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Indagationes Mathematicae xxx (xxxx) xxx

indagationes mathematicae

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Review

Large deviation principle of multidimensional multiple averages on $\mathbb{N}^d$

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Received 20 June 2021; received in revised form 6 September 2021; accepted 25 September 2021

Communicated by M. Veraar

Abstract

This paper establishes the large deviation principle (LDP) for multiple averages on $\mathbb{N}^d$. We extend the previous work of Carinci et al. (2012) to multidimensional lattice $\mathbb{N}^d$ for $d \geq 2$. The same technique is also applicable to the weighted multiple average launched by Fan (2021). Finally, the boundary conditions are imposed to the multiple sum and explicit formulae of the energy functions with respect to the boundary conditions are obtained.

Keywords: Large deviation principle; Multiple sum; Weighted multiple sum; Free energy function; Boundary condition

Contents

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1 Ban is partially supported by the Ministry of Science and Technology, ROC (Contract MOST 109-2115-M-004-002-MY2 and 108-2115-M-004-003).
2 Hu is partially supported by the National Natural Science Foundation of China (Grant 11601355).

https://doi.org/10.1016/j.indag.2021.09.010

1. Introduction

In this article, we study the large deviation rate function of the (weighted) multiple average in the multidimensional lattice $\mathbb{N}^d$. Before presenting our main results, we would like to explain below the motivation behind this study. Let (X, T) be a topological dynamical system where T is a continuous map on a compact metric space X. Defined by $\mathbb{F} = (f_1, \ldots, f_d)$ a d-tuple of functions, where $f_i : X \to \mathbb{R}$ for $1 \leq i \leq d$. The *multiple ergodic theory* is the study of the asymptotic behavior of the *multiple sum*

$$A_n\mathbb{F}(x) = \sum_{k=0}^{n-1} f_1(T_1^k(x)) f_2(T_2^k(x)) \cdots f_d(T_d^k(x)). \tag{1}$$

Such problem was initiated by Furstenberg [14] on his proof of the Szemerédi's theorem. Host and Kra [16] proved the L^2-convergence of (1) when $T_j = T^j$ ($T^j(x)$ means the jth iteration of x under T.) and $f_j \in L^\infty(\mu)$, Bourgain [4] proved the almost everywhere convergence when $d = 2$. Later, the multifractal analysis and the dimension theory of the multiple ergodic averages $\frac{A_n\mathbb{F}(x)}{N}$ in $\mathbb{N}$ (or $\mathbb{Z}$) are also interesting research subjects and have been studied in depth recently (cf. [1,2,5,12,13,17–20]). We also refer the reader to [10] for a survey and for a complete bibliography on this subject. Those works concentrate on what are known as *multiplicative subshifts*. Precisely, let $\Sigma_m = \{0, \ldots, m-1\}$ and $\Omega \subseteq \Sigma_m^{\mathbb{N}}$ be a subset. We suppose that S is a semigroup generated by primes $p_1, \ldots, p_{k-1}$, and set

$$X_\Omega^{(S)} = \{(x_k)_{k=1}^\infty \in \Sigma_m^{\mathbb{N}} : x|_{iS} \in \Omega, \forall i \in \mathbb{N},\ \gcd(i, S) = 1\}, \tag{2}$$

where $\gcd(i, S) = 1$ means that $\gcd(i, s) = 1, \forall s \in S$. It is worth noting that the investigation of $X_\Omega^{(S)}$ was started from the study of the set $X^{p_1, p_2, \ldots, p_{k-1}}$ defined below. Namely, if $p_1, \ldots, p_{k-1}$ are primes, define

$$X^{p_1, p_2, \ldots, p_{k-1}} = \{(x_i)_{i=1}^\infty \in \Sigma_m^{\mathbb{N}} : x_i x_{ip_1} \cdots x_{ip_{k-1}} = 0, \forall i \in \mathbb{N}\}. \tag{3}$$

It is clear that $X^{p_1, p_2, \ldots, p_{k-1}}$ is a special case of $X_\Omega^{(S)}$ with Ω a closed subset of $\Sigma_m^{\mathbb{N}}$. Results for Hausdorff and Minkowski dimension of (2) or (3) are obtained in [2,12,17,18] . The authors call $X_\Omega^{(S)}$ 'multiplicative subshifts' in [17] since it is invariant under *multiplicative integer action*. That is,

$$x = (x_k)_{k\geq 1} \in X_\Omega^{(S)} \ \Rightarrow\ \forall i \in \mathbb{N}, (x_{ik})_{k\geq 1} \in X_\Omega^{(S)}.$$

For $\mathbf{p}_1, \ldots, \mathbf{p}_{k-1} \in \mathbb{N}^d$, we define

$$X^{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{k-1}} = \{(x_\mathbf{i})_{\mathbf{i}\in\mathbb{N}^d} \in \Sigma_m^{\mathbb{N}^d} : x_\mathbf{i} x_{\mathbf{i}\cdot\mathbf{p}_1} \cdots x_{\mathbf{i}\cdot\mathbf{p}_{k-1}} = 0, \forall \mathbf{i} \in \mathbb{N}^d\}, \tag{4}$$

where $\mathbf{i} \cdot \mathbf{j}$ denotes the coordinate-wise product of $\mathbf{i}$ and $\mathbf{j}$, i.e., $\mathbf{i} \cdot \mathbf{j} = (i_1 j_1, \ldots, i_d j_d)$ for $\mathbf{i} = (i_l)_{l=1}^d$, $\mathbf{j} = (j_l)_{l=1}^d \in \mathbb{N}^d$. It is obvious that $X^{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{k-1}}$ is a $\mathbb{N}^d$ version of $X^{p_1, p_2, \ldots, p_{k-1}}$. Recently, Ban, Hu and Lai [1] established the Minkowski dimension of (4). Related works on the dimension theory of the multidimensional multiple sum can also be found in [5]. Let

$$m_1 \geq m_2 \geq \cdots \geq m_d \geq 2,\ \Sigma_{m_1, \ldots, m_d} = \left(\Sigma_{m_1} \times \cdots \times \Sigma_{m_d}\right)^{\mathbb{N}}$$

and define

$$X_{\Omega}^{m_1,\ldots,m_d} = \{(x_i^{(1)}, \ldots, x_i^{(d)})_{i=1}^{\infty} \in \Sigma_{m_1,\ldots,m_d} : (x_{iq^l}^{(1)}, \ldots, x_{iq^l}^{(d)}) \in \Omega, \forall q \nmid i\}. \tag{5}$$

The set (5) is called *self-affine sponges under the action of multiplicative integers*. Brunet studies the dimensions of (5) and establishes the associated Ledrappier–Young formula.

It is stressed that the problems of multifractal analysis and dimension formula of multiple average on 'multidimensional lattices' are new and difficult. The difficulty is that it is not easy to decompose the multidimensional lattices into independent sublattices according to the given 'multiple constraints', e.g., the $\mathbf{p}_i'$s in (4), and calculate its density among the entire lattice. Fortunately, the technique developed in [1] is useful and leads us to investigate the LDP for the multidimensional multiple averages launched by Carinci et al. in 2012 [6], and multidimensional weighted multiple sum mentioned in [11]. Both topics are described in the following two paragraphs.

LDP for multiple averages on $\mathbb{Z}$. Let $\mathcal{A} = \{+1, -1\}$ and denote by $\mathbb{P}_r$ the product of Bernoulli with the parameter r on $\mathcal{A}$. For $\sigma \in \mathcal{A}^{\mathbb{Z}}$, the authors [6] study the thermodynamic limit of the *free energy function* associated to the sum

$$S_N(\sigma) = \sum_{i=1}^{N} \sigma_i \sigma_{2i}, \tag{6}$$

defined as

$$F_r(\beta) = \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}_r(e^{\beta S_N}).$$

We note that if we think (6) as a Hamiltonian and the parameter β as the inverse temperature in the lattice spin systems on $\mathcal{A}^{\mathbb{Z}}$, this is the simplest version of the *multiplicative Ising model* defined in [7]. Note that the Hamiltonian (6) is long-range, non-translation invariant interaction and much more difficult to treat. In [6], the authors prove that the sequence of multiple average $\frac{S_N}{N}$ satisfies a LDP with the rate function

$$I_r(x) = \sup_{\beta\in\mathbb{R}}(\beta x - F_r(\beta)), \tag{7}$$

where

$$F_r(\beta) = \log([r(1-r)]^{\frac{3}{4}} |v^T \cdot e_+| \Lambda_+) + \mathcal{G}(\beta).$$

The reader is referred to [6] for the explicit definitions of v, e_+, Λ_+ and $\mathcal{G}(\beta)$. Roughly speaking, the LDP characterizes the limit behavior, as $\epsilon \to 0$, of a family of probability measures $\{\mu_\epsilon\}$ on a probability space $(X, \mathcal{B})$ in terms of a rate function. In [6], the *rate function* associated with the multiple average $\frac{S_N}{N}$ is defined by

$$I_r(x) = \lim_{\epsilon\to 0} \lim_{N\to\infty} -\frac{1}{N} \log \mathbb{P}_r\left(\frac{S_N}{N} \in [x-\epsilon, x+\epsilon]\right). \tag{8}$$

The authors prove that (8) exists and satisfies the *Fenchel–Legendre transform* (7) of the free energy function $F_r(\beta)$. If $F_r(\beta)$ is differentiable, then the rate function can be clearly demonstrated (Lemma 2.2.31 [8]) to be given by:

$$I_r(y) = \eta y - F_r(\eta)$$

where η is the value such that $F'(\eta) = y$.

Thus, the characterization the differentiability of the free energy function $F_r(\beta)$ is also a major subject of the LDP, and it is highly related to the phase transition phenomena of the multiplicative Ising model (cf. [15]). We refer the reader to [8,9] for the formal definitions of LDP and Fenchel–Legendre transform. The multiplicative Ising model with boundary conditions is also considered in [7]. In the first part of this work we investigate how to extend the work of [6,7] to $\mathbb{N}^d$ without and with boundary conditions. This will be done in Section 3 and Section 5 respectively. We also extend some results of the weighted multiple average [11] to the $\mathbb{N}^d$ version, and describe them below.

Multifractal analysis for weighted sums on $\mathbb{Z}$. Let (X, T) be a topological dynamical system. Fan [11] studies the multifractal analysis of the *weighted (Birkhoff) sum*

$$S_N^{(w)} f(x) = \sum_{n=1}^{N} w_n f(T^n x). \tag{9}$$

as follows. Suppose $(w_n)_{n=1}^{\infty}$ takes a finite number of values and $f_n(x) = x_n g_n(x_{n-1}, \ldots)$, where g_n depends on finite number of coordinates (see condition (C1) in Theorem 4.1) and $(w_n)_{n=1}^{\infty}$ satisfies the frequency condition (see (C2) in Theorem 4.1). The *spectrum of the Hausdorff dimension* of the level set $E(\alpha)$(defined in (44)) is obtained in Theorem 4.1. Let μ be the Möbius function, the author also considers the level set $F(\alpha)$ (defined in (46)). The dimension spectrum for the level set $F(\alpha)$ is also obtained in Theorem 4.2. In the second part of this study we establish the LDP based on the weighted multiple sum

$$S_N^{(w)} = \sum_{i=1}^{N} w_n \sigma_i \sigma_{2i} \tag{10}$$

in $\mathbb{N}^d$. Our main results are presented below.

Suppose $\mathbf{N} = (N_1, N_2, \ldots, N_d) \in \mathbb{N}^d$ and $\sigma \in \mathcal{A}^{\mathbb{N}^d}$, the (*multidimensional*) *multiple sum* is defined as

$$S^{\mathbf{p}}_{N_1 \times N_2 \times \cdots \times N_d}(\sigma) = \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \cdots \sum_{i_d=1}^{N_d} \sigma_{\mathbf{i}} \sigma_{\mathbf{p}\cdot\mathbf{i}}. \tag{11}$$

Following [6], let $\mathbb{P}_r$ be a product of Bernoulli with the parameter r over two symbols on $\mathcal{A}$. The *free energy function* associated with the sum $S^{\mathbf{p}}_{N_1 \times N_2 \times \cdots \times N_d}$ is set as[3]

$$F_r(\beta) = \lim_{\mathbf{N}\to\infty} \frac{1}{N_1 N_2 \cdots N_d} \log \mathbb{E}_r(\exp(\beta S^{\mathbf{p}}_{N_1 \times N_2 \times \cdots \times N_d})). \tag{12}$$

The associated *large deviation rate function* of the multiple average

$$\frac{S^{\mathbf{p}}_{N_1 \times N_2 \times \cdots \times N_d}}{N_1 N_2 \cdots N_d} \tag{13}$$

is defined as

$$I_r(x) = \lim_{\varepsilon\to 0} \lim_{\mathbf{N}\to\infty} -\frac{1}{N_1 N_2 \cdots N_d} \log \mathbb{P}_r\left(\frac{S^{\mathbf{p}}_{N_1 \times N_2 \times \cdots \times N_d}}{N_1 N_2 \cdots N_d} \in [x-\varepsilon, x+\varepsilon]\right). \tag{14}$$

In Theorem 3.2 an explicit formula for $F_r(\beta)$ is derived and $\beta \mapsto F_r(\beta)$ is proven to be differentiable. Furthermore, the multiple average (13) is shown to satisfy a LDP. Due to the fact

[3] To shorten notation, we write $\mathbf{N} \to \infty$ instead of $N_1, N_2, \ldots, N_d \to \infty$.

that $\beta \to F_r(\beta)$ is differentiable, an explicit expression is obtained also for $I_r(x)$. Surprisingly, the formula for $F_r(\beta)$ indicates that $I_{1/2}(x)$ is independent of the dimension $d \in \mathbb{N}$ and $\mathbf{p} \in \mathbb{N}^d$. On the other hand, let $\mathbf{w} = (w_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d}$, the *weighted multiple sum* is defined as

$$S^{\mathbf{p},\mathbf{w}}_{N_1\times N_2\times\cdots\times N_d} = \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}\cdots\sum_{i_d=1}^{N_d} w_{\mathbf{i}}\sigma_{\mathbf{i}}\sigma_{\mathbf{p}\cdot\mathbf{i}}. \tag{15}$$

We denote by $F^{\mathbf{w}}_r(\beta)$ (resp. $I^{\mathbf{w}}_r(x)$) the corresponding free energy function (resp. large deviation rate function) of the weighted multiple average

$$\frac{S^{\mathbf{p},\mathbf{w}}_{N_1\times N_2\times\cdots\times N_d}}{N_1 N_2\cdots N_d} \tag{16}$$

as in (12) (resp. (14)). The formula of $F^{\mathbf{w}}_{1/2}(\beta)$ is rigorously calculated in Theorem 4.3, the LDP for the average (16) are also established therein. It is worth to emphasize that the formula $I^{\mathbf{w}}_{1/2}$ in Theorem 4.3 is almost identical to the dimension formula (44) established in Theorem 4.1 [11], and that $I^{\mathbf{w}}_{1/2}$ does not depend on the dimension $d \in \mathbb{N}$ and the multiple constraint $\mathbf{p} \in \mathbb{N}^d$. In addition, similar results are also obtained if $(w_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d}$ is the Möbius function (Corollary 4.5). Finally, the boundary conditions on the multiple sum (11) are imposed and the corresponding energy functions are defined. The explicit formulae of these energy functions are determined in Section 5.

2. Preliminaries

In this section, we provide necessary materials and results on the decomposition of the multidimensional lattice $\mathbb{N}^d$ into independent sublattices and calculate their densities.

Given $p_1, p_2, \ldots, p_d \geq 1$ (with $p_1, p_2, \ldots, p_d$ not all equal to 1) and $N_1, N_2, \ldots, N_d \geq 1$, we let $\mathcal{M}_{\mathbf{p}} = \{(p_1^m, p_2^m, \ldots, p_d^m) : m \geq 0\}$ be the subset of $\mathbb{N}^d$, and denote by $\mathcal{M}_{\mathbf{p}}(\mathbf{i})$ a version of the lattice $\mathcal{M}_{\mathbf{p}}$ starting from $\mathbf{i} \in \mathbb{N}^d$, i.e. $\mathcal{M}_{\mathbf{p}}(\mathbf{i}) = \{(i_1 p_1^m, i_2 p_2^m, \ldots, i_d p_d^m) : m \geq 0\}$. Finally we define $\mathcal{I}_{\mathbf{p}} = \{\mathbf{i} \in \mathbb{N}^d : p_j \nmid i_j \text{ for some } 1 \leq j \leq d\}$ as the index set of $\mathbb{N}^d$.

More definitions are needed to characterize the partition of the $N_1 \times N_2 \times \cdots \times N_d$ lattice. Let $\mathcal{N}_{N_1\times N_2\times\cdots\times N_d} = \{\mathbf{i} \in \mathbb{N}^d : 1 \leq i_j \leq N_j \text{ for all } 1 \leq j \leq d\}$ be the $N_1 \times N_2 \times \cdots \times N_d$ lattice and $\mathcal{L}_{N_1\times N_2\times\cdots\times N_d}(\mathbf{i}) = \mathcal{M}_{\mathbf{p}}(\mathbf{i}) \cap \mathcal{N}_{N_1\times N_2\times\cdots\times N_d}$ be the subset of $\mathcal{M}_{\mathbf{p}}(\mathbf{i})$ in the $N_1 \times N_2 \times \cdots \times N_d$ lattice. Then we define $\mathcal{J}_{N_1\times N_2\times\cdots\times N_d;\ell} = \{\mathbf{i} \in \mathcal{N}_{N_1\times N_2\times\cdots\times N_d} : |\mathcal{L}_{N_1\times N_2\times\cdots\times N_d}(\mathbf{i})| = \ell\}$, where $|\cdot|$ denotes cardinality, as the set of points $\mathbf{i}$ in the $N_1 \times N_2 \times \cdots \times N_d$ lattice such that the cardinality of the set $\mathcal{M}_{\mathbf{p}}(\mathbf{i}) \cap \mathcal{N}_{N_1\times N_2\times\cdots\times N_d}$ is exactly ℓ. Let $\mathcal{K}_{N_1\times N_2\times\cdots\times N_d;\ell} = \{\mathbf{i} \in \mathcal{I}_{\mathbf{p}} \cap \mathcal{N}_{N_1\times N_2\times\cdots\times N_d} : |\mathcal{L}_{N_1\times N_2\times\cdots\times N_d}(\mathbf{i})| = \ell\}$ be the set of points $\mathbf{i}$ in $\mathcal{I}_{\mathbf{p}}$ that the cardinality of the set $\mathcal{M}_{\mathbf{p}}(\mathbf{i}) \cap \mathcal{N}_{N_1\times N_2\times\cdots\times N_d}$ is exactly ℓ. The following lemmas give the disjoint decomposition of $\mathbb{N}^d$ and the limit of the density of $\mathcal{K}_{N_1\times N_2\times\cdots\times N_d;\ell}$ which is the $\mathbb{N}^d$ version of Lemma 2.1 and 2.2 [1], respectively.

Lemma 2.1. *For $p_1, p_2, \ldots, p_d \geq 1$,*

$$\mathbb{N}^d = \bigsqcup_{\mathbf{i}\in\mathcal{I}_{\mathbf{p}}} \mathcal{M}_{\mathbf{p}}(\mathbf{i}).$$

Proof. We first claim that for all $\mathbf{i} \neq \mathbf{i}' \in \mathcal{I}_{\mathbf{p}}$, $\mathcal{M}_{\mathbf{p}}(\mathbf{i}) \cap \mathcal{M}_{\mathbf{p}}(\mathbf{i}') = \emptyset$. Indeed, suppose that this does not hold, then there exist $\mathbf{i} \neq \mathbf{i}' \in \mathcal{I}_{\mathbf{p}}$ such that $\mathcal{M}_{\mathbf{p}}(\mathbf{i}) \cap \mathcal{M}_{\mathbf{p}}(\mathbf{i}') \neq \emptyset$. Since $\mathbf{i} \neq \mathbf{i}'$, then there exist $m_1 \neq m_2 \geq 0$ such that $(i_1 p_1^{m_1}, \ldots, i_d p_d^{m_1}) = (i'_1 p_1^{m_2}, \ldots, i'_d p_d^{m_2})$. Without loss of generality, we assume $m_1 > m_2$, then $i_k p_k^{m_1-m_2} = i'_k$ for all $1 \le k \le d$, which gives $p_k | i'_k$ for all $1 \le k \le d$. This contradicts $\mathbf{i}' \in \mathcal{I}_{\mathbf{p}}$. It remains to show that the equality holds. For $\mathbf{i} \in \mathbb{N}^d$, then $i_k = i'_k p_k^{\alpha_k}$ with $p_k \nmid i'_k$ and $\alpha_k \ge 0$ for all $1 \le k \le d$. Take $\gamma = \min_k\{\alpha_k\}$, then $(\frac{i_1}{p_1^\gamma}, \ldots, \frac{i_d}{p_d^\gamma}) \in \mathcal{I}_{\mathbf{p}}$, which implies $\mathbf{i} \in \mathcal{M}_{\mathbf{p}}(\frac{i_1}{p_1^\gamma}, \ldots, \frac{i_d}{p_d^\gamma})$. Since the converse is clear, the proof is thus completed. □

Lemma 2.2. *For $N_1, N_2, \ldots, N_d$, and $\ell \ge 1$, we have the following assertions.*

1. $|\mathcal{J}_{N_1\times N_2\times\cdots\times N_d;\ell}| = \prod_{k=1}^d \left\lfloor \frac{N_k}{p_k^{\ell-1}} \right\rfloor - \prod_{k=1}^d \left\lfloor \frac{N_k}{p_k^{\ell}} \right\rfloor$.
2. $\lim_{\mathbf{N}\to\infty} \frac{|\mathcal{K}_{N_1\times N_2\times\cdots\times N_d;\ell}|}{|\mathcal{J}_{N_1\times N_2\times\cdots\times N_d;\ell}|} = 1 - \frac{1}{p_1 p_2 \cdots p_d}$.
3. $\lim_{\mathbf{N}\to\infty} \frac{1}{N_1\cdots N_d} \sum_{\ell=1}^{N_1\cdots N_d} |\mathcal{K}_{N_1\times\cdots\times N_d;\ell}| \log F_\ell = \sum_{\ell=1}^{\infty} \lim_{\mathbf{N}\to\infty} \frac{|\mathcal{K}_{N_1\times\cdots\times N_d;\ell}|}{N_1\cdots N_d} \log F_\ell$.

Proof.

1. Since $|\mathcal{L}_{N_1\times\cdots\times N_d}(\mathbf{i})| = \ell$, we have

$$\mathcal{J}_{N_1\times\cdots\times N_d;\ell} = \{\mathbf{i} : i_k p_k^{\ell-1} \le N_k \text{ for all } 1 \le k \le d\} \cap \left(\cup_{k=1}^d \{\mathbf{i} : i_k p_k^{\ell} > N_k\}\right).$$

Thus, the inclusion–exclusion principle infers that

$$\begin{aligned}
|\mathcal{J}_{N_1\times\cdots\times N_d;\ell}| &= \left|\bigcup_{n=1}^d \left(A \cap \{\mathbf{i} : i_n p_n^\ell > N_n\}\right)\right| \\
&= \sum_{n=1}^d \left|A \cap \{\mathbf{i} : i_n p_n^\ell > N_n\}\right| \\
&- \sum_{1\le n_1<n_2\le d} \left|A \cap \{\mathbf{i} : i_{n_1} p_{n_1}^\ell > N_{n_1} \text{ and } i_{n_2} p_{n_2}^\ell > N_{n_2}\}\right| \\
&+ \sum_{1\le n_1<n_2<n_3\le d} \left|A \cap \{\mathbf{i} : i_{n_1} p_{n_1}^\ell > N_{n_1}, i_{n_2} p_{n_2}^\ell > N_{n_2} \text{ and } i_{n_3} p_{n_3}^\ell > N_{n_3}\}\right| \\
&- \cdots + (-1)^{d-1} \left|A \cap \{\mathbf{i} : i_1 p_1^\ell > N_1, i_2 p_2^\ell > N_2, \ldots, i_d p_d^\ell > N_d\}\right|,
\end{aligned}$$

where $A = \{\mathbf{i} : i_k p_k^{\ell-1} \le N_k \text{ for all } 1 \le k \le d\}$.
It follows that

$$\begin{aligned}
|\mathcal{J}_{N_1\times\cdots\times N_d;\ell}| &= \sum_{n=1}^d \left[\left(\left\lfloor \frac{N_n}{p_n^{\ell-1}} \right\rfloor - \left\lfloor \frac{N_n}{p_n^{\ell}} \right\rfloor\right) \prod_{k\ne n} \left\lfloor \frac{N_k}{p_k^{\ell-1}} \right\rfloor\right] \\
&- \sum_{1\le n_1<n_2\le d} \left[\prod_{k_1=n_1,n_2} \left(\left\lfloor \frac{N_{k_1}}{p_{k_1}^{\ell-1}} \right\rfloor - \left\lfloor \frac{N_{k_1}}{p_{k_1}^{\ell}} \right\rfloor\right) \prod_{k_2\ne n_1,n_2} \left\lfloor \frac{N_{k_2}}{p_{k_2}^{\ell-1}} \right\rfloor\right]
\end{aligned}$$

$$+\sum_{1\le n_1<n_2<n_3\le d}\left[\prod_{k_1=n_1,n_2,n_3}\left(\left\lfloor\frac{N_{k_1}}{p_{k_1}^{\ell-1}}\right\rfloor-\left\lfloor\frac{N_{k_1}}{p_{k_1}^{\ell}}\right\rfloor\right)\prod_{k_2\neq n_1,n_2,n_3}\left\lfloor\frac{N_{k_2}}{p_{k_2}^{\ell-1}}\right\rfloor\right]$$
$$-\cdots+(-1)^{d-1}\prod_{k=1}^{d}\left(\left\lfloor\frac{N_k}{p_k^{\ell-1}}\right\rfloor-\left\lfloor\frac{N_k}{p_k^{\ell}}\right\rfloor\right).$$

Thus, we have

$$|\mathcal{J}_{N_1\times\cdots\times N_d;\ell}|=\prod_{k=1}^{d}\left\lfloor\frac{N_k}{p_k^{\ell-1}}\right\rfloor-\prod_{k=1}^{d}\left\lfloor\frac{N_k}{p_k^{\ell}}\right\rfloor.$$

2. For $m_2^{(i)}>m_1^{(i)}\ge 1$, $1\le i\le d$ and we define the rectangular lattice

$$\mathcal{R}_{m_1^{(1)},m_2^{(1)};\ldots;m_1^{(d)},m_2^{(d)}}=\left\{\mathbf{i}\in\mathbb{N}^d: m_1^{(k)}\le i_k\le m_2^{(k)}\text{ for all }1\le k\le d\right\}.$$

Clearly, the complement of $\mathcal{I}_{\mathbf{p}}$ is $\mathcal{I}_{\mathbf{p}}^c=\{\mathbf{i}: p_k\mid i_k\text{ for all }1\le k\le d\}$ and

$$\left|\mathcal{R}_{m_1^{(1)},m_2^{(1)};\ldots;m_1^{(d)},m_2^{(d)}}\cap\mathcal{I}_{\mathbf{p}}\right|=\left|\mathcal{R}_{m_1^{(1)},m_2^{(1)};\ldots;m_1^{(d)},m_2^{(d)}}\right|-\left|\mathcal{R}_{m_1^{(1)},m_2^{(1)};\ldots;m_1^{(d)},m_2^{(d)}}\cap\mathcal{I}_{\mathbf{p}}^c\right|.$$

Thus,

$$\begin{aligned}&\left|\mathcal{R}_{m_1^{(1)},m_2^{(1)};\ldots;m_1^{(d)},m_2^{(d)}}\cap\mathcal{I}_{\mathbf{p}}\right|\\ &\ge\left|\mathcal{R}_{m_1^{(1)},m_2^{(1)};\ldots;m_1^{(d)},m_2^{(d)}}\right|-\frac{1}{p_1p_2\cdots p_d}\left|\mathcal{R}_{m_1^{(1)},m_2^{(1)}+2p_1;\ldots;m_1^{(d)},m_2^{(d)}+2p_d}\right|\end{aligned}$$

and

$$\begin{aligned}&\left|\mathcal{R}_{m_1^{(1)},m_2^{(1)};\ldots;m_1^{(d)},m_2^{(d)}}\cap\mathcal{I}_{\mathbf{p}}\right|\\ &\le\left|\mathcal{R}_{m_1^{(1)},m_2^{(1)};\ldots;m_1^{(d)},m_2^{(d)}}\right|-\frac{1}{p_1p_2\cdots p_d}\left|\mathcal{R}_{m_1^{(1)},m_2^{(1)}-2p_1;\ldots;m_1^{(d)},m_2^{(d)}-2p_d}\right|.\end{aligned}$$

Then, by the Squeeze theorem,

$$\lim_{\substack{m_2^{(k)}-m_1^{(k)}\to\infty\\ 1\le k\le d}}\frac{|\mathcal{R}_{m_1^{(1)},m_2^{(1)};\ldots;m_1^{(d)},m_2^{(d)}}\cap\mathcal{I}_{\mathbf{p}}|}{|\mathcal{R}_{m_1^{(1)},m_2^{(1)};\ldots;m_1^{(d)},m_2^{(d)}}|}=1-\frac{1}{p_1p_2\cdots p_d}.$$

Consequently,

$$\lim_{\mathbf{N}\to\infty}\frac{|\mathcal{K}_{N_1\times\cdots\times N_d;\ell}|}{|\mathcal{J}_{N_1\times\cdots\times N_d;\ell}|}=\lim_{\mathbf{N}\to\infty}\frac{|\mathcal{J}_{N_1\times\cdots\times N_d;\ell}\cap\mathcal{I}_{\mathbf{p}}|}{|\mathcal{J}_{N_1\times\cdots\times N_d;\ell}|}=1-\frac{1}{p_1p_2\cdots p_d}.$$

3. Define $\bar{K}_{N_1\times\cdots\times N_d;\ell}=\begin{cases}|\mathcal{K}_{N_1\times\cdots\times N_d;\ell}| & \text{if }\ell\le N_1\cdots N_d,\\ 0 & \text{if }\ell>N_1\cdots N_d.\end{cases}$ Then

$$\lim_{\mathbf{N}\to\infty}\frac{1}{N_1\cdots N_d}\sum_{\ell=1}^{N_1\cdots N_d}|\mathcal{K}_{N_1\times\cdots\times N_d;\ell}|\log F_\ell=\lim_{\mathbf{N}\to\infty}\frac{1}{N_1\cdots N_d}\sum_{\ell=1}^{\infty}\bar{K}_{N_1\times\cdots\times N_d;\ell}\log F_\ell.$$

Hence from the Weierstrass M-test with

$$\begin{aligned}
\frac{1}{N_1\cdots N_d}\left|\bar{K}_{N_1\times\cdots\times N_d;\ell}\log F_\ell\right| &\le \frac{1}{N_1\cdots N_d}|\mathcal{J}_{N_1\times\cdots\times N_d;\ell}|\log F_\ell\\
&= \frac{1}{N_1\cdots N_d}\left(\prod_{k=1}^{d}\left\lfloor\frac{N_k}{p_k^{\ell-1}}\right\rfloor-\prod_{k=1}^{d}\left\lfloor\frac{N_k}{p_k^{\ell}}\right\rfloor\right)\log F_\ell\\
&\le \frac{1}{N_1\cdots N_d}\left(\frac{N_1\cdots N_d}{p_1^{\ell-1}p_2^{\ell-1}\cdots p_d^{\ell-1}}\right)\log F_\ell\\
&= \frac{1}{(p_1p_2\cdots p_d)^{\ell-1}}\log F_\ell
\end{aligned}$$

for all $N_1,\dots,N_d \in \mathbb{N}$ and $\sum_{\ell=1}^{\infty}\frac{\log F_\ell}{(p_1p_2\cdots p_d)^{\ell-1}} < \infty$, we deduce that $\sum_{\ell=1}^{\infty}\frac{\bar{K}_{N_1\times\cdots\times N_d;\ell}\log F_\ell}{N_1\cdots N_d}$ converges uniformly in $N_1,\dots,N_d$.

This implies

$$\begin{aligned}
\lim_{\mathbf{N}\to\infty}\frac{1}{N_1\cdots N_d}\sum_{\ell=1}^{N_1\cdots N_d}|\mathcal{K}_{N_1\times\cdots\times N_d;\ell}|\log F_\ell &= \lim_{\mathbf{N}\to\infty}\frac{1}{N_1\cdots N_d}\sum_{\ell=1}^{\infty}\bar{K}_{N_1\times\cdots\times N_d;\ell}\log F_\ell\\
&= \sum_{\ell=1}^{\infty}\lim_{\mathbf{N}\to\infty}\frac{1}{N_1\cdots N_d}\bar{K}_{N_1\times\cdots\times N_d;\ell}\log F_\ell\\
&= \sum_{\ell=1}^{\infty}\lim_{\mathbf{N}\to\infty}\frac{1}{N_1\cdots N_d}|\mathcal{K}_{N_1\times\cdots\times N_d;\ell}|\log F_\ell.
\end{aligned}$$

The proof is complete. □

3. LDP of multiple averages on $\mathbb{N}^d$

In this section, we establish the LDP for the multiple average (13), where the multiple sum $S^{\mathbf{p}}_{N_1\times N_2\times\cdots\times N_d}$ is defined in (11). The associated free energy function $F_r(\beta)$ and the large deviation rate function $I_r(x)$ are also defined in (12) and (14) respectively.

Let $N_1, N_2, \dots, N_d \in \mathbb{N}$ and $\mathbf{p} = (p_1, p_2, \dots, p_d) \in \mathbb{N}^d$. The explicit formula for $F_r(\beta)$ and the LDP of the multiple average (13) are established in Theorem 3.2. The following theorem is essential in the proofs of our results.

Theorem 3.1 (*Gärtner–Ellis [8], Theorem 2.3.6*)**.** *If the limit (12) exists and such limit is finite in a neighborhood of origin, then (14) is equal to the Fenchel–Legendre transform of (12), i.e.,*

$$I_r(x) = \sup_{\beta\in\mathbb{R}}\left(\beta x - F_r(\beta)\right).$$

Moreover, if the function (12) is differentiable, and let η be the value such that $(F_r)'(\eta) = y$, then

$$I_r(y) = \eta y - F_r(\eta).$$

Theorem 3.2. *For any $d \ge 1$ and $p_1, p_2, \dots, p_d \ge 1$, then the following statements hold true.*

1. *The explicit expression of the free energy function associated to the multiple sum $S^{\mathbf{p}}_{N_1\times N_2\times\cdots\times N_d}$ is*

$$F_r(\beta)=\frac{2p_1p_2\cdots p_d-1}{2p_1p_2\cdots p_d}\log(r(1-r))+\frac{p_1p_2\cdots p_d-1}{p_1p_2\cdots p_d}\log|v^T\cdot e_+|^2+\log\Lambda_++\mathcal{G}(\beta), \tag{17}$$

where $\Lambda_\pm$, v^T, h, e_+ and $\mathcal{G}(\beta)$ are defined in (24), (23), (21), ((26) and (29) respectively.

2. *The function $F_r(\beta)$ is differentiable with respect to $\beta\in\mathbb{R}$.*
3. *The multiple average (13) satisfies a LDP with rate function given by*

$$I_r(x)=\sup_{\beta\in\mathbb{R}}\left(\beta x-F_r(\beta)\right).$$

Furthermore, if $(F_r)'(\eta)=y$, then $I_r(y)=\eta y-F_r(\eta)$.

Proof.

1. By Lemma 2.1, we decompose the sum (11) as

$$S^{\mathbf{p}}_{N_1\times N_2\times\cdots\times N_d}=\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{p}}}\left(\sum_{\mathbf{x}\in\mathcal{L}_{N_1\times N_2\times\cdots\times N_d}(\mathbf{i})}\sigma_{\mathbf{x}}\sigma_{\mathbf{p}\cdot\mathbf{x}}\right). \tag{18}$$

For a given $\mathbf{i}\in\mathcal{I}_{\mathbf{p}}$, the $\mathbf{i}$-th term of the external sum in (18) is nothing else than the Hamiltonian of a one-dimensional nearest-neighbors Ising model, since

$$\left\{\sum_{\mathbf{x}\in\mathcal{L}_{N_1\times N_2\times\cdots\times N_d}(\mathbf{i})}\sigma_{\mathbf{x}}\sigma_{\mathbf{p}\cdot\mathbf{x}}\right\}\overset{\mathcal{D}}{=}\left\{\sum_{\ell=1}^{|\mathcal{L}_{N_1\times N_2\times\cdots\times N_d}(\mathbf{i})|}\tau^{(\mathbf{i})}_\ell\tau^{(\mathbf{i})}_{\ell+1}\right\},$$

where $\tau^{(\mathbf{i})}_\ell$ are Bernoulli with parameter r, independent for different values of $\mathbf{i}$ and for different values of ℓ and $\overset{\mathcal{D}}{=}$ denotes equality in distribution. We introduce the notation $Z(\beta,h,\ell+1)$ (as in [6]) for the sum:

$$Z(\beta,h,\ell+1)=\sum_{\tau\in\{-1,1\}^{\ell+1}}e^{\beta\sum_{i=1}^{\ell}\tau_i\tau_{i+1}+h\sum_{i=1}^{\ell+1}\tau_i} \tag{19}$$

that is the partition function of the one-dimensional Ising model with coupling strength β and external field h in the volume $\{1,\ldots,\ell\}$, with free boundary conditions. Then

$$\mathbb{E}_r\left(e^{\beta\sum_{i=1}^{\ell}\tau_i\tau_{i+1}}\right)=(r(1-r))^{\frac{\ell+1}{2}}Z(\beta,h,\ell+1), \tag{20}$$

where

$$h=\frac{1}{2}\log\left(\frac{r}{1-r}\right). \tag{21}$$

By the computation in ([3], Chapter 2), (19) becomes

$$Z(\beta,h,\ell+1)=v^T\begin{bmatrix}e^{\beta+h} & e^{-\beta}\\ e^{-\beta} & e^{\beta-h}\end{bmatrix}^\ell v=|v^T\cdot e_+|^2\Lambda_+^\ell+|v^T\cdot e_-|^2\Lambda_-^\ell, \tag{22}$$

where

$$v^T=\left(e^{\frac{h}{2}},e^{\frac{-h}{2}}\right) \tag{23}$$

and

$$\Lambda_{\pm} = e^{\beta}\left(\cosh(h) \pm \sqrt{\sinh^2(h) + e^{-4\beta}}\right) \tag{24}$$

that are the largest, resp. smallest, eigenvalues of the transition matrix $\begin{bmatrix} e^{\beta+h} & e^{-\beta} \\ e^{-\beta} & e^{\beta-h} \end{bmatrix}$ with $e_{\pm}$ the corresponding normalized eigenvectors.

The partition function (22) can be rewritten as:

$$\begin{aligned} Z(\beta, h, \ell+1) =& |v^T \cdot e_+|^2 \Lambda_+^{\ell} + (\|v\|^2 - |v^T \cdot e_+|^2)\Lambda_-^{\ell} \\ =& |v^T \cdot e_+|^2 \Lambda_+^{\ell} + (2\cosh(h) - |v^T \cdot e_+|^2)\Lambda_-^{\ell} \end{aligned} \tag{25}$$

with e_+ given by:

$$e_+ = \frac{w_+}{\|w_+\|} \text{ with } w_+ = \begin{pmatrix} -e^{-\beta} \\ e^{h+\beta} - \Lambda_+ \end{pmatrix}. \tag{26}$$

Then by (18) and (20), (12) becomes

$$F_r(\beta) = \lim_{\mathbf{N}\to\infty} \frac{1}{N_1 \cdots N_d} \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{p}}} \log(r(1-r))^{\frac{|\mathcal{L}_{N_1\times N_2\times\cdots\times N_d}(\mathbf{i})|+1}{2}} Z(\beta, h, |\mathcal{L}_{N_1\times N_2\times\cdots\times N_d}(\mathbf{i})| + 1). \tag{27}$$

Then by Lemma 2.1, Lemma 2.2 and (27), we have

$$F_r(\beta) = \sum_{\ell=1}^{\infty} \frac{(p_1p_2\cdots p_d - 1)^2}{(p_1p_2\cdots p_d)^{\ell+1}} \log(r(1-r))^{\frac{\ell+1}{2}} Z(\beta, h, \ell+1). \tag{28}$$

Combining (25) and (28) yields

$$\begin{aligned} F_r(\beta) =& \sum_{\ell=1}^{\infty} \frac{(p_1p_2\cdots p_d - 1)^2}{(p_1p_2\cdots p_d)^{\ell+1}} \log(r(1-r))^{\frac{\ell+1}{2}} \\ &+ \sum_{\ell=1}^{\infty} \frac{(p_1p_2\cdots p_d - 1)^2}{(p_1p_2\cdots p_d)^{\ell+1}} \log\left(|v^T \cdot e_+|^2 \Lambda_+^{\ell} + (2\cosh(h) - |v^T \cdot e_+|^2)\Lambda_-^{\ell}\right) \\ =& \frac{2p_1p_2\cdots p_d - 1}{2p_1p_2\cdots p_d} \log(r(1-r)) \\ &+ \sum_{\ell=1}^{\infty} \frac{(p_1p_2\cdots p_d - 1)^2}{(p_1p_2\cdots p_d)^{\ell+1}} \log|v^T \cdot e_+|^2 + \sum_{\ell=1}^{\infty} \frac{(p_1p_2\cdots p_d - 1)^2}{(p_1p_2\cdots p_d)^{\ell+1}} \ell \log \Lambda_+ + \mathcal{G}(\beta) \\ =& \frac{2p_1p_2\cdots p_d - 1}{2p_1p_2\cdots p_d} \log(r(1-r)) + \frac{p_1p_2\cdots p_d - 1}{p_1p_2\cdots p_d} \log|v^T \cdot e_+|^2 + \log\Lambda_+ + \mathcal{G}(\beta), \end{aligned}$$

where

$$\mathcal{G}(\beta) = \sum_{\ell=1}^{\infty} \frac{(p_1p_2\cdots p_d - 1)^2}{(p_1p_2\cdots p_d)^{\ell+1}} \log\left(1 + \left(\frac{2\cosh(h)}{|v^T \cdot e_+|^2} - 1\right)\left(\frac{\Lambda_-}{\Lambda_+}\right)^{\ell}\right). \tag{29}$$

2. For the proof of differentiability of $\beta \mapsto F_r(\beta)$, it is enough to show that the sum

$$\sum_{\ell=1}^{\infty} \frac{(p_1p_2\cdots p_d - 1)^2}{(p_1p_2\cdots p_d)^{\ell+1}} \left[\log\left(1 + \left(\frac{2\cosh(h)}{|v^T \cdot e_+|^2} - 1\right)\left(\frac{\Lambda_-}{\Lambda_+}\right)^{\ell}\right)\right]'$$

converges uniformly with respect to $\beta \in \mathbb{R}$, where the notation $'$ stays for derivative with respect to β. Indeed, for all $\ell \geq 1$

$$\begin{aligned}&\left[\log\left(1+\left(\frac{2\cosh(h)}{|v^T\cdot e_+|^2}-1\right)\left(\frac{\Lambda_-}{\Lambda_+}\right)^\ell\right)\right]' \\ &\quad=\frac{\left[\frac{2\cosh(h)}{(v^T\cdot e_+)^2}\right]'\left(\frac{\Lambda_-}{\Lambda_+}\right)^\ell+\left[\frac{2\cosh(h)}{(v^T\cdot e_+)^2}-1\right]\ell\left(\frac{\Lambda_-}{\Lambda_+}\right)^{\ell-1}\left(\frac{\Lambda_-}{\Lambda_+}\right)'}{1+\left(\frac{2\cosh(h)}{(v^T\cdot e_+)^2}-1\right)\left(\frac{\Lambda_-}{\Lambda_+}\right)^\ell}.\end{aligned} \tag{30}$$

Note that, for all $\ell \geq 1$

$$0 \leq \left|\frac{\Lambda_-}{\Lambda_+}\right|^\ell \leq 1. \tag{31}$$

Then (30) and (31) give

$$\begin{aligned}&\left|\left[\log\left(1+\left(\frac{2\cosh(h)}{|v^T\cdot e_+|^2}-1\right)\left(\frac{\Lambda_-}{\Lambda_+}\right)^\ell\right)\right]'\right| \\ &\quad\leq\left|\left[\frac{2\cosh(h)}{(v^T\cdot e_+)^2}\right]'\right|+\left[\frac{2\cosh(h)}{(v^T\cdot e_+)^2}-1\right]\ell\left|\left(\frac{\Lambda_-}{\Lambda_+}\right)'\right|.\end{aligned} \tag{32}$$

For any bounded closed interval $[a, b] \subseteq \mathbb{R}$, there exists a positive constant $M_{a,b}$ such that

$$0=\left[\frac{2\cosh(h)}{|v|^2}-1\right]\leq\left[\frac{2\cosh(h)}{(v^T\cdot e_+)^2}-1\right]\leq M_{a,b},\ \text{ for all } \beta\in[a,b]. \tag{33}$$

It remains to check that $\left|\left[\frac{2\cosh(h)}{(v^T\cdot e_+)^2}\right]'\right|$ and $\left|\left(\frac{\Lambda_-}{\Lambda_+}\right)'\right|$ are bounded on $[a, b]$.
Direct computation infers that

$$\left(\frac{\Lambda_-}{\Lambda_+}\right)'=\frac{(-4e^{-4\beta})\left[\cosh(h)-f(\beta)\right]\left[\frac{1-e^{-4\beta}}{f(\beta)}-\cosh(h)+f(\beta)\right]}{(1-e^{-4\beta})^2}, \tag{34}$$

where $f(\beta) = \sqrt{\sinh^2(h) + e^{-4\beta}}$. For the boundedness of (34), we apply the fact that $f(\beta)$ is bounded away from zero on the interval $[a, b]$, thus

$$\left|\left(\frac{\Lambda_-}{\Lambda_+}\right)'\right|\leq K_{a,b}, \tag{35}$$

for some constant $K_{a,b}$ and for all $\beta \in [a, b]$. Note that the L'Hôpital's rule is applied to the definition of derivative and (35) when $\beta = 0$ and β near 0 respectively. Namely, for $\beta = 0$

$$\begin{aligned}\left(\frac{\Lambda_-}{\Lambda_+}\right)'(0)&=\lim_{\epsilon\to0}\frac{\frac{\Lambda_-}{\Lambda_+}(\epsilon)-\frac{\Lambda_-}{\Lambda_+}(0)}{\epsilon}\\ &=\lim_{\epsilon\to0}\frac{\cosh(h)-f(\beta)}{\epsilon\,(\cosh(h)+f(\beta))}.\end{aligned} \tag{36}$$

Applying L'Hôpital's rule to (36), we obtain

$$\left|\left(\frac{\Lambda_-}{\Lambda_+}\right)'(0)\right|=\frac{1}{\cosh^2(h)}\leq 1. \tag{37}$$

For $|\beta| \to 0^+$, applying L'Hôpital's rule twice to (34), we have

$$\left|\left(\frac{\Lambda_-}{\Lambda_+}\right)'(0^+)\right| = \frac{1}{\cosh^2(h)}. \tag{38}$$

Then (37) and (38) imply that $\left|\left(\frac{\Lambda_-}{\Lambda_+}\right)'\right|$ is a continuous function near $\beta = 0$, which implies its boundedness when β is near to 0.

It is easy to check that

$$e^{h-2\beta} > 0 \text{ and } \sinh(h) - \sqrt{\sinh^2(h) + e^{-4\beta}} < 0, \text{ for all } \beta \in [a, b]. \tag{39}$$

Then (39) gives

$$v^T \cdot e_+ \neq 0, \text{ for all } \beta \in [a, b]. \tag{40}$$

Combining (40) and the fact that $0 < C < f(\beta)$ for some C, we have

$$\left|\left[\frac{2\cosh(h)}{(v^T \cdot e_+)^2}\right]'\right| \leq N_{a,b}, \tag{41}$$

for some constant $N_{a,b}$ and for all $\beta \in [a, b]$.

The uniform convergence of $\beta \mapsto F_r(\beta)$ on $[a, b]$ is thus obtained by the Weierstrass M-test using (32), (35) and (41). Since a and b are arbitrary, we obtain the differentiability of $F_r(\beta)$ on $\mathbb{R}$.

3. Theorem 3.2 (3) follows from Theorem 3.1 and (1), (2) of Theorem 3.2. □

Remark 3.3. We note that when $r = \frac{1}{2}$, we have

$$I_{\frac{1}{2}}(y) = \eta\frac{e^\eta - e^{-\eta}}{e^\eta + e^{-\eta}} - \log(e^\eta + e^{-\eta}) + \log 2,$$

where

$$y = \frac{e^\eta - e^{-\eta}}{e^\eta + e^{-\eta}}.$$

Thus, $I_{\frac{1}{2}}$ is independent of the dimension $d \in \mathbb{N}$ and the multiple constraint vector $\mathbf{p} = (p_1, \ldots, p_d) \in \mathbb{N}^d$.

Corollary 3.4. *For any $p_1, p_2 \geq 1$,*

1. *The free energy function associated to the sum $S^{\mathbf{p}}_{N_1\times N_2}$ is*

$$F_r(\beta) = \frac{2p_1p_2 - 1}{2p_1p_2}\log(r(1-r)) + \frac{p_1p_2 - 1}{p_1p_2}\log\left|v^T \cdot e_+\right|^2 + \log \Lambda_+ + \mathcal{G}(\beta),$$

where

$$\mathcal{G}(\beta) = \sum_{\ell=1}^{\infty} \frac{(p_1p_2 - 1)^2}{(p_1p_2)^{\ell+1}} \log\left(1 + \left(\frac{2\cosh(h)}{\left|v^T \cdot e_+\right|^2} - 1\right)\left(\frac{\Lambda_-}{\Lambda_+}\right)^\ell\right).$$

2. *In addition, if we let $\mathbf{p} = (2, 1)$, then the free energy function associated to the sum*

$$S^{(2,1)}_{N_1\times N_2} = \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} \sigma_{(i_1,i_2)}\sigma_{(2i_1,i_2)}$$

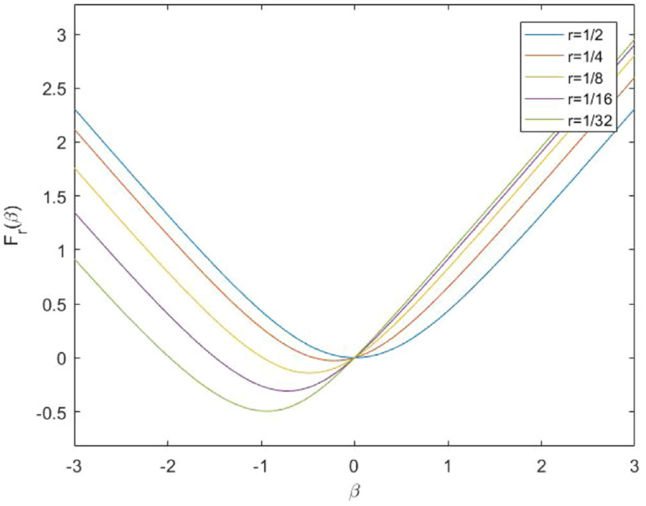

Fig. 1. Plot of the $F_r(\beta)$ with $p_1 = 2$ and $p_2 = 1$ for different r values.

is

$$F_r(\beta) = \log\left((r(1-r))^{\frac{3}{4}}|v^T \cdot e_+|\Lambda_+\right) + \mathcal{G}(\beta), \tag{42}$$

where

$$\mathcal{G}(\beta) = \frac{1}{2}\sum_{\ell=1}^{\infty}\frac{1}{2^\ell}\log\left(1 + \left(\frac{2\cosh(h)}{|v^T \cdot e_+|^2} - 1\right)\left(\frac{\Lambda_-}{\Lambda_+}\right)^\ell\right).$$

Remark 3.5.

1. Formula (42) is obtained in [6] for the multiple sum (6). Therefore (17) is a multidimensional version of the free energy function on $\mathbb{N}^d$.
2. When $r = 1/2$, we have $h = 0$, $\Lambda_+ = e^\beta + e^{-\beta}$ and $|v^T \cdot e_+|^2 = \|v\|^2 = 2$. This implies $\mathcal{G}(\beta) = 0$ and
$$F_{\frac{1}{2}}(\beta) = \log\left(\frac{1}{2}(e^\beta + e^{-\beta})\right). \tag{43}$$
Thus, $F_{\frac{1}{2}}$ is independent of the dimension $d \in \mathbb{N}$ and the choice of $\mathbf{p} \in \mathbb{N}^d$.
3. When $r \neq 1/2$, we have $h = \frac{1}{2}\log(r/1-r) \neq 0$. Then for all $\beta \in \mathbb{R}$, the vector v is not parallel to the vector w_+ and so is e_+, that gives
$$|v^T \cdot e_+|^2 < \|v\|^2\|e_+\|^2 = (r(1-r))^{-\frac{1}{2}}.$$
This gives the effect of multiple constraint.
4. Figs. 1 and 2 illustrate the free energy functions for different $r \in (0, 1)$ which is obtained from Theorem 3.2 by truncating the sum to the first 100 terms. Fig. 1 is the graph obtained for the case $d = 2$ with $p_1 = 2$ and $p_2 = 1$. In fact this coincides with the one-dimensional result shown in [6]. Fig. 2 shows the free energy behavior for the multidimensional case $d = 5$ with $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$ and $p_5 = 11$.

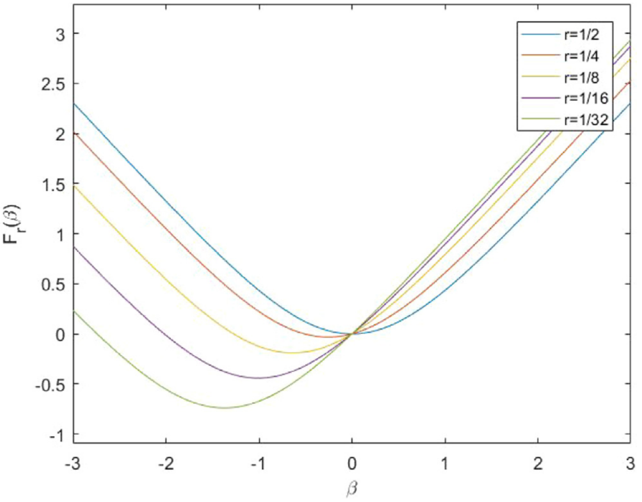

Fig. 2. Plot of the $F_r(\beta)$ with $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7$ and $p_5 = 11$ for different r values.

4. LDP of weighted multiple averages on $\mathbb{N}^d$

Let (X, T) be a topological dynamical system. Fan [11] studies the multifractal analysis of the following *weighted sum*

$$S_N^{(w)} f(x) = \sum_{n=1}^{N} w_n f_n(T^n x) = \sum_{n=1}^{N} w_n f_n(x_n, x_{n+1}, \ldots),$$

where $f = (f_n) \subseteq C(X)$. Define the level set $E(\alpha)$ according to the weighted average $\frac{S_N^{(w)} f(x)}{N}$ as

$$E(\alpha) = \left\{ x \in X : \lim_{N\to\infty} \frac{S_N^{(w)} f(x)}{N} = \alpha \right\}. \tag{44}$$

The Hausdorff dimension of (44) is given in the following theorem.

Theorem 4.1 (*A. Fan, [11], Theorem 1.5*)**.** *Let $\mathcal{A} = \{-1, 1\}$. Suppose the function f_n be of the form $f_n(x) = x_n g(x_{n+1}, x_{n+2}, \ldots)$ for some g_n satisfying the following condition:*

(C1) for all $n \geq 1$, $g_n(x_{n+1}, \ldots)$ takes values in $\mathcal{A}$ and only depends on a finite number of coordinates.
Moreover assume the collection of weights $(w_n)_{n\in\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, w_n takes values in a finite set $\{v_1, v_2, \ldots, v_m\}$ and satisfies the following condition:
(C2) for all $1 \leq j \leq m$, the following frequencies exist

$$P_j := \lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : w_n = v_j\}}{N}.$$

Then for $\alpha \in (-\sum P_j|v_j|, \sum P_j|v_j|)$,

$$\dim E(\alpha) = \frac{1}{\log 2} \sum_{j=1}^{m} P_j \left(\log(e^{\lambda_\alpha v_j} + e^{-\lambda_\alpha v_j}) - \lambda_\alpha v_j \frac{e^{\lambda_\alpha v_j} - e^{-\lambda_\alpha v_j}}{e^{\lambda_\alpha v_j} + e^{-\lambda_\alpha v_j}} \right), \tag{45}$$

where λ_α is the unique solution of the equation

$$\sum_{j=1}^{m} P_j v_j \frac{e^{\lambda_\alpha v_j} - e^{-\lambda_\alpha v_j}}{e^{\lambda_\alpha v_j} + e^{-\lambda_\alpha v_j}} = \alpha.$$

Let $\mu : \mathbb{N} \to \{-1, 0, 1\}$ be the Möbius function, the author also considers the level set

$$F(\alpha) = \left\{ (x_n)_{n=1}^{\infty} \in \{-1, 1\}^{\mathbb{N}} : \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) x_n x_{n+1} = \alpha \right\} \tag{46}$$

for which the dimension spectrum is given in the following theorem.

Theorem 4.2 (*A. Fan, [11], Theorem 1.6*)**.** *For $\alpha \in (-\frac{\pi^2}{6}, \frac{\pi^2}{6})$,*

$$\dim F(\alpha) = 1 - \frac{6}{\pi^2} + \frac{6}{\pi^2 \log 2} H\left(\frac{1}{2} + \frac{\pi^2}{12}\alpha\right),$$

where $H(x) = -x\log x - (1-x)\log(1-x)$.

In this section, we establish the LDP based on the weighted multiple average (16). The main results of this section are presented below.

Let $N_1, N_2, \ldots, N_d, p_1, p_2, \ldots, p_d \geq 1$. Assume the weights $\mathbf{w} = (w_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d}$ take a finite number of values $v_1, v_2, \ldots, v_m$ and the following frequencies exist

$$P_k := \lim_{\mathbf{N}\to\infty} \frac{\#\{\mathbf{x} \in \mathcal{L}_{N_1\times N_2\times\cdots\times N_d}(\mathbf{i}) : w_{\mathbf{x}} = v_k\}}{|\mathcal{L}_{N_1\times N_2\times\cdots\times N_d}(\mathbf{i})|} \tag{47}$$

for all $\mathbf{i} \in \mathcal{I}_{\mathbf{p}}$ and $1 \leq k \leq m$.

Theorem 4.3. *Let $d \geq 1$ and $p_1, p_2, \ldots, p_d \geq 1$, and let the $\mathbf{w} = (w_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d}$ be a collection of weights satisfying the frequency condition (47), then the following statements hold true.*

1. *The free energy function associated to the sum (15) is equal to*

$$F_{\frac{1}{2}}^{\mathbf{w}}(\beta) = \sum_{k=1}^{m} P_k \log(e^{\beta v_k} + e^{-\beta v_k}) - \log 2.$$

2. *The function $F_{\frac{1}{2}}^{\mathbf{w}}(\beta)$ is differentiable with respect to $\beta \in \mathbb{R}$.*
3. *The multiple average (16) satisfies a LDP with rate function*

$$I_{\frac{1}{2}}^{\mathbf{w}}(x) = \sup_{\beta\in\mathbb{R}} \left(\beta x - \sum_{k=1}^{m} P_k \log(e^{\beta v_k} + e^{-\beta v_k}) + \log 2 \right)$$

that is equal to:

$$I_{\frac{1}{2}}^{\mathbf{w}}(y) = \sum_{k=1}^{m} P_k \left(\eta v_k \frac{e^{\eta v_k} - e^{-\eta v_k}}{e^{\eta v_k} + e^{-\eta v_k}} - \log(e^{\eta v_k} + e^{-\eta v_k}) \right) + \log 2,$$

where

$$y = \sum_{k=1}^{m} P_k \frac{v_k (e^{\eta v_k} - e^{-\eta v_k})}{e^{\eta v_k} + e^{-\eta v_k}}.$$

Proof.

1. Observe that the transition matrices commute, i.e.,

$$\begin{bmatrix} e^{\beta v_i} & e^{-\beta v_i} \\ e^{-\beta v_i} & e^{\beta v_i} \end{bmatrix}\begin{bmatrix} e^{\beta v_j} & e^{-\beta v_j} \\ e^{-\beta v_j} & e^{\beta v_j} \end{bmatrix} = \begin{bmatrix} e^{\beta v_j} & e^{-\beta v_j} \\ e^{-\beta v_j} & e^{\beta v_j} \end{bmatrix}\begin{bmatrix} e^{\beta v_i} & e^{-\beta v_i} \\ e^{-\beta v_i} & e^{\beta v_i} \end{bmatrix}, \tag{48}$$

for all $1 \leq i < j \leq m$. According to (48), we exchange the order of matrix products on the sublattices $\mathcal{L}_{N_1\times N_2\times\cdots\times N_d}(\mathbf{i})$ for all $\mathbf{i} \in \mathcal{I}_{\mathbf{p}}$ so that the products have the following forms

$$\begin{bmatrix} e^{\beta v_1} & e^{-\beta v_1} \\ e^{-\beta v_1} & e^{\beta v_1} \end{bmatrix}^{k_1}\begin{bmatrix} e^{\beta v_2} & e^{-\beta v_2} \\ e^{-\beta v_2} & e^{\beta v_2} \end{bmatrix}^{k_2}\cdots\begin{bmatrix} e^{\beta v_m} & e^{-\beta v_m} \\ e^{-\beta v_m} & e^{\beta v_m} \end{bmatrix}^{k_m} \tag{49}$$

and then we choose the same eigenvectors for all $\begin{bmatrix} e^{\beta v_i} & e^{-\beta v_i} \\ e^{-\beta v_i} & e^{\beta v_i} \end{bmatrix}$, $1 \leq i \leq m$. Then by Lemmas 2.1 and 2.2 and (47), we have

$$\begin{aligned} F^{\mathbf{w}}_{\frac{1}{2}}(\beta) &= \sum_{\ell=1}^{\infty} \frac{(p_1p_2\cdots p_d-1)^2}{(p_1p_2\cdots p_d)^{\ell+1}} \log\frac{1}{2^{\ell+1}} \\ &\times \begin{pmatrix} 1 & 1 \end{pmatrix} \prod_{k=1}^{m} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e^{\beta v_k}+e^{-\beta v_k} & 0 \\ 0 & e^{\beta v_k}-e^{-\beta v_k} \end{bmatrix}^{P_k\ell} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \sum_{\ell=1}^{\infty} \frac{(p_1p_2\cdots p_d-1)^2}{(p_1p_2\cdots p_d)^{\ell+1}} \log\frac{1}{2^{\ell}} \prod_{k=1}^{m} (e^{\beta v_k}+e^{-\beta v_k})^{P_k\ell} \\ &= \sum_{k=1}^{m} P_k \sum_{\ell=1}^{\infty} \frac{(p_1p_2\cdots p_d-1)^2}{(p_1p_2\cdots p_d)^{\ell+1}} \left[-\ell\log 2 + \ell\log(e^{\beta v_k}+e^{-\beta v_k})\right] \\ &= \sum_{k=1}^{m} P_k \log(e^{\beta v_k}+e^{-\beta v_k}) - \log 2. \end{aligned}$$

2. The formula in Theorem 4.3 (1) implies

$$(F^{\mathbf{w}}_{\frac{1}{2}})'(\beta) = \sum_{k=1}^{m} P_k \frac{v_k(e^{\beta v_k}-e^{-\beta v_k})}{e^{\beta v_k}+e^{-\beta v_k}}$$

and

$$(F^{\mathbf{w}}_{\frac{1}{2}})''(\beta) = \sum_{k=1}^{m} P_k \frac{4v_k^2}{(e^{\beta v_k}+e^{-\beta v_k})^2} > 0.$$

3. The result (3) of Theorem 4.3 follows from Theorem 3.1 and items (1) and (2) of Theorem 4.3. □

Remark 4.4. Note that the rate function $I^{\mathbf{w}}_{\frac{1}{2}}$ is dependent only on the frequency P_k, defined in (47), and not on the dimension $d \in \mathbb{N}$ and the multiple constraint vector $\mathbf{p} = (p_1, \ldots, p_d) \in \mathbb{N}^d$.

Note that when $w_{\mathbf{i}} = 1$ for all $\mathbf{i} \in \mathbb{N}^d$, $F^{\mathbf{w}}_{\frac{1}{2}}(\beta)$ is equal to the free energy function of the sum $S_N = \sum_{i=1}^{N} \sigma_i\sigma_{2i}$ in [6].

We say $\mathbf{w} = (w_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d}$ is a *Möbius weight* if it is Möbius on each spin, for instance, in the spin

$$(\alpha_1, \alpha_2, \dots, \alpha_d), (\alpha p_1, \alpha_2 p_2, \dots, \alpha_d p_d), \dots, (\alpha p_1^r, \alpha_2 p_2^r, \dots, \alpha_d p_d^r),$$

we put $w_{(\alpha_1 p_1^{i-1}, \alpha_2 p_2^{i-1}, \dots, \alpha_d p_d^{i-1})} = \mu(i)$ for all $1 \le i \le r+1$. Then we have the following corollary

Corollary 4.5. *Let $d \ge 1$, $p_1, p_2, \dots, p_d \ge 1$, and let $\mathbf{w} = (w_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d}$ be a Möbius weight, then the free energy function associated to sum $S^{\mathbf{p},\mathbf{w}}_{N_1\times N_2\times\cdots\times N_d}$ is equal to*

$$F^{\mathbf{w}}_{\frac{1}{2}}(\beta) = \frac{6}{\pi^2}\log\left(\frac{1}{2}(e^\beta + e^{-\beta})\right).$$

Proof. Since

$$\lim_{r\to\infty}\frac{1}{r}\sum_{n=1}^{r}|\mu(n)| = \frac{6}{\pi^2},$$

the frequency of 1, -1 and 0 denoted by P_1, P_{-1} and P_0 respectively (defined in (47)), satisfy $P_1 + P_{-1} = \frac{6}{\pi^2}$ and $P_0 = 1 - \frac{6}{\pi^2}$.

Then by Theorem 4.3, we have

$$\begin{aligned} F^{\mathbf{w}}_{\frac{1}{2}}(\beta) =& \sum_{k=1}^{m} P_k \log(e^{\beta v_k} + e^{-\beta v_k}) - \log 2 \\ =&(P_0 - 1)\log 2 + (P_1 + P_{-1})\log(e^\beta + e^{-\beta}) \\ =&\frac{6}{\pi^2}\log\left(\frac{1}{2}(e^\beta + e^{-\beta})\right). \end{aligned}$$

The proof is complete. □

5. Boundary conditions on multiple sums

In [7] the authors consider the 'multiplicative Ising model' on the lattice $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ with parameters β the *inverse temperature* , J the *coupling strength*, and h the *magnetic field*. The Hamiltonian is given by

$$H(\sigma) = -\beta\left(\sum_{i\in\mathbb{N}} J\sigma_i\sigma_{2i} + h\sum_{i\in\mathbb{N}}\sigma_i\right) \tag{50}$$

where the spin configuration $\sigma = (\sigma_i)_{i\in\mathbb{N}_0}$ lives in $\mathcal{A}^{\mathbb{N}_0}$.

In [7] the authors consider also the Hamiltonian in the finite lattice $[1, 2N]\cap\mathbb{N}$ with boundary condition $\eta = (\eta_i)_{i=1}^{\infty}$, $\eta_i \in \{-1, 1\}$ that is defined as follows

$$H_N^\eta(\sigma_{[1,2N]}) = -\beta\left(\sum_{i=1}^{N} J\sigma_i\sigma_{2i} + \sum_{i=1}^{2N} h\sigma_i \pm \sum_{i=N+1}^{2N}\sigma_i\eta_{2i}\right). \tag{51}$$

In this circumstance, $H_N^\emptyset(\sigma_{[1,2N]}) = -\beta\left(\sum_{i=1}^{N} J\sigma_i\sigma_{2i} + \sum_{i=1}^{2N} h\sigma_i\right)$ denotes the Hamiltonian with free boundary conditions.

Following [7], we impose the boundary conditions on the multiple sum (11) when $r = \frac{1}{2}$. To avoid the cumbersome computation, we restrict ourselves on the case when $d = 2$ since the case of general dimensions can be treated in the same fashion.

For $N_1, N_2 \geq 1$, define the *Dirichlet boundary condition Type 1* (BC1) by, putting $+1$ on the boundary. That is,

$$\begin{aligned}
&\sigma_{(1,1)}, \ldots, \sigma_{(1,N_2)} = +1, \sigma_{(1,N_2)}, \ldots, \sigma_{(N_1,N_2)} = +1,\\
&\sigma_{(1,1)}, \ldots, \sigma_{(N_1,1)} = +1, \sigma_{(N_1,1)}, \ldots, \sigma_{(N_1,N_2)} = +1.
\end{aligned}$$

The *Dirichlet boundary conditions Type 2* (BC2) are defined by "all $+1$ except (i_1, i_2) with $1 < i_1 < N_1$ and $1 < i_2 < N_2$". The *Periodic boundary condition* (BCp) is defined by "all spins have same sign on its starting point and end. That is, for any $(i_1, i_2) \in \mathcal{I}_{p_1,p_2} \cap \mathcal{N}_{N_1 \times N_2}$,

$$\sigma_{(i_1,i_2)} = \sigma_{(j_1,j_2)}$$

where (j_1, j_2) is the smallest point (coordinate-wise) in $\mathcal{L}_{N_1 \times N_2}(i_1, i_2)$ such that $j_1 \geq N_1$ and $j_2 \geq N_2$".

Define the *energy function* corresponding to the BCi boundary conditions by

$$F^{(\mathrm{BCi})}(\beta) := \lim_{N_1,N_2\to\infty} \frac{1}{N_1 N_2} \log \mathbb{E}_{\frac{1}{2};(\mathrm{BCi})} \left(e^{\beta S^{(p_1,p_2)}_{N_1 \times N_2}} \right), \ \mathrm{i} \in \{1, 2, \mathrm{p}\}. \tag{52}$$

Theorem 5.1.

1. *The explicit formulas for the energy functions $F^{(\mathrm{BCi})}(\beta)$, for $\mathrm{i} \in \{1, 2, \mathrm{p}\}$, are given by:*

$$\begin{aligned}
F^{(\mathrm{BC1})}(\beta) &= \log(e^{\beta} + e^{-\beta}) - \log 2,\\
F^{(\mathrm{BC2})}(\beta) &= \log(e^{\beta} + e^{-\beta}) - \left(\frac{2p_1p_2 - 1}{p_1p_2}\right) \log 2,\\
F^{(\mathrm{BCp})}(\beta) &= \log(e^{\beta} + e^{-\beta}) - \left(\frac{2p_1p_2 - 1}{p_1p_2}\right) \log 2\\
&\quad + \sum_{\ell=1}^{\infty} \frac{(p_1p_2 - 1)^2}{(p_1p_2)^{\ell+1}} \log \left[1 + \left(\frac{e^{\beta} - e^{-\beta}}{e^{\beta} + e^{-\beta}} \right)^{\ell} \right].
\end{aligned}$$

2. *The functions $F^{(\mathrm{BCi})}(\beta)$ for $\mathrm{i} \in \{1, 2, \mathrm{p}\}$ are differentiable with respect to $\beta \in \mathbb{R}$.*
3. *The multiple average (13) with boundary conditions* BCi *satisfies the LDP with rate function*

$$I(x) = \sup_{\beta\in\mathbb{R}} \left(\beta x - F^{(\mathrm{BCi})}(\beta) \right), \ \mathrm{i} \in \{1, 2, \mathrm{p}\}.$$

Furthermore, if $(F^{(\mathrm{BCi})})'(\eta) = y$, then $I(y) = \eta y - F^{(\mathrm{BCi})}(\eta)$, $\mathrm{i} \in \{1, 2, \mathrm{p}\}$.

Proof.

1. Applying the decomposition in Section 3 to (52), we have

$$F^{(\mathrm{BC1})}(\beta) = \lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \left\{ \sum_{\ell=1}^{N_1N_2} A_{N_1,N_2;\ell} \log \frac{1}{2^{\ell+1}} \left|V^\ell\right| \right.$$
$$+ \sum_{\ell=1}^{N_1N_2} B_{N_1,N_2;\ell} \log \frac{1}{2^{\ell+1}} \left[(V^\ell)_{11} + (V^\ell)_{21}\right]$$
$$+ \sum_{\ell=1}^{N_1N_2} C_{N_1,N_2;\ell} \log \frac{1}{2^{\ell+1}} \left[(V^\ell)_{11} + (V^\ell)_{12}\right]$$
$$\left. + \sum_{\ell=1}^{N_1N_2} D_{N_1,N_2;\ell} \log \frac{1}{2^{\ell+1}} (V^\ell)_{11} \right\},$$

where

$$V^\ell = \begin{bmatrix} \frac{1}{2}(e^\beta + e^{-\beta})^\ell + \frac{1}{2}(e^\beta - e^{-\beta})^\ell & \frac{1}{2}(e^\beta + e^{-\beta})^\ell - \frac{1}{2}(e^\beta - e^{-\beta})^\ell \\ \frac{1}{2}(e^\beta + e^{-\beta})^\ell - \frac{1}{2}(e^\beta - e^{-\beta})^\ell & \frac{1}{2}(e^\beta + e^{-\beta})^\ell + \frac{1}{2}(e^\beta - e^{-\beta})^\ell \end{bmatrix}$$

and the coefficients $A_{N_1,N_2;\ell}$, $B_{N_1,N_2;\ell}$, $C_{N_1,N_2;\ell}$ and $D_{N_1,N_2;\ell}$ are the numbers of sublattices $\mathcal{L}_{N_1\times N_2}(i_1,i_2), (i_1,i_2) \in \mathcal{I}_{p_1,p_2}$ with cardinality ℓ which intersects the boundaries

$$(1,1),\ldots,(1,N_2),\ (N_1,1),\ldots,(N_1,N_2),\ (1,1),\ldots,(N_1,1),\ (1,N_2),\ldots,(N_1,N_2)$$

empty, exact starting point, exact end and two points respectively.
For any $\beta \in \mathbb{R}$, we have

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{\ell=1}^{N_1N_2} A_{N_1,N_2;\ell} \log \frac{1}{2^{\ell+1}} \left|V^\ell\right| \le F^{(\mathrm{BC1})}(\beta) \tag{53}$$

and

$$F^{(\mathrm{BC1})}(\beta) \le \lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{\ell=1}^{N_1N_2} |\mathcal{K}_{N_1\times N_2;\ell}| \log \frac{1}{2^{\ell+1}} \left|V^\ell\right|, \tag{54}$$

where $\mathcal{K}_{N_1\times N_2;\ell}$ is defined in Section 2.
Indeed,

$$|\mathcal{K}_{N_1\times N_2;\ell}| - 2\left(\left\lfloor \frac{N_1}{p_1^{\ell-1}} \right\rfloor - \left\lfloor \frac{N_1}{p_1^{\ell}} \right\rfloor + \left\lfloor \frac{N_2}{p_2^{\ell-1}} \right\rfloor - \left\lfloor \frac{N_2}{p_2^{\ell}} \right\rfloor\right) \le A_{N_1,N_2;\ell}, \tag{55}$$

since the boundaries

$$(1,1),\ldots,(1,N_2) \text{ and } (N_1,1),\ldots,(N_1,N_2)$$

intersect at most $\left\lfloor \frac{N_2}{p_2^{\ell-1}} \right\rfloor - \left\lfloor \frac{N_2}{p_2^{\ell}} \right\rfloor$ sublattices with length ℓ, and the boundaries

$$(1,1),\ldots,(N_1,1) \text{ and } (1,N_2),\ldots,(N_1,N_2)$$

intersect at most $\left\lfloor \frac{N_1}{p_1^{\ell-1}} \right\rfloor - \left\lfloor \frac{N_1}{p_1^{\ell}} \right\rfloor$ sublattices with length ℓ.

Then (53) and (55) give

$$\lim_{N_1,N_2\to\infty}\frac{1}{N_1N_2}\sum_{\ell=1}^{N_1N_2}\left(|\mathcal{K}_{N_1\times N_2;\ell}|-2E_{N_1,N_2;\ell}\right)\log\frac{1}{2^{\ell+1}}\left|V^{\ell}\right|\leq F^{(\mathrm{BC1})}(\beta), \tag{56}$$

where $E_{N_1,N_2;\ell}=\left\lfloor\frac{N_1}{p_1^{\ell-1}}\right\rfloor-\left\lfloor\frac{N_1}{p_1^{\ell}}\right\rfloor+\left\lfloor\frac{N_2}{p_2^{\ell-1}}\right\rfloor-\left\lfloor\frac{N_2}{p_2^{\ell}}\right\rfloor$.
On the other hand, combining (54) and Theorem 3.2,

$$F^{(\mathrm{BC1})}(\beta)\leq F_{\frac{1}{2}}(\beta), \tag{57}$$

where $F_{\frac{1}{2}}(\beta)=\log\left(\frac{1}{2}(e^{\beta}+e^{-\beta})\right)$.
It remains to show that

$$\lim_{N_1,N_2\to\infty}\frac{1}{N_1N_2}\sum_{\ell=1}^{N_1N_2}2E_{N_1,N_2;\ell}\log\frac{1}{2^{\ell+1}}\left|V^{\ell}\right|=0. \tag{58}$$

It is enough to rewrite (58) as

$$\lim_{N_1,N_2\to\infty}\frac{C}{N_1N_2}\sum_{\ell=1}^{\infty}\ell\left(N_1\left(\frac{1}{p_1^{\ell-1}}-\frac{1}{p_1^{\ell}}\right)+N_2\left(\frac{1}{p_2^{\ell-1}}-\frac{1}{p_2^{\ell}}\right)\right), \tag{59}$$

where C is a constant dependent only on the maximum eigenvalue of V. Indeed, (59) is equal to 0 by direct computation.
By the Squeeze Theorem with (56)–(58), we obtain

$$F^{(\mathrm{BC1})}(\beta)=\log\left(\frac{1}{2}(e^{\beta}+e^{-\beta})\right).$$

2. Since the free energy function corresponding to (BC2) is

$$\begin{aligned}F^{(\mathrm{BC2})}(\beta)=\lim_{N_1,N_2\to\infty}\frac{1}{N_1N_2}&\left\{\sum_{\ell=1}^{N_1N_2}A_{N_1,N_2;\ell}\log\frac{1}{2^{\ell+1}}[(V^{\ell})_{11}+(V^{\ell})_{21}]\right.\\&+\sum_{\ell=1}^{N_1N_2}B_{N_1,N_2;\ell}\log\frac{1}{2^{\ell+1}}[(V^{\ell-1})_{11}V_{11}+(V^{\ell-1})_{21}V_{11}]\\&+\sum_{\ell=1}^{N_1N_2}C_{N_1,N_2;\ell}\log\frac{1}{2^{\ell+1}}[(V^{\ell-1})_{11}V_{11}+(V^{\ell-1})_{12}V_{21}]\\&\left.+\sum_{\ell=1}^{N_1N_2}D_{N_1,N_2;\ell}\log\frac{1}{2^{\ell+1}}(V^{\ell-1})_{11}V_{11}\right\}\end{aligned}$$

similarly to what we obtained in the proof of $F^{(\mathrm{BC1})}$, we have that

$$F^{(\mathrm{BC2})}(\beta)=\lim_{N_1,N_2\to\infty}\frac{1}{N_1N_2}\sum_{\ell=1}^{N_1N_2}|\mathcal{K}_{N_1\times N_2;\ell}|\log\frac{1}{2^{\ell+1}}[(V^{\ell})_{11}+(V^{\ell})_{21}].$$

By Lemmas 2.1 and 2.2, we have

$$\begin{aligned}F^{(\mathrm{BC2})}(\beta) &= \log\left(\frac{1}{2}(e^{\beta}+e^{-\beta})\right)+\sum_{\ell=1}^{\infty}\frac{(p_1p_2-1)^2}{(p_1p_2)^{\ell+1}}\log\left(\frac{1}{2}\right)\\ &= \log\left(\frac{1}{2}(e^{\beta}+e^{-\beta})\right)-\left(\frac{p_1p_2-1}{p_1p_2}\right)\log 2\\ &= \log(e^{\beta}+e^{-\beta})-\left(\frac{2p_1p_2-1}{p_1p_2}\right)\log 2.\end{aligned}$$

3. Using a method similar to the one used in the proof of item 2, we obtain

$$F^{(\mathrm{BCp})}(\beta)=\lim_{N_1,N_2\to\infty}\frac{1}{N_1N_2}\sum_{\ell=1}^{N_1N_2}|\mathcal{K}_{N_1\times N_2;\ell}|\log\frac{1}{2^{\ell+1}}\mathrm{tr}(V^{\ell}).$$

The proof is completed by Lemma 2.1, Lemma 2.2 and direct computation.

4. Theorem 5.1 (2) follows from the direct calculation of the formula established in Theorem 5.1 (1). Theorem 5.1 (3) follows from Theorem 3.1. This completes the proof. □

Remark 5.2. The reason for what F(BC1) is equal to (43) lies in the fact that (BC1) affects at most $2(N1+N2)$ sublattices and then $\frac{2(N_1+N_2)}{N_1N_2}$ tends to zero as N_1 and N_2 tend to infinity. On the other hand, (BC2) and (BCp) affect almost all sublattices in $N_1 \times N_2$ lattice. This makes the difference between $F^{(\mathrm{BC1})}$ and $F^{(\mathrm{BCi})}$, i $\in$ {2, p}.

6. Conclusion and some open problems

6.1. Conclusion

In Section 3, we obtained an explicit formula for the free energy function $F_r(\beta)$ associated to the multiple sum (11). Then we established a LDP for the multiple average (13) with rate function $I_r(y) = \eta y - F_r(\eta)$, with η so that $F_r'(\eta) = y$. This is obtained using the differentiability of $F_r(\beta)$ and Theorem 3.1. In Section 4, the LDP of the weighted multiple average (16) is also established for $r = 1/2$. Note that, when $r \neq 1/2$, the free energy function is difficult to compute since the matrices do not commute in general. The boundary conditions (BCi), i$\in$ {1, 2, p} are imposed to the multiple sum (11) in Section 5. Rigorous formulae for the free energy functions associated with the boundary conditions are derived. Consequently, the LDP results follow as well. The following problem remains open.

Problem 1. In this article, we only consider the 2-multiple sum (11). The k-multiple sum is defined similarly. Namely, for $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{k-1} \in \mathbb{N}^d$, define

$$S_{N_1\times N_2\times\cdots\times N_d}^{\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_{k-1}}:=\sum_{i_i=1}^{N_1}\sum_{i_2=1}^{N_2}\cdots\sum_{i_d=1}^{N_d}\sigma_{\mathbf{i}}\sigma_{\mathbf{p}_1\cdot\mathbf{i}}\sigma_{\mathbf{p}_2\cdot\mathbf{i}}\cdots\sigma_{\mathbf{p}_{k-1}\cdot\mathbf{i}}.$$

Theorem 3.2 (or Theorem 4.3) demonstrates the absence of the phase transition phenomenon for the multiplicative Ising model with 2-multiple sum. Does the phase transition phenomenon occur with respect to the k-multiple sum for $k \geq 3$? Such a problem depends on explicitly calculating the free energy function $F_r(\beta)$, and this is an extremely difficult task.

Acknowledgments

We would like to sincerely thank the anonymous referee for providing inspiring comments and helpful suggestions for the first draft of this article. These significantly improve the readability and solidify the validity of the theorems in the paper.

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