph on the right side at the figure (2.4) show the states of walks that one end at the rightmost vertex in the topper subgraph, another one in the topper subgraph too, but not the leftmost vertex. Otherwise the other vertices on the topper subgraph won't be on this Hamiltonian walk.

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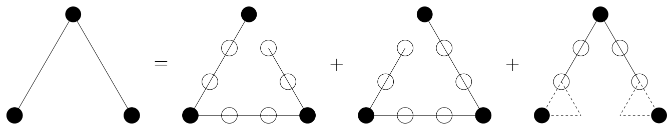

Figure 2.4: Illustration for the expression of $f_3(n+1)$. In the third sense,both of the left and right second-largest triangle have Hamiltonian walks with one end at the topmost vertex, but both the same sides outmost vertices are not the another end.

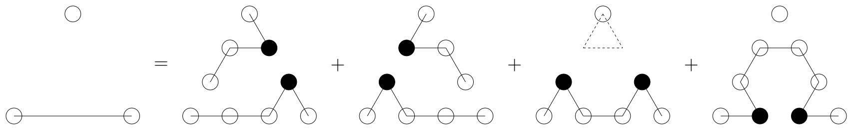

Figure 2.5: Illustration for the expression of $g_1(n+1)$

Finally, consider the quantities $f_3(n+1)$, $g_1(n+1)$ and $g_2(n+1)$, according to the figures (2.4), (2.5) and (2.6), we have the following recursions as

$$\begin{aligned} f_3(n+1) &= 3\left(2f_1^2(n)g_2(n) + (f_1(n))^2(h_0(n) - f_1(n))^2\right) \\ &= 3\left(2g_2(n) + (h_0(n)-1)^2\right), \end{aligned} \tag{2.2.4}$$

$$g_1(n+1) = 3f_1^2(n)g_1(n) + f_1^2(n)h_0(n) = 3g_1(n) + h_0(n), \tag{2.2.5}$$

and

$$\begin{aligned} & g_2(n+1) \\ =& f_1(n)g_1(n)(h_0(n)-1) + 2f_1(n)g_1^2(n) + 2f_1^2(n)(g_1(n)+g_2(n)) \\ &+ f_1(n)h_0(n)(h_0(n)-1) + f_1(n)g_1^2(n) + f_1^2(n)g_2(n) + f_1(n)g_1(n)(h_0(n)-1) \\ =& h_0^2(n) - h_0(n) + 2h_0(n)g_1(n) + 3g_1^2(n) + 3g_2(n). \end{aligned} \tag{2.2.6}$$

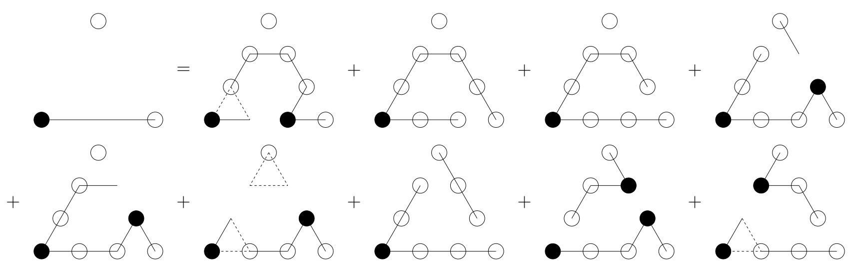

Figure 2.6: Illustration for the expression of $g_2(n+1)$

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In summary, we have a set of recursions that is

$$\begin{cases} h_0(n+1) = 2(h_0(n) + g_1(n)) \\ f_3(n+1) = 6g_2(n) + 3(h_0(n) - 1)^2 \\ g_1(n+1) = 3g_1(n) + h_0(n) \\ g_2(n+1) = h_0^2(n) - h_0(n) + 2h_0(n)g_1(n) + 3g_1^2(n) + 3g_2(n). \end{cases} \quad (2.2.7)$$

2.3 Number of Hamiltonian walks

In this section, we will solve the recursions in the previous section and get the number of Hamiltonian walks. Consider the recursions (2.2.3) and (2.2.5), we have

$$g_1(n+2) - 5g_1(n+1) + 4g_1(n) = 0. \quad (2.3.1)$$

By the definition of $g_1(n)$, we can easily get $g_1(1) = 1$ and $g_1(2) = 5$. So, the solution of the recursion (2.3.1) can be solved as

$$g_1(n) = \frac{1}{3}(4^n - 1). \quad (2.3.2)$$

At the same time, we can get the solution of $h_0(n)$ through (2.2.5).

Theorem 2.3.1. *The number of Hamiltonian walks on H_3^n with one end at the topmost vertex is given by*

$$h_0(n) = g_1(n+1) - 3g_1(n) = \frac{1}{3}(4^n + 2). \quad (2.3.3)$$

Hence, we have

$$f_2(n) = h_0(n) - 2 = \frac{1}{3}(4^n - 4). \quad (2.3.4)$$

Therefore, we can also get a recursion of $g_2(n)$ that is

$$\begin{aligned} g_2(n+1) &= h_0^2(n) - h_0(n) + 2h_0(n)g_1(n) + 3g_1^2(n) + 3g_2(n) \\ &= \frac{2}{3}16^n - \frac{1}{3}4^n - \frac{1}{3} + 3g_2(n). \end{aligned} \quad (2.3.5)$$

Such that

$$\frac{g_2(n+1)}{3^{n+1}} - \frac{g_2(n)}{3^n} = \frac{2}{9}\left(\frac{16}{3}\right)^n - \frac{1}{9}\left(\frac{4}{3}\right)^n - \frac{1}{9}\left(\frac{1}{3}\right)^n. \quad (2.3.6)$$

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So, we have

$$g_2(n) = \frac{2}{39}16^n - \frac{1}{3}4^n + \frac{3}{26}3^n + \frac{1}{6}, \tag{2.3.7}$$

because of $g_2(1) = 0$, which obtained

$$\begin{aligned} f_3(n+1) &= 3(2g_2(n) + (h_0(n) - 1)^2) \\ &= \frac{25}{39}16^n - \frac{8}{3}4^n + \frac{9}{13}3^n + \frac{4}{3}. \end{aligned} \tag{2.3.8}$$

So, we can get the solution of $f_3(n)$ as the following theorem.

Theorem 2.3.2. *The number of Hamiltonian walks on H_3^n with no end at any of the outmost vertices is given by 0 for n=1 and*

$$f_3(n) = \frac{25}{624}16^n - \frac{2}{3}4^n + \frac{3}{13}3^n + \frac{4}{3} \tag{2.3.9}$$

for $n \geq 2$.

Since (2.1.2), we have the following theorem to show the number of Hamiltonian walks on H_3^n.

Theorem 2.3.3. *The number of Hamiltonian walks on H_3^n is 3 for n=1 and*

$$h(n) = \frac{25}{624}16^n + \frac{1}{3}4^n + \frac{3}{13}3^n + \frac{1}{3} \tag{2.3.10}$$

for $n \geq 2$.

Counterpart to this conclusion, the number of Hamiltonian walks on the Sierpinski gasket had been derived by Shu-Chiuan Chang and Lung-Chi Chen [3].

Remark 2.3.4. *If we defined the number of Hamiltonian walks on the two-dimensional Sierpinski gasket $SG(n)$ to be $HSG(n)$, then it is 12 for $n = 1$ and*

$$\begin{aligned} HSG(n) = \frac{(2\sqrt{3})^2}{6} \Bigg\{ &(\frac{5^2 \times 7^2 \times 17^2}{2^{12} \times 3^5 \times 13})16^n + (\frac{7 \times 13 \times 17}{2^3 \times 3^5})4^n \\ &+ (\frac{11 \times 257}{2^3 \times 3^5 \times 13})3^n + \frac{4391}{2^3 \times 3^5} - (\frac{1}{2^4 \times 3^5})\delta_{n2} \Bigg\} \end{aligned} \tag{2.3.11}$$

for $n \geq 2$ where δ_{ij} is the Kronecker delta function.

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Considering the asymptotic behaviour when n is large, let us use the symbol $f(n) \sim g(n)$ to denote $lim_{n\to\infty} f(n)/g(n) = 1$.

Theorem 2.3.5. *When n is large, the asymptotic behaviour of the number of Hamiltonian walks on H_3^n is given by*

$$h(n) \sim \frac{25 \times 16^n}{624}. \tag{2.3.12}$$

The logarithmic ratio of the number of Hamiltonian walks on the Hanoi graph wit that one end at the topmost vertex, when n is large, is given by

$$\frac{\ln(h(n))}{\ln(h_0(n))} \sim \frac{\ln(\frac{25}{624}) + 2n\ln(4)}{\ln(\frac{1}{3}) + n\ln(4)} \sim 2. \tag{2.3.13}$$

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Chapter 3

Hamiltonian paths in weighted Hanoi graph

In last chapter, we've counted the number of Hamiltonian paths in a Hanoi graph without being weighted as (2.3.10). In this chapter, we will calculate the weighted sum of Hamiltonian paths in a weighted Hanoi graph whose edges in three directions is weighted by x, y, z.

3.1 Preliminaries

Definition 3.1.1. *Similarly as H_3^n, let H'^n_3 be a weighted Hanoi graph for a puzzle with n disks on 3 towers which is weighted on edges as follows.*
(i) Weighted as x if the edge on H'^n_3 which is parallel to the base of H'^n_3.
(ii) Weighted as y if the edge on H'^n_3 which is parallel to the right leg of H'^n_3.
(iii) Weighted as z if the edge on H'^n_3 which is parallel to the left leg of H'^n_3.

If the weight of each Hamiltonian path is defined as the production of the weights of its edges, to calculate the weighted sum of Hamiltonian walks on H'^n_3, let us define some quantities as follows.

Definition 3.1.2. *Consider the weighted Hanoi graph H'^n_3 for a puzzle with n disks on 3 towers.*
(i) Define $h'(n)$ as the weighted sum of Hamiltonian walks on H'^n_3.
(ii) Define $h'_0(0, n)$ as the weighted sum of Hamiltonian walks on H'^n_3 with one end at the the topmost vertex.
(iii) Define $h'_0(1, n)$ as the weighted sum of Hamiltonian walks on H'^n_3 with one end at the the

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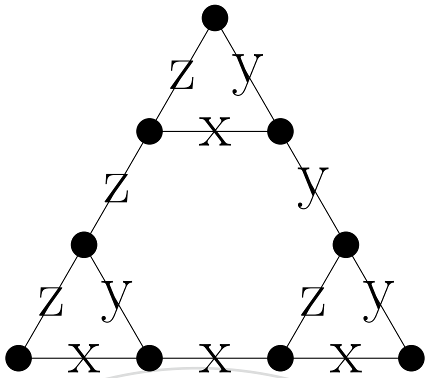

Figure 3.1: The weighted Hanoi graph, ${H'}_3^2$

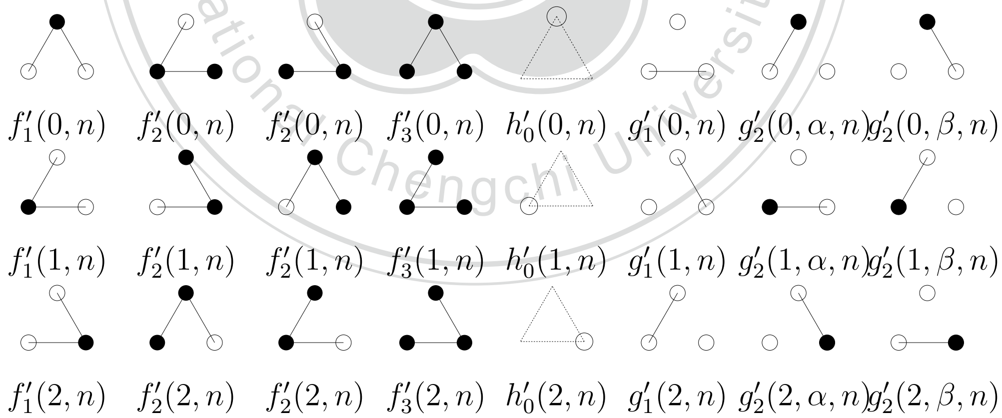

Figure 3.2: Illustration for the quantities $f'_1(i,n)$,$f'_2(i,n)$,$f'_3(i,n)$,$g'_1(i,n))$,$g'_2(i,\alpha,n))$,$g'_2(i,\beta,n)$ and $h'_0(i,n)$ for $i=0,1,2$. $f'_2(0,n)$ is the weighted sum of both the second graph and the third one, which similarly as $f'_2(1,n)$ and $f'_2(2,n)$.

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leftmost vertex.

(iv) Define $h'_0(2,n)$ *as the weighted sum of Hamiltonian walks on* H'^n_3 *with one end at the the rightmost vertex.*

(v) Define $f'_1(n)$ *as the weighted sum of Hamiltonian walks on* H'^n_3 *with both two ends at outmost vertices.*

(vi) Define $f'_1(0,n)$ *as the weighted sum of Hamiltonian walks on* H'^n_3 *with one end at the rightmost vertex, another one at the leftmost vertex.*

(vii) Define $f'_1(1,n)$ *as the weighted sum of Hamiltonian walks on* H'^n_3 *with one end at the rightmost vertex, another one at the topmost vertex.*

(viii) Define $f'_1(2,n)$ *as the weighted sum of Hamiltonian walks on* H'^n_3 *with one end at the lefttmost vertex, another one at the topmost vertex.*

(ix) Define $f'_2(n)$ *as the weighted sum of Hamiltonian walks on* H'^n_3 *with one end at an outmost vertex, another one not at any outmost vertex.*

(x) Define $f'_2(0,n)$ *as the weighted sum of Hamiltonian walks on* H'^n_3 *with one end at the topmost vertex, another one not at rightmost vertex nor left one.*

(xi) Define $f'_2(1,n)$ *as the weighted sum of Hamiltonian walks on* H'^n_3 *with one end at the leftmost vertex, another one not at other outmost vertices.*

(xii) Define $f'_2(2,n)$ *as the weighted sum of Hamiltonian walks on* H'^n_3 *with one end at the rightmost vertex, another one not at others.*

(xiii) Define $f'_3(n)$ *as the weighted sum of Hamiltonian walks on* H'^n_3 *with no end at any of the outmost vertices.*

(xiv) Define $f'_3(0,n)$ *as the weighted sum of Hamiltonian walks on* H'^n_3 *with no end at any of the outmost vertices. And the path on leftmost vertex to the rightmost one is path thought of the topmost one.*

(xv) Define $f'_3(1,n)$ *as the weighted sum of Hamiltonian walks on* H'^n_3 *with no end at any of the outmost vertices. And the path on topmost vertex to the leftmost one is path thought of the leftmost one.*

(xvi) Define $f'_3(2,n)$ *as the weighted sum of Hamiltonian walks on* H'^n_3 *with no end at any of the outmost vertices. And the path on topmost vertex to the rightmost one is path thought of the rightmost one.*

In figure 3.2, An open circle denoting an outmost vertex corresponds to an end of a walk;

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a solid one corresponds it to be a middle point; if a vertex is not denoted by any of them, that means any of these would be done. In addition, two outmost vertices connected by a solid line belong to a same path without another outmost vertex; if an outmost vertex is not connected with any others, then it's not connected to other outmost vertices; if two outmost vertices connected by a solid line, that means they are connected but the another outmost vertex may in the middle of them.

These definitions follow

$$h'(n) = f'_1(n) + f'_2(n) + f'_3(n), \tag{3.1.1}$$

$$f'_1(n) = f'_1(0,n) + f'_1(1,n) + f'_1(2,n), \tag{3.1.2}$$

$$f'_2(n) = f'_2(0,n) + f'_2(1,n) + f'_2(2,n), \tag{3.1.3}$$

$$f'_3(n) = f'_3(0,n) + f'_3(1,n) + f'_3(2,n) \tag{3.1.4}$$

And

$$\begin{cases} h'_0(0,n) = f'_1(1,n) + f'_1(2,n) + f'_2(0,n) \\ h'_0(1,n) = f'_1(0,n) + f'_1(2,n) + f'_2(1,n) \\ h'_0(2,n) = f'_1(0,n) + f'_1(1,n) + f'_2(2,n). \end{cases} \tag{3.1.5}$$

In addition, for the convenience of calculation, we define a function $h'_0(n)$ as

$$h'_0(n) := h'_0(0,n) + h'_0(1,n) + h'_0(2,n) \tag{3.1.6}$$

To get the solutions of $f'_1(n)$, $f'_2(n)$ and $f'_3(n)$, we need following definitions.

Definition 3.1.3. *Consider the weighted Hanoi graph ${H'}^n_3$ for a puzzle with n disks on 3 towers.*
(i) Define $g'_1(n)$ as the weighted sum of spanning subgraphs with two walks such that an outmost vertex be the end of one walk and the other two outmost vertices be ends of another walk on ${H'}^n_3$.
(ii) Define $g'_1(0,n)$ as the weighted sum of spanning subgraphs with two walks such that the

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topmost vertex be the end of one walk and the other two outmost vertices be ends of another walk on ${H'}_3^n$.

(iii) Define $g'_1(1,n)$ *as the weighted sum of spanning subgraphs with two walks such that the leftmost vertex be the end of one walk and the other two outmost vertices be ends of another walk on* ${H'}_3^n$.

(iv) Define $g'_1(2,n)$ *as the weighted sum of spanning subgraphs with two walks such that the rightmost vertex be the end of one walk and the other two outmost vertices be ends of another walk on* ${H'}_3^n$.

(v) Define $g'_2(n)$ *as the weighted sum of spanning subgraphs with two walks such that an outmost vertex be the end of one walk and the other two outmost vertices belong to another walk, and only one be an end.*

(vi) Define $g'_2(0,n)$ *as the weighted sum of spanning subgraphs with the topmost is not the end of either walk in the cases of* $g'_2(n)$.

(vii) Define $g'_2(0,\alpha,n)$ *as the weighted sum of spanning subgraphs connecting the leftmost vertex and topmost vertex in the cases of* $g'_2(0,n)$.

(viii) Define $g'_2(0,\beta,n)$ *as the weighted sum of spanning subgraphs connecting the rightmost vertex and topmost vertex in the cases of* $g'_2(0,n)$.

(ix) Define $g'_2(1,n)$ *as the weighted sum of spanning subgraphs with the leftmost is not the end of either walk in the cases of* $g'_2(n)$.

(x) Define $g'_2(1,\alpha,n)$ *as the weighted sum of spanning subgraphs connecting the leftmost vertex and rightmost vertex in the cases of* $g'_2(2,n)$.

(xi) Define $g'_2(1,\beta,n)$ *as the weighted sum of spanning subgraphs connecting the topmost vertex and rightmost vertex in the cases of* $g'_2(2,n)$.

(xii) Define $g'_2(2,n)$ *as the weighted sum of spanning subgraphs with the rightmost is not the end of either walk in the cases of* $g'_2(n)$.

(xiii) Define $g'_2(2,\alpha,n)$ *as the weighted sum of spanning subgraphs connecting the rightmost vertex and topmost vertex in the cases of* $g'_2(1,n)$.

(xiv) Define $g'_2(2,\beta,n)$ *as the weighted sum of spanning subgraphs connecting the rightmost vertex and leftmost vertex in the cases of* $g'_2(1,n)$.

It follows

$$g'_1(n) = g'_1(0,n) + g'_1(1,n) + g'_1(2,n) \tag{3.1.7}$$

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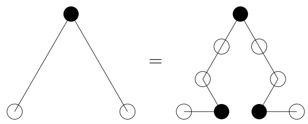

Figure 3.3: Illustration for the expression of $f_1'(0, n+1)$

and

$$
\begin{cases}
g_2'(n) = g_2'(0,n) + g_2'(1,n) + g_2'(2,n) \\
g_2'(0,n) = g_2'(0,\alpha,n) + g_2'(0,\beta,n) \\
g_2'(1,n) = g_2'(1,\alpha,n) + g_2'(1,\beta,n) \\
g_2'(2,n) = g_2'(2,\alpha,n) + g_2'(2,\beta,n),
\end{cases}
\tag{3.1.8}
$$

while we can easily get

$$
\begin{cases}
f_1'(0,1) = yz \\
f_1'(1,1) = xz \\
f_1'(2,1) = xy,
\end{cases}
\tag{3.1.9}
$$

$$
\begin{cases}
g_1'(0,1) = x \\
g_1'(1,1) = y \\
g_1'(2,1) = z,
\end{cases}
\tag{3.1.10}
$$

$$
g_2'(1) = f_2'(1) = f_3'(1) = 0
$$

and

$$
\begin{cases}
g_1'(0,2) = 3x^3y^2z^2 + x^2y^3z^2 + x^2y^2z^3 \\
g_1'(1,2) = x^3y^2z^2 + 3x^2y^3z^2 + x^2y^2z^3 \\
g_1'(2,2) = x^3y^2z^2 + x^2y^3z^2 + 3x^2y^2z^3
\end{cases}
\tag{3.1.11}
$$

as the initial values.

3.2 Building recursions to calculate the weighted sum of Hamiltonian walks

In this section, we will enumerate the relationship between these variables by building some recursions.

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Consider the quantities $f_1'(i,n+1), i=0,1,2$, according to the figure 3.3, we have

$$\begin{cases} f_1'(0,n+1) = yz\prod_{i=0}^{2} f_1'(i,n) \\ f_1'(1,n+1) = xz\prod_{i=0}^{2} f_1'(i,n) \\ f_1'(2,n+1) = xy\prod_{i=0}^{2} f_1'(i,n). \end{cases} \tag{3.2.1}$$

So we can get

$$\prod_{i=0}^{2} f_1'(i,n+1) = x^2y^2z^2\prod_{i=0}^{2}(f_1'(i,n))^3$$

which follows

$$\prod_{i=0}^{2} f_1'(i,n) = (xyz)^{3^n-1}$$

with

$$\begin{cases} f_1'(0,n) = \frac{1}{x}(xyz)^{3^{n-1}} \\ f_1'(1,n) = \frac{1}{y}(xyz)^{3^{n-1}} \\ f_1'(2,n) = \frac{1}{z}(xyz)^{3^{n-1}} \\ f_1'(n) = (\frac{1}{x}+\frac{1}{y}+\frac{1}{z})(xyz)^{3^{n-1}}. \end{cases} \tag{3.2.2}$$

In order to facilitate calculate, let $\tilde{F}(n) := (xyz)^{-3^{n-1}}F(n)$ for all function $F(n)$. So that we have

$$\begin{cases} \tilde{f}_1'(0,n) = \frac{1}{x} \\ \tilde{f}_1'(1,n) = \frac{1}{y} \\ \tilde{f}_1'(2,n) = \frac{1}{z} \\ \tilde{f}_1'(n) = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}. \end{cases} \tag{3.2.3}$$

Consider the quantities $f_2'(i,n+1), i=0,1,2$, which has two symmetrical equivalence,

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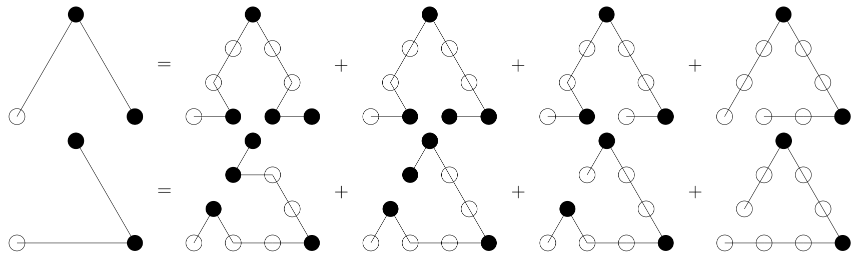

Figure 3.4: Illustration for the expression of $f'_2(1, n+1)$

according to the figure 3.4, we have

$$\begin{aligned} f'_2(1,n+1) =& xyf'_1(0,n)f'_1(2,n)(f'_1(0,n)+f'_2(2,n)) \\ &+yzf'_1(0,n)f'_1(2,n)(f'_1(2,n)+f'_2(0,n)) \\ &+xyzf'_1(0,n)f'_1(2,n)(g'_1(0,n)+g'_1(2,n)) \end{aligned} \tag{3.2.4}$$

as similar as

$$\begin{aligned} f'_2(0,n+1) =& xzf'_1(1,n)f'_1(2,n)(f'_1(2,n)+f'_2(1,n)) \\ &+xyf'_1(1,n)f'_1(2,n)(f'_1(1,n)+f'_2(2,n)) \\ &+xyzf'_1(1,n)f'_1(2,n)(g'_1(1,n)+g'_1(2,n)), \end{aligned} \tag{3.2.5}$$

and

$$\begin{aligned} f'_2(2,n+1) =& yzf'_1(0,n)f'_1(1,n)(f'_1(1,n)+f'_2(0,n)) \\ &+xzf'_1(0,n)f'_1(1,n)(f'_1(0,n)+f'_2(1,n)) \\ &+xyzf'_1(0,n)f'_1(1,n)(g'_1(0,n)+g'_1(1,n)), \end{aligned} \tag{3.2.6}$$

while we can get some recursions of quantities $h'_0(i, n+1), i = 0, 1, 2$ by referencing (3.1.5) and (3.2.3) as

$$\begin{cases} \frac{1}{x}\tilde{h}'_0(0,n+1) = \frac{1}{y}\tilde{h}'_0(1,n) + \frac{1}{z}\tilde{h}'_0(2,n) + \tilde{g}'_1(1,n) + \tilde{g}'_1(2,n) \\ \frac{1}{y}\tilde{h}'_0(1,n+1) = \frac{1}{z}\tilde{h}'_0(2,n) + \frac{1}{x}\tilde{h}'_0(0,n) + \tilde{g}'_1(0,n) + \tilde{g}'_1(2,n) \\ \frac{1}{z}\tilde{h}'_0(2,n+1) = \frac{1}{x}\tilde{h}'_0(0,n) + \frac{1}{y}\tilde{h}'_0(1,n) + \tilde{g}'_1(0,n) + \tilde{g}'_1(1,n). \end{cases} \tag{3.2.7}$$

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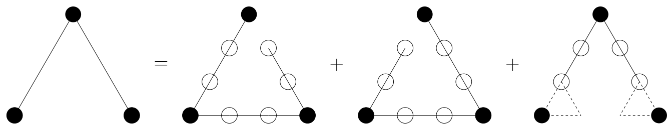

Figure 3.5: Illustration for the expression of $f'_3(0, n+1)$. In the third sence,both of the left and right second-largest triangle have Hamiltonian walks with one end at the topmost vertex, but both the same sides outmost vertices are not the another end.

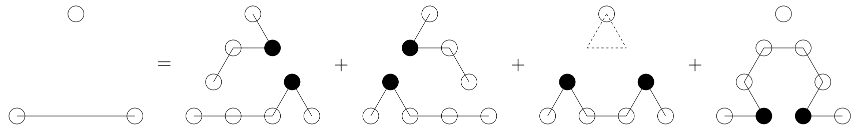

Figure 3.6: Illustration for the expression of $g'_1(0, n+1)$

Consider the quantities $f'_3(i, n+1), i = 0, 1, 2$, according to the figure 3.5, we have

$$\begin{aligned} f'_3(0, n+1) &= xyzf'_1(0,n)f'_1(2,n)g'_2(1,\beta,n) + xyzf'_1(0,n)f'_1(1,n)g'_2(2,\alpha,n) \\ &\quad + yzf'_1(0,n)(h'_0(0,n) - f'_1(1,n))(h'_0(0,n) - f'_1(2,n)) \\ \Rightarrow \tilde{f}'_3(0, n+1) &= \frac{1}{x}(y\tilde{h}'_0(0,n) - 1)(z\tilde{h}'_0(0,n) - 1) + y\tilde{g}'_2(1,\beta,n) + z\tilde{g}'_2(2,\alpha,n). \end{aligned} \tag{3.2.8}$$

So, we have the recursions as the rotation types of (3.2.8) as

$$\begin{cases} \tilde{f}'_3(0, n+1) = \frac{1}{x}(y\tilde{h}'_0(0,n) - 1)(z\tilde{h}'_0(0,n) - 1) + y\tilde{g}'_2(1,\beta,n) + z\tilde{g}'_2(2,\alpha,n) \\ \tilde{f}'_3(1, n+1) = \frac{1}{y}(x\tilde{h}'_0(1,n) - 1)(z\tilde{h}'_0(1,n) - 1) + x\tilde{g}'_2(0,\alpha,n) + z\tilde{g}'_2(2,\beta,n) \\ \tilde{f}'_3(2, n+1) = \frac{1}{z}(x\tilde{h}'_0(2,n) - 1)(y\tilde{h}'_0(2,n) - 1) + x\tilde{g}'_2(0,\beta,n) + y\tilde{g}'_2(1,\alpha,n). \end{cases} \tag{3.2.9}$$

Let

$$\tilde{g}'_3(n) = x\tilde{g}'_2(0,n) + y\tilde{g}'_2(1,n) + z\tilde{g}'_2(2,n),$$

we can get

$$\begin{aligned} \tilde{f}'_3(n+1) &= \frac{yz}{x}(\tilde{h}'_0(0,n))^2 + \frac{xz}{y}(\tilde{h}'_0(1,n))^2 + \frac{xy}{z}(\tilde{h}'_0(2,n))^2 \\ &\quad - \frac{y+z}{x}\tilde{h}'_0(0,n) - \frac{x+z}{y}\tilde{h}'_0(1,n) - \frac{x+y}{z}\tilde{h}'_0(2,n) \\ &\quad + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \tilde{g}'_3(n). \end{aligned} \tag{3.2.10}$$

Now, we have got the recursions of $\tilde{f}'_i(j, n+1), i = 1, 2, 3$ and $j = 0, 1, 2$. To get the

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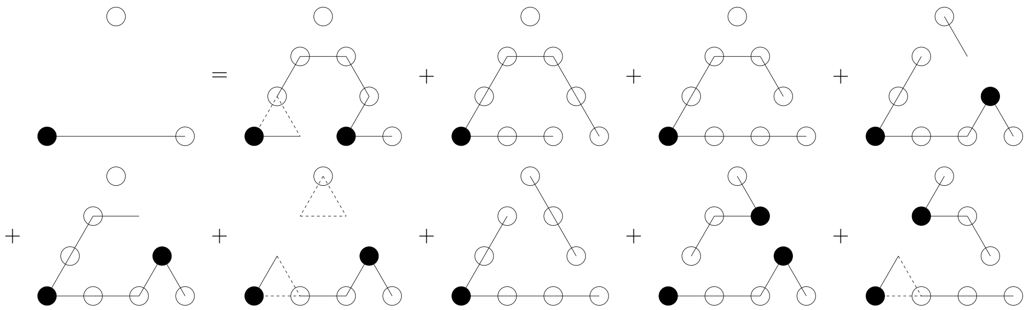

Figure 3.7: Illustration for the expression of $g_2'(1,\alpha,n+1)$

solutions of them, we need some recursions of $\tilde{g}_1'(i,n+1) and \tilde{g}_2'(i,j,n+1), i=0,1,2 and j=\alpha,\beta$.

Consider the quantities $g_1'(i,n+1), i=0,1,2$, according to the figure 3.6, we have

$$\begin{aligned} g_1'(0,n+1) &= xzf_1'(0,n)f_1'(2,n)g_1'(0,n)+xyf_1'(0,n)f_1'(1,n)g_1'(0,n) \\ &\quad + x(f_1'(0,n))^2h_0'(0,n)+yzf_1'(1,n)f_1'(2,n)g_1'(0,n) \\ \Rightarrow \tilde{g}_1'(0,n+1) &= 3\tilde{g}_1'(0,n)+\frac{1}{x}\tilde{h}_0'(0,n). \end{aligned} \tag{3.2.11}$$

E.g.

$$\begin{cases} \tilde{g}_1'(0,n+1) = 3\tilde{g}_1'(0,n)+\frac{1}{x}\tilde{h}_0'(0,n) \\ \tilde{g}_1'(1,n+1) = 3\tilde{g}_1'(1,n)+\frac{1}{y}\tilde{h}_0'(1,n) \\ \tilde{g}_1'(2,n+1) = 3\tilde{g}_1'(2,n)+\frac{1}{z}\tilde{h}_0'(2,n). \end{cases} \tag{3.2.12}$$

Consider the quantity $g_2'(i,j,n+1), i=0,1,2, \quad j=\alpha,\beta$, according to the figure 3.7,

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we have

$$
\begin{aligned}
g_2'(1,\alpha,n+1) =& yzf_1'(1,n)g_1'(0,n)(h_0'(0,n)-f_1'(2,n)) \\
&+xyzf_1'(1,n)g_1'(0,n)g_1'(1,n)+xyzf_1'(1,n)(g_1'(0,n))^2 \\
&+xzf_1'(0,n)f_1'(1,n)(g_2'(2,n)+g_1'(1,n)+g_1'(0,n)) \\
&+xf_1'(0,n)h_0'(0,n)(h_0'(2,n)-f_1'(0,n))+xyzf_1'(1,n)g_1'(0,n)g_1'(1,n) \\
&+xzf_1'(0,n)f_1'(2,n)g_2'(1,\alpha,n)+xyf_1'(1,n)g_1'(0,n)(h_0'(2,n)-f_1'(0,n)) \\
\Rightarrow y\tilde{g}_2'(1,\alpha,n+1) =& y\tilde{g}_2'(1,\alpha,n)+z\tilde{g}_2'(2,n)+z\tilde{g}_1'(1,n)-y\tilde{g}_1'(0,n) \\
&+2xyz\tilde{g}_1'(0,n)\tilde{g}_1'(1,n)+xyz(\tilde{g}_1'(0,n))^2+xy\tilde{g}_1'(0,n)\tilde{h}_0'(2,n) \\
&+yz\tilde{g}_1'(0,n)\tilde{h}_0'(0,n)+y\tilde{h}_0'(0,n)\tilde{h}_0'(2,n)-\frac{y}{x}\tilde{h}_0'(0,n).
\end{aligned}
\tag{3.2.13}
$$

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E.g.

$$
\begin{cases}
x\tilde{g}_2'(0,\alpha,n+1) = & x\tilde{g}_2'(0,\alpha,n) + y\tilde{g}_2'(1,n) + y\tilde{g}_1'(0,n) - x\tilde{g}_1'(2,n) \\
& +2xyz\tilde{g}_1'(0,n)\tilde{g}_1'(2,n) + xyz(\tilde{g}_1'(2,n))^2 + xz\tilde{g}_1'(2,n)\tilde{h}_0'(1,n) \\
& +xy\tilde{g}_1'(2,n)\tilde{h}_0'(2,n) + x\tilde{h}_0'(1,n)\tilde{h}_0'(2,n) - \frac{x}{z}\tilde{h}_0'(2,n) \\
x\tilde{g}_2'(0,\beta,n+1) = & x\tilde{g}_2'(0,\beta,n) + z\tilde{g}_2'(2,n) + z\tilde{g}_1'(0,n) - x\tilde{g}_1'(1,n) \\
& +2xyz\tilde{g}_1'(0,n)\tilde{g}_1'(1,n) + xyz(\tilde{g}_1'(1,n))^2 + xy\tilde{g}_1'(1,n)\tilde{h}_0'(2,n) \\
& +xz\tilde{g}_1'(1,n)\tilde{h}_0'(1,n) + x\tilde{h}_0'(1,n)\tilde{h}_0'(2,n) - \frac{x}{y}\tilde{h}_0'(1,n) \\
y\tilde{g}_2'(1,\alpha,n+1) = & y\tilde{g}_2'(1,\alpha,n) + z\tilde{g}_2'(2,n) + z\tilde{g}_1'(1,n) - y\tilde{g}_1'(0,n) \\
& +2xyz\tilde{g}_1'(0,n)\tilde{g}_1'(1,n) + xyz(\tilde{g}_1'(0,n))^2 + xy\tilde{g}_1'(0,n)\tilde{h}_0'(2,n) \\
& +yz\tilde{g}_1'(0,n)\tilde{h}_0'(0,n) + y\tilde{h}_0'(0,n)\tilde{h}_0'(2,n) - \frac{y}{x}\tilde{h}_0'(0,n) \\
y\tilde{g}_2'(1,\beta,n+1) = & y\tilde{g}_2'(1,\beta,n) + x\tilde{g}_2'(1,n) + x\tilde{g}_1'(1,n) - y\tilde{g}_1'(2,n) \\
& +2xyz\tilde{g}_1'(1,n)\tilde{g}_1'(2,n) + xyz(\tilde{g}_1'(2,n))^2 + yz\tilde{g}_1'(2,n)\tilde{h}_0'(0,n) \\
& +xy\tilde{g}_1'(2,n)\tilde{h}_0'(2,n) + y\tilde{h}_0'(0,n)\tilde{h}_0'(2,n) - \frac{y}{z}\tilde{h}_0'(2,n) \\
z\tilde{g}_2'(2,\alpha,n+1) = & z\tilde{g}_2'(2,\alpha,n) + x\tilde{g}_2'(0,n) + x\tilde{g}_1'(2,n) - z\tilde{g}_1'(1,n) \\
& +2xyz\tilde{g}_1'(1,n)\tilde{g}_1'(2,n) + xyz(\tilde{g}_1'(1,n))^2 + yz\tilde{g}_1'(1,n)\tilde{h}_0'(0,n) \\
& +xz\tilde{g}_1'(1,n)\tilde{h}_0'(1,n) + z\tilde{h}_0'(0,n)\tilde{h}_0'(1,n) - \frac{z}{y}\tilde{h}_0'(1,n) \\
z\tilde{g}_2'(2,\beta,n+1) = & z\tilde{g}_2'(2,\beta,n) + y\tilde{g}_2'(1,n) + y\tilde{g}_1'(2,n) - z\tilde{g}_1'(0,n) \\
& +2xyz\tilde{g}_1'(0,n)\tilde{g}_1'(2,n) + xyz(\tilde{g}_1'(0,n))^2 + xz\tilde{g}_1'(0,n)\tilde{h}_0'(1,n) \\
& +yz\tilde{g}_1'(0,n)\tilde{h}_0'(0,n) + z\tilde{h}_0'(0,n)\tilde{h}_0'(1,n) - \frac{z}{x}\tilde{h}_0'(0,n).
\end{cases}
\tag{3.2.14}
$$

3.3 Get the weighted sum

In this section, we will solve the recursions in the previous section and get the weighted sum of these weighted Hamiltonian walks. Considering functions (3.2.7) and (3.2.12), we have

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$$
\begin{aligned}
&\tilde{g}_1'(0,n+2)-2\tilde{g}_1'(0,n+1)-2\tilde{g}_1'(0,n)\\
=&\tilde{g}_1'(1,n+2)-2\tilde{g}_1'(1,n+1)-2\tilde{g}_1'(1,n)\\
=&\tilde{g}_1'(2,n+2)-2\tilde{g}_1'(2,n+1)-2\tilde{g}_1'(2,n)\\
=&\tilde{g}_1'(n+1)-2\tilde{g}_1'(n).
\end{aligned} \tag{3.3.1}
$$

So that

$$\tilde{g}_1'(n+2)-5\tilde{g}_1'(n+1)+4\tilde{g}_1'(n)=0, \tag{3.3.2}$$

where we have

$$\tilde{g}_1'(1)=\frac{x+y+z}{xyz} \quad \textit{and} \quad \tilde{g}_1'(2)=\frac{5(x+y+z)}{xyz}$$

as the initial values.

So, we can get the solution of recursion (3.3.2), that is

$$\tilde{g}_1'(n)=\frac{1}{3}(4^n-1)\frac{x+y+z}{xyz}. \tag{3.3.3}$$

In this way, we have a recursion of $\tilde{g}_1'(0,n)$ as

$$
\begin{aligned}
&\tilde{g}_1'(0,n+2)-2\tilde{g}_1'(0,n+1)-2\tilde{g}_1'(0,n)\\
=&\tilde{g}_1'(1,n+2)-2\tilde{g}_1'(1,n+1)-2\tilde{g}_1'(1,n)\\
=&\tilde{g}_1'(2,n+2)-2\tilde{g}_1'(2,n+1)-2\tilde{g}_1'(2,n)\\
=&\tilde{g}_1'(n+1)-2\tilde{g}_1'(n)\\
=&\frac{1}{3}(2\cdot 4^n+1)\frac{x+y+z}{xyz}.
\end{aligned} \tag{3.3.4}
$$

Noticed that $1\pm\sqrt{3}$ is the solutions of $x^2-2x-2=0$. By Vandermonde's, assume that

$$\tilde{g}_1'(0,n)=A(1+\sqrt{3})^n+B(1-\sqrt{3})^n+C\cdot 4^n+D,$$

where A,B,C,D is fixed by x,y,z. Noticed that

$$\tilde{g}_1'(0,0)=0, \quad \tilde{g}_1'(0,1)=\frac{x}{xyz}, \quad \tilde{g}_1'(0,2)=\frac{3x+y+z}{xyz}$$

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and

$$\tilde{g}_1'(0,3) = 2\tilde{g}_1'(0,2) + 2\tilde{g}_1'(0,1) + \frac{3(x+y+z)}{xyz} = \frac{11x+5y+5z}{xyz}.$$

We have

$$\begin{cases} A+B+C+D=0 \\ (A+B)+\sqrt{3}(A-B)+4C+D = \dfrac{x}{xyz} \\ 4(A+B)+2\sqrt{3}(A-B)+16C+D = \dfrac{3x+y+z}{xyz} \\ 10(A+B)+6\sqrt{3}(A-B)+64C+D = \dfrac{11x+5y+5z}{xyz}. \end{cases} \tag{3.3.5}$$

The solution of these equations is

$$\begin{cases} A = \dfrac{\sqrt{3}}{18}(\dfrac{2x-y-z}{xyz}) \\ B = -\dfrac{\sqrt{3}}{18}(\dfrac{2x-y-z}{xyz}) \\ C = \dfrac{x+y+z}{9xyz} \\ D = -\dfrac{x+y+z}{9xyz}. \end{cases} \tag{3.3.6}$$

So, we can get

$$\begin{cases} \tilde{g}_1'(0,n) = \dfrac{\sqrt{3}}{18}(\dfrac{2x-y-z}{xyz})(1+\sqrt{3})^n - \dfrac{\sqrt{3}}{18}(\dfrac{2x-y-z}{xyz})(1-\sqrt{3})^n \\ \qquad + (\dfrac{x+y+z}{9xyz})4^n - \dfrac{x+y+z}{9xyz} \\ \tilde{g}_1'(1,n) = \dfrac{\sqrt{3}}{18}(\dfrac{2y-x-z}{xyz})(1+\sqrt{3})^n - \dfrac{\sqrt{3}}{18}(\dfrac{2y-x-z}{xyz})(1-\sqrt{3})^n \\ \qquad + (\dfrac{x+y+z}{9xyz})4^n - \dfrac{x+y+z}{9xyz} \\ \tilde{g}_1'(2,n) = \dfrac{\sqrt{3}}{18}(\dfrac{2z-x-y}{xyz})(1+\sqrt{3})^n - \dfrac{\sqrt{3}}{18}(\dfrac{2z-x-y}{xyz})(1-\sqrt{3})^n \\ \qquad + (\dfrac{x+y+z}{9xyz})4^n - \dfrac{x+y+z}{9xyz}. \end{cases} \tag{3.3.7}$$

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At the same time, we can get the solution of $\tilde{h}_0'(i,n), \quad i=0,1,2$ as

$$\begin{cases}\tilde{h}_0'(0,n) = (\frac{3-2\sqrt{3}}{18})(\frac{2x-y-z}{yz})(1+\sqrt{3})^n + (\frac{3+2\sqrt{3}}{18})(\frac{2x-y-z}{yz})(1-\sqrt{3})^n \\ \qquad + (\frac{x+y+z}{9yz})4^n + \frac{2(x+y+z)}{9yz} \\ \tilde{h}_0'(1,n) = (\frac{3-2\sqrt{3}}{18})(\frac{2y-x-z}{xz})(1+\sqrt{3})^n + (\frac{3+2\sqrt{3}}{18})(\frac{2y-x-z}{xz})(1-\sqrt{3})^n \\ \qquad + (\frac{x+y+z}{9xz})4^n + \frac{2(x+y+z)}{9xz} \\ \tilde{h}_0'(2,n) = (\frac{3-2\sqrt{3}}{54})(\frac{2z-x-y}{xy})(1+\sqrt{3})^n + (\frac{3+2\sqrt{3}}{18})(\frac{2z-x-y}{xy})(1-\sqrt{3})^n \\ \qquad + (\frac{x+y+z}{9xy})4^n + \frac{2(x+y+z)}{9xy} \end{cases} \tag{3.3.8}$$

while $n \geq 2$, where

$$\tilde{h}_0'(n) = \frac{1}{x}\tilde{h}_0'(0,n) + \frac{1}{y}\tilde{h}_0'(1,n) + \frac{1}{z}\tilde{h}_0'(2,n).$$

So, we have

$$\tilde{h}_0'(n) = \frac{x+y+z}{3xyz}(4^n+2). \tag{3.3.9}$$

Therefore, by (3.1.5), we have

$$\begin{cases}\tilde{f}_2'(0,n) = (\frac{3-2\sqrt{3}}{18})(\frac{2x-y-z}{yz})(1+\sqrt{3})^n + (\frac{3+2\sqrt{3}}{18})(\frac{2x-y-z}{yz})(1-\sqrt{3})^n \\ \qquad + (\frac{x+y+z}{9yz})4^n + \frac{2(x+y+z)}{9yz} - \frac{y+z}{yz} \\ \tilde{f}_2'(1,n) = (\frac{3-2\sqrt{3}}{18})(\frac{2y-x-z}{xz})(1+\sqrt{3})^n + (\frac{3+2\sqrt{3}}{18})(\frac{2y-x-z}{xz})(1-\sqrt{3})^n \\ \qquad + (\frac{x+y+z}{9xz})4^n + \frac{2(x+y+z)}{9xz} - \frac{x+z}{xz} \\ \tilde{f}_2'(2,n) = (\frac{3-2\sqrt{3}}{54})(\frac{2z-x-y}{xy})(1+\sqrt{3})^n + (\frac{3+2\sqrt{3}}{18})(\frac{2z-x-y}{xy})(1-\sqrt{3})^n \\ \qquad + (\frac{x+y+z}{9xy})4^n + \frac{2(x+y+z)}{9xy} - \frac{x+y}{xy}. \end{cases} \tag{3.3.10}$$

Defines u, w as

$$u := \frac{x^2+y^2+z^2-xy-xz-xy}{xyz}, \quad w := \frac{(x+y+z)^2}{xyz},$$

we have $\tilde{f}_1'(n) = \frac{1}{3}(w-u)$.

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So, the solution of $\tilde{f}'_2(n)$ is as follows.

Theorem 3.3.1. *The weighted sum of Hamiltonian walks on ${H'}_3^n$ with one end at an outmost vertex, another one not at any outmost vertex is given by 0 for n=1 and*

$$\tilde{f}'_2(n) = (\frac{3-2\sqrt{3}}{9})u(1+\sqrt{3})^n + (\frac{3+2\sqrt{3}}{9})u(1-\sqrt{3})^n + (\frac{w}{9})4^n - \frac{2}{9}(2w-3u) \quad (3.3.11)$$

for $n \geq 2$.

Now, we've got the solutions of $f'_1(n)$ and $f'_2(n)$. In order to get the solution of $f'_3(n)$, we can consider some modifications to recursions of $g'_2(i,j,n), \quad i = 0,1,2, \quad j = \alpha, \beta$.

Add up all the equation of (3.2.14), we have

$$\tilde{g}'_3(n+1) = 3\tilde{g}'_3(n) + \tilde{K}(n), \quad (3.3.12)$$

in which

$$\begin{aligned}
\tilde{K}(n) := &-\left(\frac{y+z}{x}\tilde{h}'_0(0,n) + \frac{x+z}{y}\tilde{h}'_0(1,n) + \frac{x+y}{z}\tilde{h}'_0(2,n)\right) \\
&+ 2\left(x\tilde{h}'_0(1,n)\tilde{h}'_0(2,n) + y\tilde{h}'_1(0,n)\tilde{h}'_0(2,n) + z\tilde{h}'_0(0,n)\tilde{h}'_0(1,n)\right) \\
&+ 2xyz\left(\frac{1}{x}\tilde{h}'_0(0,n)\tilde{g}'_1(0,n) + \frac{1}{y}\tilde{h}'_0(1,n)\tilde{g}'_1(1,n) + \frac{1}{z}\tilde{h}'_0(2,n)\tilde{g}'_1(2,n)\right) \\
&+ xyz\left(\frac{1}{x}\tilde{h}'_0(0,n)\tilde{g}'_1(1,n) + \frac{1}{y}\tilde{h}'_0(1,n)\tilde{g}'_1(2,n) + \frac{1}{z}\tilde{h}'_0(2,n)\tilde{g}'_1(0,n)\right. \\
&\left.+\frac{1}{y}\tilde{h}'_0(1,n)\tilde{g}'_1(0,n) + \frac{1}{z}\tilde{h}'_0(2,n)\tilde{g}'_1(1,n) + \frac{1}{x}\tilde{h}'_0(0,n)\tilde{g}'_1(2,n)\right) \\
&+ 2xyz\left((\tilde{g}'_1(0,n))^2 + (\tilde{g}'_1(1,n))^2 + (\tilde{g}'_1(0,n))^2\right) \\
&+ 4xyz\left(\tilde{g}'_1(0,n)\tilde{g}'_1(1,n) + \tilde{g}'_1(0,n)\tilde{g}'_1(2,n) + \tilde{g}'_1(1,n)\tilde{g}'_1(2,n)\right).
\end{aligned} \quad (3.3.13)$$

According to (3.3.7) and (3.3.8), we have

$$\begin{aligned}
\tilde{K}(n) = &-(\frac{9-5\sqrt{3}}{18})u(4+2\sqrt{3})^n - (\frac{9+5\sqrt{3}}{18})u(4-2\sqrt{3})^n - \frac{u}{9}(-2)^n \\
&+ (\frac{3-2\sqrt{3}}{9})u(1+\sqrt{3})^n + (\frac{3+2\sqrt{3}}{9})u(1-\sqrt{3})^n \\
&+ (\frac{4w}{9})16^n - (\frac{2w}{9})4^n - \frac{2w}{9},
\end{aligned} \quad (3.3.14)$$

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Noticed that $\tilde{g}_3'(1) = 0$, we have

$$\begin{aligned}
\tilde{g}_3'(n) &= 3^n \sum_{k=1}^{n-1} \frac{1}{3^k} \tilde{K}(k) \\
&= (\frac{39 - 23\sqrt{3}}{2 \times 9 \times 11}) u (4 + 2\sqrt{3})^n + (\frac{39 + 23\sqrt{3}}{2 \times 9 \times 11}) u (4 - 2\sqrt{3})^n \\
&+ \frac{u}{45}(-2)^n + (\frac{\sqrt{3}u}{9})(1 + \sqrt{3})^n - (\frac{\sqrt{3}u}{9})(1 - \sqrt{3})^n \\
&+ (\frac{4w}{9 \times 13}) 16^n - (\frac{2w}{9}) 4^n + cu3^n + (\frac{w}{13}) 3^n + \frac{w}{9}
\end{aligned} \tag{3.3.15}$$

as the constant $c = -\dfrac{206}{495}$.

According to (3.2.10), we have

$$\begin{aligned}
\tilde{f}_3'(n+1) &= \frac{yz}{x}(\tilde{h}_0'(0,n))^2 + \frac{xz}{y}(\tilde{h}_0'(1,n))^2 + \frac{xy}{z}(\tilde{h}_0'(2,n))^2 \\
&- \frac{y+z}{x}\tilde{h}_0'(0,n) - \frac{x+z}{y}\tilde{h}_0'(1,n) - \frac{x+z}{x}\tilde{h}_0'(2,n) \\
&+ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \tilde{g}_3'(n) \\
&= (\frac{116 - 67\sqrt{3}}{2 \times 9 \times 11}) u (4 + 2\sqrt{3})^n + (\frac{116 + 67\sqrt{3}}{2 \times 9 \times 11}) u (4 - 2\sqrt{3})^n \\
&- \frac{4u}{45}(-2)^n + (\frac{3 - \sqrt{3}}{9}) u (1 + \sqrt{3})^n + (\frac{3 + \sqrt{3}}{9}) u (1 - \sqrt{3})^n \\
&+ (\frac{25w}{27 \times 13}) 16^n - (\frac{8w}{27}) 4^n + cu3^n + (\frac{w}{13}) 3^n + \frac{4w}{27} - \frac{u}{3},
\end{aligned} \tag{3.3.16}$$

in which we can get the solution of $f_3'(n)$ as the following theorem.

Theorem 3.3.2. *the weighted sum of Hamiltonian walks on H'^n_3 with no end at any of the outmost vertex is given by 0 for n=1 and*

$$\begin{aligned}
f_3'(n) &= (xyz)^{3^{n-1}} [(\frac{433 - 250\sqrt{3}}{4 \times 9 \times 11}) u (4 + 2\sqrt{3})^n + (\frac{433 + 250\sqrt{3}}{4 \times 9 \times 11}) u (4 - 2\sqrt{3})^n \\
&+ \frac{2u}{45}(-2)^n - (\frac{3 - 2\sqrt{3}}{9}) u (1 + \sqrt{3})^n - (\frac{3 + 2\sqrt{3}}{9}) u (1 - \sqrt{3})^n \\
&+ (\frac{25w}{16 \times 27 \times 13}) 16^n - (\frac{2w}{27}) 4^n + (\frac{cu}{3}) 3^n + (\frac{w}{39}) 3^n + \frac{4w}{27} - \frac{u}{3}]
\end{aligned} \tag{3.3.17}$$

as the constant $c = -\dfrac{206}{495}$ while $u = \dfrac{x^2 + y^2 + z^2 - xy - xz - xy}{xyz}$ and $w = \dfrac{(x+y+z)^2}{xyz}$ for $n \geq 2$.

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According to (3.1.1), (3.2.2), (3.3.11) and (3.3.17), we have the following solution to show the weighted sum of Hamiltonian walks on H'^n_3 we need.

Theorem 3.3.3. *The weighted sum of weighted Hamiltonian walks on H'^n_3 is given by $xy+xz+yz$ for $n=1$ and*

$$\begin{aligned} h'(n) &= f'_1(n) + f'_2(n) + f'_3(n) \\ &= (xyz)^{3^{n-1}}[(\frac{433-250\sqrt{3}}{4\times 9\times 11})u(4+2\sqrt{3})^n + (\frac{433+250\sqrt{3}}{4\times 9\times 11})u(4-2\sqrt{3})^n \\ &+ \frac{2u}{45}(-2)^n + (\frac{25w}{16\times 27\times 13})16^n + (\frac{w}{27})4^n + (\frac{cu}{3})3^n + (\frac{w}{39})3^n + \frac{w}{27}] \end{aligned} \quad (3.3.18)$$

as the constant $c = -\frac{206}{495}$ while $u = \frac{x^2+y^2+z^2-xy-xz-xy}{xyz}$ and $w = \frac{(x+y+z)^2}{xyz}$ for $n \geq 2$.

Especially, if $x=y=z=1$, e.g. $u=0$ and $w=9$. We can get

$$h'(n) = h(n).$$

Consider the asymptotic behaviour of $h'(n)$, we have

Theorem 3.3.4. *When n is large, the asymptotic behaviour of the weighted sum of Hamiltonian walks on H'^n_3 is given by*

$$h'(n) \sim \frac{25w}{16\times 27\times 13}16^n(xyz)^{3^{n-1}}. \quad (3.3.19)$$

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