國立政治大學應用數學系

博士學位論文

實數標號的反魔幻圖形

Graphs with $\mathbb{R}$-Antimagic Labeling

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中文摘要

設 G 是一個圖，且 A 是複數的子集，其中 $|A| = |E(G)|$，且 $E(G)$ 為圖 G 的邊所成集合。標號在集合 A 裡頭的邊標記，是從 $E(G)$ 映射到 A 的函數。設 B 是複數的子集，且 $|B| \geq |E(G)|$。若對於集合 B 的每個子集 A，滿足 $|A| = |E(G)|$，而且標號在 A 裡頭的邊標記，使得不同頂點它們連接的邊標記之總和是不同的，則圖 G 被稱為 B-反魔幻。一般文獻中，若 G 是 $\{1, 2, \ldots, |E(G)|\}$-反魔幻，則稱圖 G 是反魔幻的。反魔幻圖的概念是由 Hartsfield and Ringel [11] 在 1990 年提出的。他們猜測至少有兩條邊的連通圖都是反魔幻的。這個猜想還沒有完全解決。許多研究人員在反魔圖領域做出了一些努力。

設 $\mathbb{R}$ 表所有實數所成集合，且 $\mathbb{C}$ 表所有複數所成集合。我們將反魔圖的定義延伸推廣至 $\mathbb{R}$-反魔幻圖。在第二章，我們證明了每個 $\mathbb{R}$-反魔幻圖都是 $\mathbb{C}$-反魔幻。我們也證明了若圖 G 為正則圖，則 $\mathbb{R}^+$-反魔幻圖就是 $\mathbb{R}$-反魔幻。另外，我們也發現了有一類正則圖是 $\mathbb{R}$-反魔幻。

在第三章中，我們證明了環及點數大於等於 3 的完全圖是 $\mathbb{R}$-反魔幻。假設圖 G 是環或點數大於 3 的完全圖，我們可以依照每個頂點邊標記總和的大小，將點以 $u_1, u_2, \cdots, u_n$ 排序，無關乎標號的選取，這樣的性質我們就稱為均勻 $\mathbb{R}$-反魔幻。明顯地，每個均勻 $\mathbb{R}$-反魔幻, 都是 $\mathbb{R}$-反魔幻。我們也證明了 $G_1 \square G_2 \square \cdots \square G_n$ $(n \geq 2)$ 是均勻 $\mathbb{R}$-反魔幻，其中每個 G_i 是環或點數大於等於 3 的完全圖。

在第四章，我們證明了輪子，爪子及點數大於等於 6 的路徑是 $\mathbb{R}$-反魔幻。最後，我們在第五章作研究結果總結及討論，並提出未來研究方向。

關鍵字：$\mathbb{R}$-反魔幻圖, 正則圖, 笛卡爾乘積圖, 均勻 $\mathbb{R}$-反魔幻

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Abstract

Let G be a finite graph, and $A \subseteq \mathbb{C}$. An edge labeling of graph G with labels in A is an injection from $E(G)$ to A, where $E(G)$ is the edge set of G, and A is a subset of $\mathbb{C}$. Suppose that B is a set of complex numbers with $|B| \geq |E(G)|$. If for every $A \subseteq B$ with $|A| = |E(G)|$, there is an edge labeling of G with labels in A such that the sums of the labels assigned to edges incident to distinct vertices are different, then G is said to be B-antimagic. A graph G is an antimagic graph in the literature, if G is $\{1, 2, \ldots, |E(G)|\}$-antimagic.

The concept of antimagic graphs was introduced by Hartsfield and Ringel [11] in 1990. They conjectured that every connected graph with at least two edges was antimagic. The conjecture has not been completely solved yet.

We propose the concept of $\mathbb{R}$-antimagic graphs in this thesis. In Chapter 2, we prove that every $\mathbb{R}$-antimagic graph is $\mathbb{C}$-antimagic. We also show that every $\mathbb{R}^+$-antimagic graph is also $\mathbb{R}$-antimagic if the graph is regular. Additionally, we discover a special class of regular graphs that are $\mathbb{R}$-antimagic (see Theorem 2.3.5). One of the graphs in this class is the Peterson graph.

In Chapter 3, we show that cycles and complete graphs of order ≥ 3 are $\mathbb{R}$-antimagic. Assume that G is a complete graph or a cycle with $V(G) = \{u_1, u_2, \cdots, u_n\}$ ($n \geq 3$). We have found that all the vertices of G can be listed as $u_1, u_2, \cdots, u_n$ such that for every $A \subseteq \mathbb{R}$ with $|A| = |E(G)|$, there is an edge labeling f of G with labels in A such that $f^+(u_1) < f^+(u_2) < \cdots < f^+(u_n)$. The property we call uniformly $\mathbb{R}$-antimagic property which is independent of the choice of the subset A of $\mathbb{R}$. Clearly, every uniformly $\mathbb{R}$-antimagic is $\mathbb{R}$-antimagic. We prove that Cartesian products $G_1 \square G_2 \square \cdots \square G_n$ $(n \geq 2)$ are uniformly $\mathbb{R}$-antimagic, where each G_i is a complete graph of order ≥ 2 or a cycle.

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In Chapter 4, we prove that wheels, paws, and paths of order ≥ 6 are $\mathbb{R}$-antimagic. Finally, we summarize the findings and recommend future research in Chapter 5.

Keywords: $\mathbb{R}$-antimagic graphs, Regular graphs, Cartesian product of graphs, Uniformly $\mathbb{R}$-antimagic graphs

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Chapter 1

Introduction

We begin this chapter by introducing some fundamental definitions and notations that will be used throughout this thesis. Following that, we look at the literature on antimagic labeling and give an overview of this study.

1.1 Fundamental definitions and notations

This section will go over some fundamental definitions and notations in graph theory which are used in this thesis. We primarily adhere to the standard terminologies and notations found in West's graph theory textbook [21].

A *graph* G is represented by the ordered pair $G = (V, E)$, where V is a collection of elements referred to as *vertices* and E is a collection of unordered pairs of vertices referred to as *edges*. The set V (or $V(G)$) is referred to as the *vertex set* of G, and the set E (or $E(G)$) is referred to as the *edge set* of G. In a graph G, an edge is a two-element subset of V denoted by $e = xy$, where $e \in E$, x and y are referred to as the *endpoints* of e. A vertex x is said to be *adjacent* to the vertex y if there is an edge between x and y, and an edge e is said to be *incident* to the vertex x if x is an endpoint of e.

The set of edges incident to v in graph G is denoted by $E_G(v)$. In a graph, a *clique* is a set of vertices that are adjacent to each other, and a set of pairwise nonadjacent vertices in a graph is called an *independent set*. If the vertices of a graph can be partitioned into a clique and an independent set, it is a *split graph*. The *order* of graph G, denoted by $|V(G)|$, is defined as the cardinality of the set V. The cardinality of the set E, denoted by $|E(G)|$, is the *size* of the graph

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G. If both the vertex set and the edge set of a graph are finite, the graph is considered to be *finite*.

If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G, and we denote as $H \subseteq G$. We call H is a *spanning subgraph* of a graph G if $V(H) = V(G)$. A subgraph $H = (V', E')$ of $G = (V, E)$ is called an *induced subgraph* of G if E' consists of all edges of G that join vertices in V'. For $S \subseteq V(G)$, we denote $G[S]$ as the *induced subgraph* of G, which is the subgraph induced by S. The graph $G-S$ is the subgraph of G induced by $V(G)-S$. For $S = \{v\}$, we denote $G-S$ by $G-v$. Similarly, $G-uv$ is the graph obtained from G by deleting the edge uv, and also $G + uv$ denotes the graph obtained from G by adding edge uv.

In a graph G, the *neighborhood* of a vertex v, written as $N_G(v)$ or $N(v)$, is the set of all vertices adjacent to v. In the graph G, the number of edges incident to v is called the degree of a vertex v, denoted by $deg_G(v)$ or $deg(v)$. The maximum degree of G, denoted by $\Delta(G)$, is the maximum of $deg_G(v)$ over all vertices v in $V(G)$, and the minimum degree of G, written as $\delta(G)$, is the minimum of $deg_G(v)$ over all vertices v in $V(G)$, i.e., $\Delta(G) \geq deg_G(v) \geq \delta(G)$, for all $v \in V(G)$. A vertex v is called an *isolated* vertex if it has no neighbor in G. A graph is said to be a k-regular graph if all its vertices have the same degree k.

A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints. A *simple* graph is a graph having no loops or multiple edges. Next we define some particular families of graphs studied in this thesis. A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. P_n denotes a path of order n. A *u, v-path* is a path whose vertices of degree 1 (its endpoints) are u and v; the others are internal vertices. The *length* of u, v-path is the size of u, v-path. The *distance* between vertices u and v on the u, v-path which is denoted by $d_G(u, v)$, is the least length of u, v-path. A *connected* graph G is a graph in which there is an $u, v-$path whenever $u, v \in V(G)$. Otherwise, a graph is called *disconnected*. A closed path, a u, v-path with $u = v$, is called a *cycle*. A cycle of order n, denoted by C_n. A *tree* is a graph that is connected and has no cycles.

A *complete graph* K_n of order n is a graph where every two distinct vertices are adjacent. The set of pairwise nonadjacent vertices in a graph is an *independent set*. A graph G is *bipartite* if $V(G)$ is the union of two disjoint independent sets, and each independent set is called a *partite set*. A *complete bipartite graph* is a simple bipartite graph such that two vertices are adjacent

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if and only if they are in different partite sets. When the partite sets have sizes r and s, the complete bipartite graph is denoted by $K_{r,s}$. The graph $K_{1,n}(n \geq 2)$ is represented by the star S_n, and mS_n is a star forest made up of disjointed $m(m \geq 1)$ copies of S_n. A *spider graph* is a tree with at most one vertex of degree greater than two. A *paw* is $K_{1,3} + e$. A *wheel* W_n, with n spokes, is a graph with a center v connected to all the n vertices in cycle C_n. The graphs mentioned above are all connected.

The *Cartesian product* of graphs G and H, denoted by $G \square H$, is the graph that has vertex set $V(G) \times V(H) = \{(u, v) \mid u \in V(G), v \in V(H)\}$; and two vertices (u, v) and (u', v') are adjacent in $G \square H$ if and only if (1) $u = u'$ and $(v, v') \in E(G)$, or (2) $v = v'$ and $(u, u') \in E(H)$.

1.2 Antimagicness of graphs

All graphs considered in this dissertation are finite, simple, and without isolated vertices. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{C}$ the set of complex numbers. Assume that G is a graph. We have the following definitions:

Definition 1.2.1. *The edge labeling of G with the labels in A is a bijection from $E(G)$ to A where A is a subset of $\mathbb{C}$ with $|A| = |E(G)|$.*

Definition 1.2.2. *If f is an edge labeling of G with labels in A, then we use $f^+(v)$ to denote $\sum_{e \in E_G(v)} f(e)$ for any vertex v of G, and $f^+(v)$ is called the vertex sum at v.*

Definition 1.2.3. *If B is a subset of $\mathbb{C}$ with $|B| \geq |E(G)|$ such that for each subset A of B with $|A| = |E(G)|$, there is an edge labeling f of G with labels in A such that $f^+(u)$ is not equal to $f^+(v)$ for any two distinct vertices u, v of G, then we say that f is a B-antimagic labeling of G.*

A graph G is called *B-antimagic* if G has a B-antimagic labeling.

In the literature, a graph G is *antimagic* if G is $\{1, 2, \cdots, |E(G)|\}$-antimagic. The concept of antimagic graphs was introduced by Hartsfield and Ringel [11] in 1990. They conjectured that every connected graph with at least two edges was antimagic. This conjecture has not been completely solved yet. In the field of antimagic graphs, graph theory researchers have made some efforts. Some partial results are listed below.

Graphs with maximum degree $|V(G)| - 1$ are antimagic. Alon et al. [1] used probabilistic methods and analytic number theory to show that there is an absolute constant C such that

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graphs with minimum degree $\delta(G) \geq C \log |V(G)|$ are antimagic. They also proved that complete partite graphs (other than K_2) and graphs with maximum degree at least $|V(G)| - 2$ are antimagic. Shang proved that all spiders are antimagic [15]. Later, Shang et al. [16] gave a criterion for $mS_2 \cup S_n (n \geq 3)$ to be antimagic.

The antimagicness for some special types of regular graphs was verified by Cranston [9], Cranston et al. [10], and Liang and Zhu [13]. According to Cranston [9], every regular bipartite graph (with a degree of at least two) is antimagic. Cranston relied heavily on the Marriage Theorem to prove this. Later, Cranston et al. [10] proved that all regular graphs with odd degree are antimagic. And, it has been proved by Chang et al. [5] that all regular graphs with even degree are antimagic. Hence, k-regular graphs are antimagic where $k \geq 2$.

Some studies have addressed the antimagicness of Cartesian products. Wang [18] proved that any Cartesian product of two or more cycles is antimagic. The general result also shows that $C_n \Box H$ is antimagic, where $n \geq 3$, and H is an antimagic k-regular graph ($k > 1$). Wang and Hsiao [19] later introduced new classes of antimagic graphs that were constructed using Cartesian products. They proved that $P_m \Box P_n$ ($m \geq n \geq 2$)and $G \Box P_n$ ($n \geq 2$) are antimagic, where G is a regular antimagic graph. Cheng independently proved more generalized results in [7, 8], which are the Cartesian products of two paths, as well as the Cartesian products of two or more regular graphs, are antimagic. Moreover, Zhang and Sun [22] proved that if a regular graph G is antimagic, then for any connected graph H, the Cartesian product $G \Box H$ is antimagic.

Assume that $\mathbb{R}^+$ denotes the set of positive numbers. Matamala et al. [14] proposed the concept of *universal antimagic* graphs, and a graph G is universal antimagic if G is $\mathbb{R}^+$-antimagic. They proved that paths, cycles, and graphs whose connected components are cycles or paths of odd lengths are universal antimagic. Split graphs, as well as any graph containing a complete bipartite graph as a spanning subgraph, are shown to be universal antimagic in their paper.

1.3 Overview of the thesis

In this thesis, we generalize further and define $\mathbb{R}$-antimagic graphs.

In Chapter 2, we prove that every $\mathbb{R}$-antimagic graph is $\mathbb{C}$-amtimagic. We also show that every $\mathbb{R}^+$-antimagic graph is also $\mathbb{R}$-antimagic if the graph is regular. Additionally, we propose

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that a class of regular graphs is $\mathbb{R}$-antimagic.

In Chapter 3, we show that Cartesian products $G_1 \Box G_2 \Box \cdots \Box G_n$ $(n \geq 2)$ are $\mathbb{R}$-antimagic, where each G_i is a complete graph of order ≥ 2 or a cycle. The methods of labeling on Cartesian products of cycles used in this paper are similar in [7, 18]. We present efficient algorithms for finding edge labelings of Cartesian products of cycles and complete graphs in Chapter 3.

In Chapter 4, we show that wheels, paws, and paths of order ≥ 6 are $\mathbb{R}$-antimagic.

In Chapter 5, we summarize our results and make suggestions for future studies.

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Chapter 2

$\mathbb{R}$-antimagic regular graphs

In this chapter, we will prove that $\mathbb{R}^+$-antimagic regular graphs are $\mathbb{R}$-antimagic. Furthermore, we also prove that a spacial class of regular graphs is $\mathbb{R}$-antimagic.

2.1 $\mathbb{R}^+ \cup \{0\}$-antimagic graphs

The concept of universal antimagic graphs is proposed in [14]. In this section, we introduce $\mathbb{R}^+ \cup \{0\}$-antimagic graphs. It is easy to see that S_n ($n \geq 3$) is $\mathbb{R}^+ \cup \{0\}$-antimagic. Moreover, we have the following results.

Theorem 2.1.1. *If G is a connected graph of order ≥ 3 with $\Delta(G) = |V(G)| - 1$ and $G \neq S_2$, then G is $\mathbb{R}^+ \cup \{0\}$-antimagic.*

Proof. Assume $|V(G)| = n$, $|E(G)| = m$ and v is a vetex of G with degree $n-1$ (see Figure 2.1). Let $r_1 > r_2 > r_3 > \cdots > r_{n-1} > r_n > \cdots > r_m$ be the arbitrarily given nonnegative numbers. First, we arbitrarily assign labels in $\{r_n, r_{n+1}, r_{n+2}, \ldots, r_m\}$ to the edges in $G-v$. Denote this labeling of $G-v$ by g, and $g^+(w)$ is the vertex sum of w under the labeling g for each vertex w of $G-v$. We order the vertices of $G-v$ as $v_1, v_2, \ldots, v_{n-1}$ in such a way that $g^+(v_1) \geq g^+(v_2) \geq \cdots \geq g^+(v_{n-1})$. Then we assign the remaining $n-1$ real numbers to the edges $vv_1, vv_2, \ldots, vv_{n-1}$ in decreasing order, i.e., assign r_i to vv_i. We define an edge labeling f of G with labels in $\{r_1, r_2, \ldots, r_{n-1}, r_n, \ldots, r_m\}$ by $f(e) = g(e)$ if $e \in E(G-v)$, and $f(vv_i) = r_i$, $i = 1, 2, \ldots, n-1$. Then the vertex sum of v_i, $f^+(v_i) = g^+(v_i) + r_i$ for $i = 1, 2, \ldots, n-1$, and $f^+(v) = \sum_{1 \leq i \leq n-1} r_i$. Since $deg(v) \geq deg(v_i)$, $i = 1, 2, \ldots, n-1$, we

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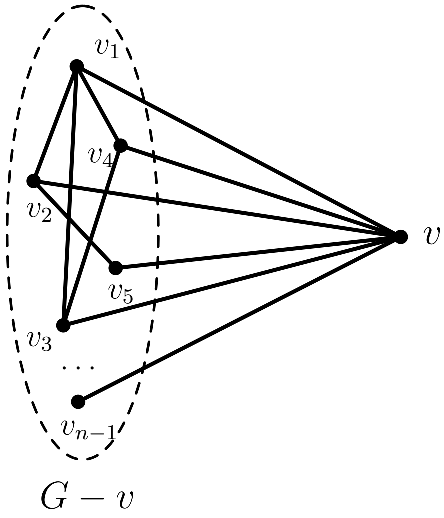

Figure 2.1: $\Delta(G) = |V(G)| - 1$

obtain $f^+(v) > f^+(v_1) > f^+(v_2) > \cdots > f^+(v_{n-1})$. The labeling is $\mathbb{R}^+ \cup \{0\}$-antimagic labeling. □

We prove that stars are $\mathbb{R}^+ \cup \{0\}$-antimagic, but not $\mathbb{R}$-antimagic.

Remark 2.1.2. *S_n $(n \geq 3)$ is $\mathbb{R}^+ \cup \{0\}$-antimagic, but not $\mathbb{R}$-antimagic.*

Proof. Let S_n be the star with $V(S_n) = \{v_1, v_2, \cdots, v_n\} \cup \{v\}$ and $E(S_n) = \{vv_i | \ i = 1, 2, \cdots, n\}$. Since $\Delta(S_n) = n = |V(G)| - 1$, we have, by Theorem 2.1.1, that S_n is $\mathbb{R}^+ \cup \{0\}$-antimagic.

Now, we prove S_n $(n \geq 3)$ is not $\mathbb{R}$-antimagic. Let $r_1 < r_2 < r_3 < \cdots < r_n$ be real numbers with $r_1 + r_2 + \cdots + r_{n-1} = 0$. Let f be an arbitrary edge labeling of S_n with labels in $\{r_1, r_2, r_3, \cdots, r_n\}$. Without loss of generality, f is defined by $f(vv_i) = r_i$ for $i = 1, 2, \cdots, n$ (see Figure 2.2). We see that $f^{\,+}(v_n) = r_n = r_1 + r_2 + \cdots + r_{n-1} + r_n = f^{\,+}(v)$. Accordingly, S_n is not $\{r_1, r_2, r_3, \cdots, r_n\}$-antimagic, which results in S_n not $\mathbb{R}$-antimagic. □

As a result, stars, complete graphs, and wheels are $\mathbb{R}^+ \cup \{0\}$-antimagic. We illustrate that wheels and complete graphs are $\mathbb{R}$-antimagic in Chapters 3 and 4.

2.2 $\mathbb{R}$-antimagic graphs and $\mathbb{C}$-antimagic graphs

Assume that G is a graph. The main result of this section is that G is $\mathbb{R}$-antimagic if and only if G is $\mathbb{C}$-antimagic. We begin with the following lemmas:

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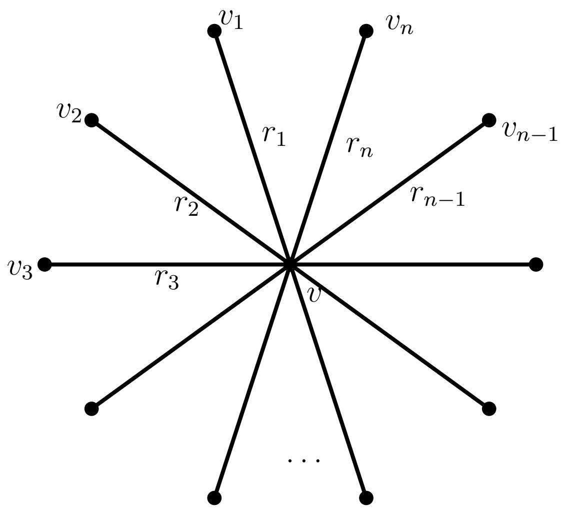

Figure 2.2: Edge labeling of star

Lemma 2.2.1. *Assume G is a graph and $A \subseteq B \subseteq \mathbb{C}$ with $|A| \geq |E(G)|$. If G is B-antimagic, then G is A-antimagic.*

Lemma 2.2.1 implies that if a graph G is $\mathbb{C}$-antimagic, it is $\mathbb{R}$-antimagic, and if G is $\mathbb{R}$-antimagic, it is $\mathbb{R}^+ \cup \{0\}$-antimagic.

Lemma 2.2.2. *Suppose that $a_1 + b_1 i$, $a_2 + b_2 i$, $\dots$,$a_m + b_m i$ $(m \geq 2)$ are distinct complex numbers, where a_1, a_2, $\dots$, a_m, b_1, b_2, $\dots$, b_m are real numbers. Then there exists $r \in \mathbb{R}$ such that $a_1 + rb_1$, $a_2 + rb_2$, $\dots$,$a_m + rb_m$ are all distinct.*

Proof. Let $f_1(x) = a_1 + b_1 x$, $f_2(x) = a_2 + b_2 x, \dots, f_m(x) = a_m + b_m x$ be linear real functions. From the assumption, we see that $f_1(x)$, $f_2(x)$, $\dots$, $f_m(x)$ are distinct linear real functions. Then, there exists $r \in \mathbb{R}$ such that $f_1(r), f_2(r), \dots, f_m(r)$ are distinct, i.e., $a_1 + rb_1$, $a_2 + rb_2$, $\dots$,$a_m + rb_m$ are all distinct. □

Lemma 2.2.3. *Let a, b, a', b', r be real numbers. Suppose that $a + rb \neq a' + rb'$. Then, $a + bi \neq a' + b'i$.*

Proof. Suppose, on the contrary, that $a + bi = a' + b'i$. Since a, b, a', b', r be real numbers, we have $a = a'$ and $b = b'$, which implies $a + rb = a' + rb'$, contradicting the assumption. This confirms the lemma. □

Theorem 2.2.4. *A graph G is $\mathbb{R}$-antimagic if and only if G is $\mathbb{C}$-antimagic.*

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Proof. The "if" part follows from Lemma 2.2.1. Now, we prove the "only if" part.

We show that G is $\mathbb{C}$-antimagic if G is $\mathbb{R}$-antimagic. Let $\mid E(G) \mid = m$. Arbitrarily give m distinct complex numbers $a_1 + b_1 i$, $a_2 + b_2 i$, $\ldots$, $a_m + b_m i$, where a_1, a_2, $\ldots$, a_m, b_1, b_2, $\ldots$, b_m are real numbers. By Lemma 2.2.2, there exists $r \in \mathbb{R}$ such that $a_1 + rb_1$, $a_2 + rb_2$, $\ldots$, $a_m + rb_m$ are all distinct. Since G is $\mathbb{R}$-antimagic, there exists an edge labeling f of G with labels in $\{a_1 + rb_1, a_2 + rb_2, \ldots, a_m + rb_m\}$ such that the vertex sums of G are all distinct.

Let f^* be an edge labeling of G with labels in $\{a_1 + b_1 i, a_2 + b_2 i, \ldots, a_m + b_m i\}$ defined by $f^*(e) = a_j + b_j i$, if $f(e) = a_j + rb_j (1 \le j \le m)$ where $e \in E(G)$. We see that if the vertex sum of f^* at a vertex v is $A + Bi$, then the vertex sum of f at a vertex v is $A + rB$. Since the vertex sums of f are all distinct, we have, by Lemma 2.2.3, the vertex sums of f^* are all distinct. This completes the proof. □

2.3 A class of $\mathbb{R}$-antimagic regular graphs

Some results of the $\mathbb{R}$-antimagicness of regular graphs are obtained in this section.

Let $A \subseteq \mathbb{C}$ and $\alpha \in \mathbb{C}$. We denote the set $\{\alpha a \mid a \in A\}$ by αA, and denote $(-1)A$ by $-A$. We have the following lemma:

Lemma 2.3.1. *Assume that G is a graph, and A is the subset of $\mathbb{C}$, and $\alpha \in \mathbb{C} - \{0\}$. If G is A-antimagic, then*

(1) *G is αA-antimagic.*

(2) *G is $-A$-antimagic.*

Proof. (1) Assume that $|E(G)| = m$. Arbitrarily give m distinct complex numbers αz_1, αz_2, $\ldots$, αz_m, where z_1, z_2, $\ldots$, z_m are in $A \subseteq \mathbb{C}$, and $\alpha \neq 0$. We see that $\alpha z_i \neq \alpha z_j$ if and only if $z_i \neq z_j$. Because z_1, z_2, $\ldots$, and z_m are all distinct, αz_1, αz_2, $\ldots$, and αz_m are all distinct as well. Since G is A-antimagic, there exists an edge labeling f of G with labels in $\{z_1, z_2, \ldots, z_m\}$ such that the vertex sums f^+ of G are all distinct.

Let $g(e) = \alpha f(e)$ for all $e \in E(G)$. Then g is an edge labeling of G with labels in $\{\alpha z_1, \alpha z_2, \ldots, \alpha z_m\}$. Because the vertex sums f^+ of G are all distinct, the vertex sums g^+ of G are also distinct. This completes the proof.

(2) Since $-A = (-1)A$, (2) follows from (1). □

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Lemma 2.3.2. *Assume that G is a regular graph and $|E(G)| = m$. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be distinct complex numbers, and α be a nonzero complex number. If G is $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$-antimagic, then G is $\{\alpha_1 + \alpha, \alpha_2 + \alpha, \ldots, \alpha_m + \alpha\}$-antimagic.*

Proof. Assume that G is a k-regular graph and G is $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$-antimagic. Then there exist an edge labeling f, such that the vertex sums $f^+(v) = \sum_{e \in E_G(v)} f(e)$ of G are all distinct. Let $B = \{\alpha_1 + \alpha, \alpha_2 + \alpha, \ldots, \alpha_m + \alpha\}$, and $g(e) = f(e) + \alpha$. Then g is an edge labeling with labels in B. Since G is k-regular and f^+ are all distinct, we obtain that $g^+(v) = \sum_{e \in E_G(v)} g(e) = \sum_{e \in E_G(v)} f(e) + k\alpha$ are all distinct. Hence, G is B-antimagic. □

Theorem 2.3.3. *Assume that G is a regular graph. Let $a, b \in \mathbb{R}$ and $a < b$. The following statements are equivalent.*

(1) *G is $\mathbb{R}$-antimagic.*

(2) *G is $\mathbb{R}^+$-antimagic.*

(3) *G is (a, b)-antimagic.*

Proof. Assume that G is a regular graph of size m.

(1) ⇒ (3).

Since $(a, b) \subseteq \mathbb{R}$ and G is $\mathbb{R}$-antimagic, G is (a, b)-antimagic by Lemma 2.2.1.

(3) ⇒ (2).

Let $r_1, r_2, \ldots, r_m$ be the arbitrarily given positive numbers where $0 < r_1 < r_2 < \cdots < r_m$. For $a < b$, assume that $t_i = a + (b - a)\frac{r_i}{r_m + 1}$ for $i = 1, 2, \ldots, m$. Since $0 < r_1 < r_i < r_m < r_m + 1$ and $a < b$, we can have

$$a < a + (b - a)\frac{r_i}{r_m + 1} = t_i < b$$

for $i = 1, 2, \ldots, m$. By Lemma 2.2.1, if G is (a, b)-antimagic, then G is $\{t_1, t_2, \ldots, t_m\}$-antimagic. Therefore, G is $\{r_1, r_2, \ldots, r_m\}$-antimagic by Lemmas 2.3.1 and 2.3.2. Since $\{r_1, r_2, \ldots, r_m\} \subseteq \mathbb{R}^+$ is arbitrary, we obtain that G is $\mathbb{R}^+$-antimagic.

(2) ⇒ (1).

Let $r_1, r_2, \ldots, r_m$ be the arbitrarily given real numbers where $r_1 < r_2 < \cdots < r_m$. For $\alpha > 0$, assume that $t_i = r_i - r_1 + \alpha$ for $i = 1, 2, \ldots, m$. Then, we can have $t_i = r_i - r_1 + \alpha > 0$.

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By Lemma 2.2.1, if G is $\mathbb{R}^+$-antimagic, then G is $\{t_1, t_2, \ldots, t_m\}$-antimagic. Therefore, G is $\{r_1, r_2, \ldots, r_m\}$-antimagic by Lemma 2.3.2. Since $\{r_1, r_2, \ldots, r_m\} \subseteq \mathbb{R}$ is arbitrary, we obtain that G is $\mathbb{R}$-antimagic. □

Let G be a graph and A be a subset of $\mathbb{R}$ with $|A| = |E(G)|$. If g is an edge labeling of G with labels in A and K, L are nonempty subsets of $E(G)$ such that $g(x) < g(y)$ for all $x \in K$, $y \in L$, then we write $K \prec L$ *under* g. It is easy to see that the relation $\prec$ is transitive (i.e., if K, L, M are nonempty subsets of $E(G)$, and $K \prec L$, $L \prec M$, then $K \prec M$).

When proving Theorems 2.3.5, the following useful lemma will be utilized.

Lemma 2.3.4. *Let G be an arbitrary graph and A be a subset of $\mathbb{R}$ with $|A| = |E(G)|$. Let f be an edge labeling of G with labels in A. Suppose that A_1, A_2, B_1, B_2 are pairwise disjoint nonempty subsets of the edge set $E(G)$ with $|A_2| + |B_1| \leq k$, $|A_1| + |A_2| = |B_1| + |B_2| = k$ such that $A_1 \prec B_1 \cup B_2$ and $A_2 \prec B_2$ under f. Then*

$$\sum_{e \in A_1 \cup A_2} f(e) < \sum_{e \in B_1 \cup B_2} f(e). \tag{2.3.1}$$

Proof. Since A_1, A_2, B_1, B_2 are pairwise disjoint, and $|A_1| + |A_2| = |B_1| + |B_2| = k$. Consider that $A_1 \cup A_2 = \{a_1, a_2, \ldots, a_k\}$ and $B_1 \cup B_2 = \{b_1, b_2, \ldots, b_k\}$. Since $|A_2| + |B_1| \leq k$, we can assume that $|A_2| = t$ and $|B_1| \leq k - t$. Let $s = |B_2| = k - |B_1|$ and $A_2 = \{a_1, a_2, \ldots, a_t\}$ and $B_2 = \{b_1, b_2, \ldots, b_s\}$. Hence, $s = |B_2| = k - |B_1| \geq t = |A_2|$, i.e., $0 \leq t \leq s \leq k$. Since A_1, A_2, B_1, B_2 are pairwise disjoint, we have

$$\sum_{e \in A_1 \cup A_2} f(e) = \sum_{e \in A_2} f(e) + \sum_{e \in A_1} f(e), \tag{2.3.2}$$

and

$$\sum_{e \in B_1 \cup B_2} f(e) = \sum_{e \in B_2} f(e) + \sum_{e \in B_1} f(e) = \sum_{1 \leq i \leq s} f(b_i) + \sum_{s+1 \leq i \leq k} f(b_i) = \sum_{1 \leq i \leq t} f(b_i) + \sum_{t+1 \leq i \leq k} f(b_i). \tag{2.3.3}$$

Since $A_2 \prec B_2$ under f and $|A_2| = t$, we have $f(a) < f(b)$ for all $a \in A_2$, $b \in B_2$. Hence

$$\sum_{e \in A_2} f(e) < \sum_{1 \leq i \leq t} f(b_i). \tag{2.3.4}$$

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Also note that $A_1 \prec B_1 \cup B_2$ under f, we have $f(a) < f(b)$ for all $a \in A_1, b \in B_1 \cup B_2$. Then

$$\sum_{e \in A_1} f(e) = \sum_{t+1 \leq i \leq k} f(a_i) < \sum_{t+1 \leq i \leq k} f(b_i). \tag{2.3.5}$$

Combining (2.3.2)∼(2.3.5), we obtain

$$\sum_{e \in A_1 \cup A_2} f(e) < \sum_{e \in B_1 \cup B_2} f(e). \tag{2.3.6}$$

□

Assume G is a graph and $H \subseteq V(G)$ and $K \subseteq V(G)$. We define $G[H, K]$ as the bipartite subgraph of G induced by edges between H and K. The order of edge sets labeled in Theorem 2.3.5 is identical to that in [13]. Because of the difference in labels, we have the following theorem.

Theorem 2.3.5. *Let G be a k-regular graph, and v_0 be an arbitrarily given vertex of G. Assume that every vertex of G has distance at most p to v_0. Let $L_j = \{u \mid d_G(u, v_0) = j\}$ for $0 \leq j \leq p$. For all $x \in L_{j-1}$ and $y \in L_j$ ($j \geq 1$), if either*

1. *x is not adjacent to y and $deg_{G[L_{j-1},L_i]}(x) + deg_{G[L_{j-1},L_j]}(y) \leq k$ or*
2. *x is adjacent to y and $deg_{G[L_{j-1},L_j]}(x) + deg_{G[L_{j-1},L_j]}(y) \leq k + 1$,*

then G is $\mathbb{R}$-antimagic.

Proof. Let G be a k-reqular graph, and v_0 be a vertex of G. Assume that $V(G) = L_0 \cup L_1 \cup \cdots \cup L_p$ where $L_p \neq \varnothing$. For each $u \in V(G)$, $u \in L_j$ ($j = 1, 2, \ldots, p$), we arbitrarily choose an edge joining u and a vertex in L_{j-1}, and denote this edge by $\tau(u)$. Clearly, $\tau(u) \in G[L_{j\text{-}1}, L_j]$. We shall denote by $G[L_j]$ the subgraph of G induced by L_j. Note that each vertex in L_j is incident to at least one edge of $G[L_{j\text{-}1}, L_j]$. Let $E_j = E(G[L_j])$, $E'_j = \{\tau(u) \mid u \in L_j\}$, and $E''_j = E(G[L_{j\text{-}1}, L_j]) - E'_j$ for $j = 1, 2, \ldots, p$. We see that the edge set $E(G)$ is the union of E_j, E'_j and E''_j ($j = 1, 2, \ldots, p$). Also note that E_j, E'_j and E''_j are pairwise disjoint. Now we prove that G is $\mathbb{R}$-antimagic. Let $A \subseteq \mathbb{R}$ with $|A| = |E(G)|$ be arbitrarily given. Let f to be an edge labeling of G with labels in A. The edge sets will be labeled sequentially as follows:

$$E_p, E''_p, E'_p, E_{p-1}, E''_{p-1}, E'_{p-1}, \ldots, E_1, E''_1, E'_1$$

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For $v \in L_j$, $j \geq 1$, we define

$$s(v) = \sum_{e \in E(G[L_j, L_{j+1}]) \cap E_G(v)} f(e) + \sum_{e \in E_j \cap E_G(v)} f(e) + \sum_{e \in E''_j \cap E_G(v)} f(e)$$

where $E_G(v)$ denotes the set of edge incident to v. Let $|L_j| = n_j$ for $j = 1, 2, \cdots, p$. The labels of the edges in E_j and E''_j are arbitrary. Then the labeling f of G with labels in A satisfies Rules 2.3.1∼2.3.3:

Rule 2.3.1. For $j = 1, 2, \cdots, p$, $E_j \prec E''_j \prec E'_j$.

Rule 2.3.2. For $j = 2, 3, \cdots, p$, $E'_j \prec E_{j-1}$.

Rule 2.3.3. For $j = 1, 2, \cdots, p$, if $s(u) \leq s(v)$ then $f(\tau(u)) < f(\tau(v))$ for all $u, v \in L_j$, $u \neq v$.

Claim 1. *For $j = 1, 2, \cdots, p$, $u, v \in L_j$, $u \neq v$, then $f^{\,+}(u) \neq f^{\,+}(v)$.*

Check of Claim 1. For $j = 1, 2, \cdots, p$,

$$\begin{aligned} f^{\,+}(u) &= \sum_{e \in E(G[L_j, L_{j+1}]) \cap E_G(u)} f(e) + \sum_{e \in E_j \cap E_G(u)} f(e) + \sum_{e \in E''_j \cap E_G(u)} f(e) + f(\tau(u)) \\ &= s(u) + f(\tau(u)), \end{aligned} \tag{2.3.7}$$

and

$$\begin{aligned} f^{\,+}(v) &= \sum_{e \in E(G[L_j, L_{j+1}]) \cap E_G(v)} f(e) + \sum_{e \in E_j \cap E_G(v)} f(e) + \sum_{e \in E''_j \cap E_G(v)} f(e) + f(\tau(v)) \\ &= s(v) + f(\tau(v)). \end{aligned} \tag{2.3.8}$$

Without loss of generality, we may assume $s(u) \leq s(v)$. By Rule 2.3.3, since $s(u) \leq s(v)$, we have $f(\tau(u)) < f(\tau(v))$. Therefore, $f^{\,+}(u) < f^{\,+}(v)$ for $j = 1, 2, \cdots, p$. This completes the check of Claim 1.

Claim 2. *For $j = 1, 2, \cdots, p$, $f^{\,+}(y) < f^{\,+}(x)$ for all $x \in L_{j-1}, y \in L_j$.*

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Check of Claim 2. Let

$$
\begin{aligned}
A_1 &= (E(G\,[L_j, L_{j+1}]) \cup E_j) \cap E_G(y), \\
A_2 &= E(G\,[L_{j-1}, L_j]) \cap E_G(y), \\
B_1 &= E(G\,[L_{j-1}, L_j]) \cap E_G(x), \\
B_2 &= (E_{j-1} \cup E(G\,[L_{j-2}, L_{j-1}])) \cap E_G(x).
\end{aligned} \tag{2.3.9}
$$

We can see that

$$
\begin{aligned}
f^{\,+}(y) = &\sum_{e \in E(G[L_j, L_{j+1}] \cap E_G(y)} f(e) + \sum_{e \in E_j \cap E_G(y)} f(e) + \sum_{e \in E''_j \cap E_G(y)} f(e) \\
&+ f(\tau(y))
\end{aligned} \tag{2.3.10}
$$

and

$$
\begin{aligned}
f^{\,+}(x) = &\sum_{e' \in E(G[L_{j-1}, L_j] \cap E_G(x)} f(e') + \sum_{e' \in E_{j-1} \cap E_G(x)} f(e') + \sum_{e' \in E''_{j-1} \cap E_G(x)} f(e') \\
&+ f(\tau(x))
\end{aligned} \tag{2.3.11}
$$

Since $\tau(y) \in E'_j$ and $\tau(x) \in E'_{j-1}$, we rewrite

$$
f^{\,+}(y) = \sum_{e \in A_1 \cup A_2} f(e), \tag{2.3.12}
$$

and

$$
f^{\,+}(x) = \sum_{e' \in B_1 \cup B_2} f(e'), \tag{2.3.13}
$$

We distinguish two cases:

Case 1. *x is not adjacent to y (see Figure 2.3).*

Since G be a k-reqular graph, and A_1, A_2, B_1, B_2 are pairwise disjoint nonempty subsets of the edge set $E(G)$, we have $|A_1| + |A_2| = |B_1| + |B_2| = k$. Since $deg_{G[L_{j-1}, L_j]}(x) + deg_{G[L_{j-1}, L_j]}(y) \leq k$ for all $x \in L_{j-1}$ and $y \in L_j$, $j \geq 1$, we have $|A_2| + |B_1| \leq k$. From Rule 2.3.1 and Rule 2.3.2, since

$$
E''_{j+1} \prec E'_{j+1} \prec E_j \prec E''_j \prec E'_j \prec E_{j-1} \prec E''_{j-1} \prec E'_{j-1},
$$

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we have $A_1 \prec B_1 \cup B_2$ and $A_2 \prec B_2$ under f. By Lemma 2.3.4, we obtain

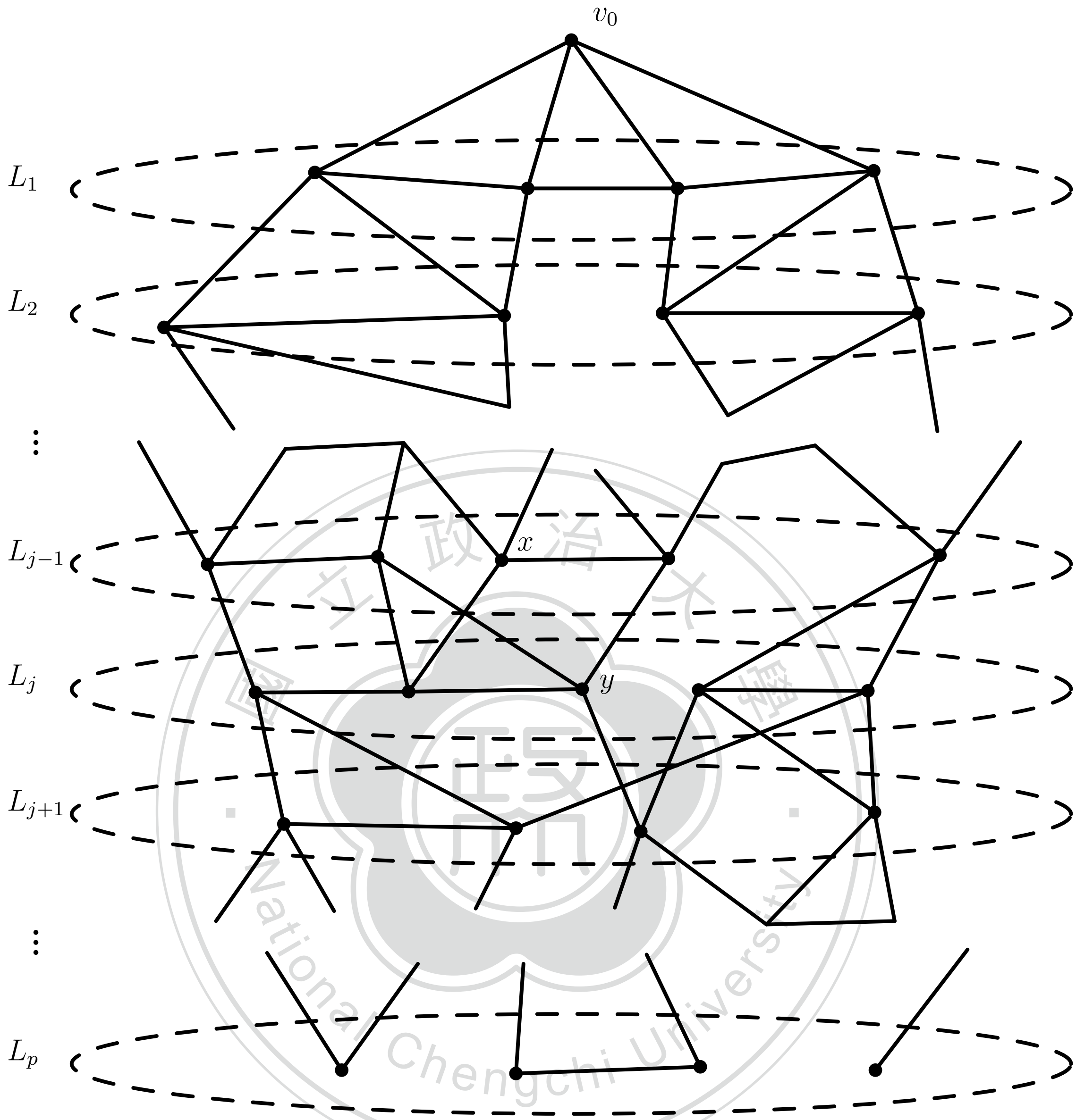

Figure 2.3: x is not adjacent to y

$$f^+(y) = \sum_{e \in A_1 \cup A_2} f(e) < \sum_{e' \in B_1 \cup B_2} f(e') = f^+(x). \tag{2.3.14}$$

Case 2. *x is adjacent to y (see Figure 2.4).*

Let $\epsilon = xy$. We see that $A_2 \cap B_1 = \{\epsilon\}$. Let $C = A_2 - \{\epsilon\}$ and $D = B_1 - \{\epsilon\}$. Then A_1, C, D, B_2 are pairwise disjoint nonempty subsets of the edge set $E(G)$, we have $|A_1| + |C| = |D| + |B_2| = k - 1$. Since $deg_{G[L_{j-1},L_j]}(x) + deg_{G[L_{j-1},L_j]}(y) \leq k + 1$ for all

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$x \in L_{j-1}$ and $y \in L_j$, $j \geq 1$, we have $|C| + |D| \leq k - 1$. From Rule 2.3.1 and Rule 2.3.2, since

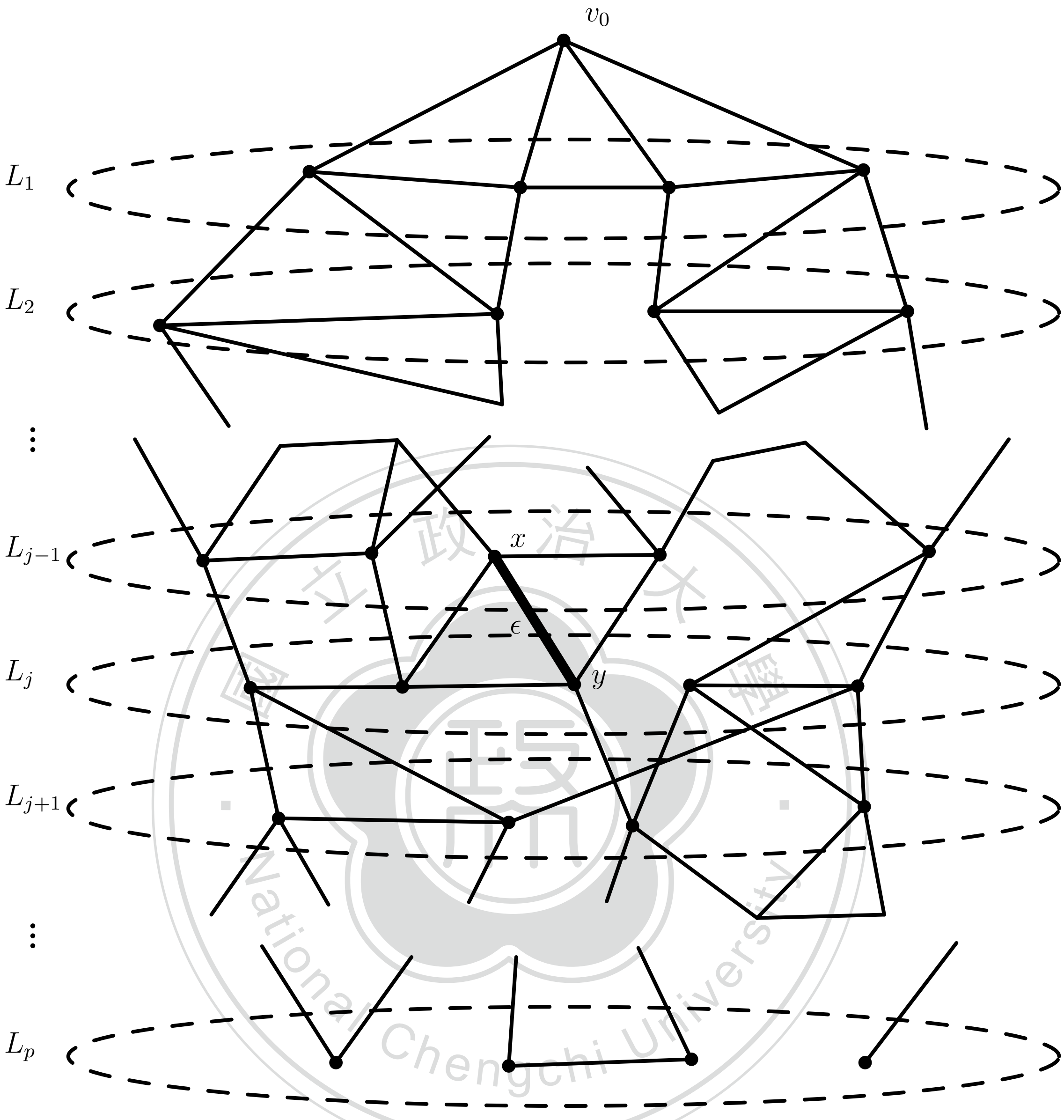

Figure 2.4: x is adjacent to y

$$E''_{j+1} \prec E'_{j+1} \prec E_j \prec E''_j \prec E'_j \prec E_{j-1} \prec E''_{j-1} \prec E'_{j-1},$$

we have $A_1 \prec D \cup B_2$ and $C \prec B_2$ under f. By Lemma 2.3.4, we obtain

$$\sum_{e \in A_1 \cup C} f(e) < \sum_{e' \in D \cup B_2} f(e'). \tag{2.3.15}$$

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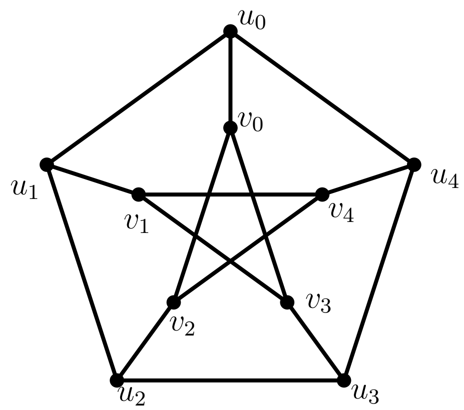

Figure 2.5: Peterson Graph

That implies

$$\begin{aligned} f^{+}(y) &= \sum_{e \in A_1 \cup A_2} f(e) = \sum_{e \in A_1 \cup C} f(e) + f(\epsilon) \\ &< \sum_{e' \in D \cup B_2} f(e') + f(\epsilon) = \sum_{e' \in B_1 \cup B_2} f(e') = f^{+}(x). \end{aligned} \tag{2.3.16}$$

This completes the check of Claim 2.

From Claim 1 and Claim 2, we can have G is $\mathbb{R}$-antimagic. □

As a result, all cycles are $\mathbb{R}$-antimagic. We will discuss cycles in Chapter 3. Additionally, they have another property of $\mathbb{R}$-antimagic.

Petersen graphs are graphs with the vertex set $\{u_0, u_1, \ldots, u_4, v_0, v_1, \ldots, v_4, \}$ and the edge set $\{u_i u_{i+1} \mid i = 0, 1, 2, 3, 4\} \cup \{v_i v_{i+2} \mid i = 0, 1, 2, 3, 4\} \cup \{u_i v_i \mid i = 0, 1, 2, 3, 4\}$ with indices taken by modulo 5 (see Figure 2.5). The Petersen graph is without a doubt one of the most well-known objects encountered by graph theorists. The following corollary shows that the Peterson graph is $\mathbb{R}$-antimagic.

Corollary 2.3.1. *Peterson graph is $\mathbb{R}$-antimagic.*

Proof. Let G be the Peterson graph. We draw the Petersen graph in another way (see Figure 2.6). Let

$$\begin{aligned} V_0 &= \{u_1\}, \\ V_1 &= \{u_2, u_6, u_5\}, \\ V_2 &= \{u_3, u_4, u_7, u_8, u_9, u_{10}\}. \end{aligned} \tag{2.3.17}$$

Therefore, V_0, V_1, and V_2 form a partition of $V(G)$, and $V_i = \{u \mid d_G(u, u_1) = i\}$. Note that

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$G[V_1, V2]$ is the bipartite subgraph of G induced by edges between V_1 and V_2. Since Peterson graph is 3-regular, and $deg_{G[V_1,V2]}(x) + deg_{G[V_1,V2]}(y) = 3$ for all $x \in V_1$ and $y \in V_2$. By Theorem 2.3.5, we can obtain that G is $\mathbb{R}$-antimagic. Then, we complete the proof. $\square$

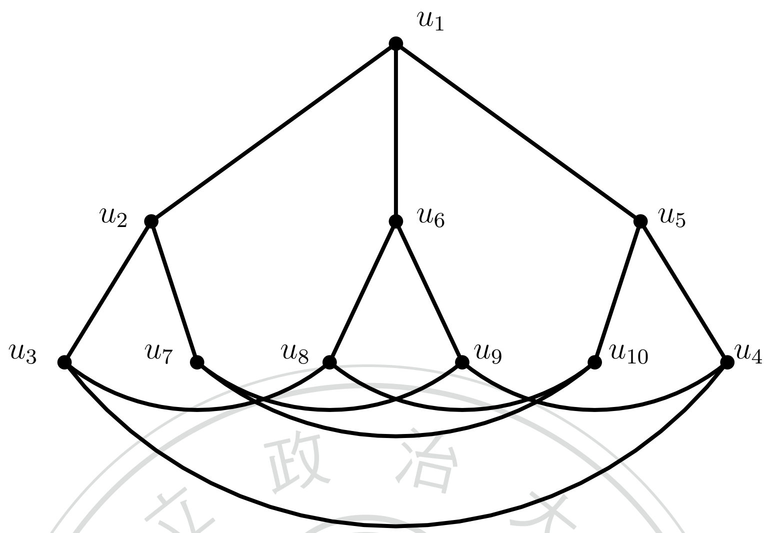

Figure 2.6: Another form of Peterson graph

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Chapter 3

Uniformly $\mathbb{R}$-antimagic graphs

The main result in this chapter is that the Cartesian products $G_1 \Box G_2 \Box \cdots \Box G_n$ $(n \geq 2)$ are $\mathbb{R}$-antimagic, where each G_i is a complete graph of order ≥ 2 or a cycle.

3.1 Cycles and complete graphs

To prove the results in this section, we need the concept of *uniformly $\mathbb{R}$-antimagic graphs*, which is defined below.

Definition 3.1.1. *Let G be a graph. Suppose that all the vertices of G can be listed as $u_1, u_2, \cdots, u_m$ such that for every $A \subseteq \mathbb{R}$ with $|A| = |E(G)|$, there is an edge labeling f of G with labels in A such that $f^+(u_1) < f^+(u_2) < \cdots < f^+(u_m)$. Then we say that G is uniformly $\mathbb{R}$-antimagic, and that the sequence of vertices $u_1, u_2, \cdots, u_m$ has the uniformly $\mathbb{R}$-antimagic property.*

Note that in this definition, the ordering of the vertices $u_1, u_2, \cdots, u_m$ satisfying the property $f^+(u_1) < f^+(u_2) < \cdots < f^+(u_m)$ is independent of the choice of the subset A of $\mathbb{R}$. Obviously, every uniformly $\mathbb{R}$-antimagic graph is $\mathbb{R}$-antimagic.

We are the first to define $\mathbb{R}$-antimagic graphs and to propose the uniformly $\mathbb{R}$-antimagic property. Some of our results are shown in [6]. Before proving our main result, we describe uniformly $\mathbb{R}$-antimagic property on cycles and complete graphs.

Theorem 3.1.2. *[6] Every cycle is uniformly $\mathbb{R}$-antimagic.*

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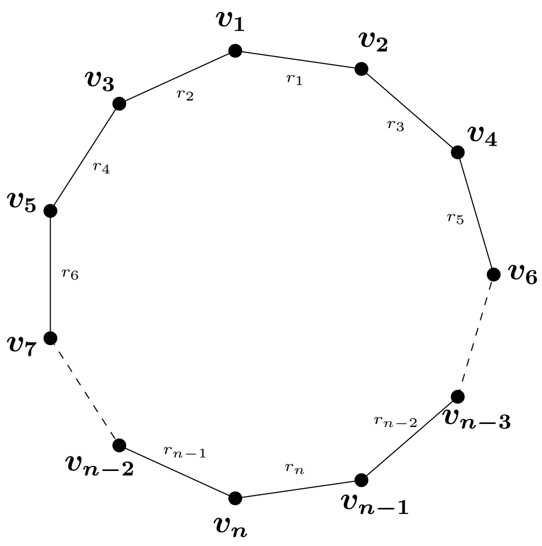

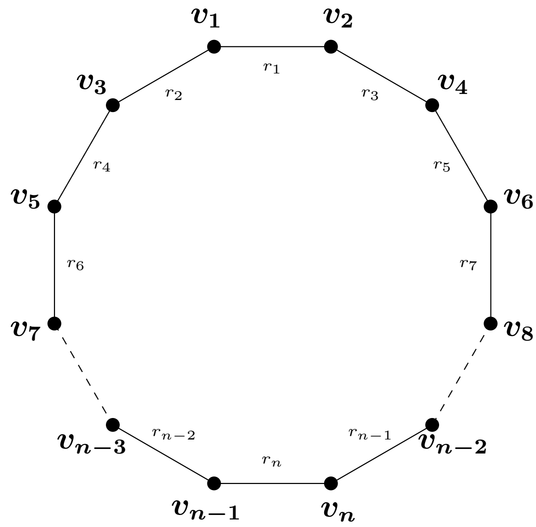

Figure 3.1: Edge labeling of C_n

Proof. Let C_n be the cycle with vertex set $\{v_1, v_2, \cdots, v_n\}$ and edge set $\{v_1v_2\} \cup \{v_iv_{i+2} | \ i = 1, 2, \cdots, n-2\} \cup \{v_{n-1}v_n\}$. Let $r_1 < r_2 < r_3 < \cdots < r_n$ be the arbitrarily given n real numbers. We define an edge labeling f of C_n with labels in $\{r_1, r_2, \cdots, r_n\}$ by $f(v_1v_2) = r_1$, $f(v_iv_{i+2}) = r_{i+1}$ for $i = 1, 2, \cdots, n-2$, and $f(v_{n-1}v_n) = r_n$ (see Figure 3.1).

Then $f^+(v_1) = r_1+r_2$, $f^+(v_i) = r_{i-1}+r_{i+1}$ for $i = 2, \cdots, n-1$, and $f^+(v_n) = r_{n-1}+r_n$. Since $r_1+r_2 < r_1+r_3 < r_2+r_4 < r_3+r_5 < r_4+r_6 < \cdots < r_{n-3}+r_{n-1} < r_{n-2}+r_n < r_{n-1}+r_n$, we have $f^+(v_1) < f^+(v_2) < \cdots < f^+(v_n)$. We see that the listing of vertices $v_1, v_2, \cdots, v_n$ with the property $f^+(v_1) < f^+(v_2) < \cdots < f^+(v_n)$ is independent of the arbitrarily given $r_1 < r_2 < r_3 < \cdots < r_n$. Thus, C_n is uniformly $\mathbb{R}$-antimagic. $\square$

Theorem 3.1.3. *[6] The complete graph K_n $(n \geq 3)$ is uniformly $\mathbb{R}$-antimagic.*

Proof. Let K_n be the complete graph with vertex set $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(K_n) = \{v_iv_j | \ 1 \leq i < j \leq n\}$. Let $r_1 < r_2 < r_3 < \cdots < r_{\binom{n}{2}}$ be the arbitrarily given real numbers.

Let f be an edge labeling of K_n with labels in $\{r_1, r_2, r_3, \cdots, r_{\binom{n}{2}}\}$ such that for $i = 1, 2, \cdots, n-2$, $f(v_iv_{i+1}) < f(v_iv_{i+2}) < f(v_iv_{i+3}) < \cdots < f(v_iv_n) < f(v_{i+1}v_{i+2})$. Hence $f(v_1v_2) < f(v_1v_3) < \cdots < f(v_1v_n) < f(v_2v_3) < f(v_2v_4) < \cdots < f(v_2v_n) < f(v_3v_4) <$

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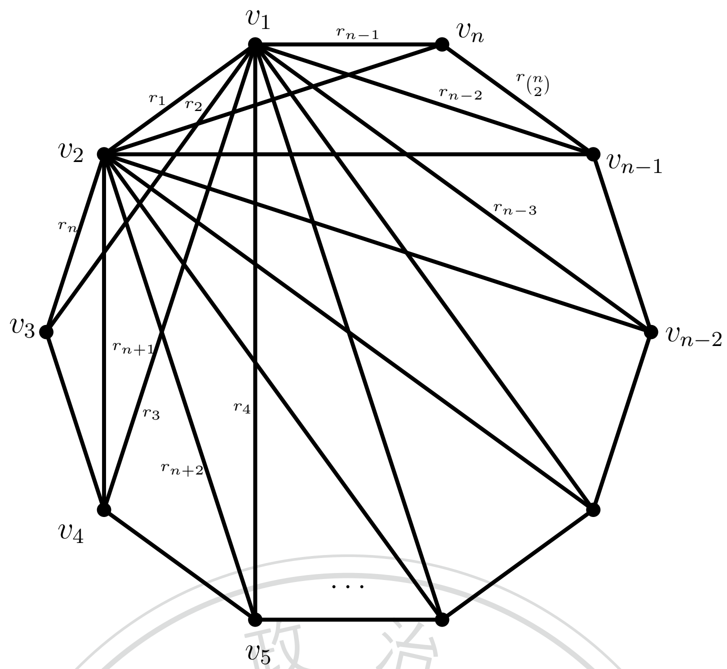

Figure 3.2: Edge labeling of K_n

$\cdots < f(v_{n-1}v_n)$ (see Figure 3.1).

For $1 \leq i \leq n-1$, we have

$$\begin{aligned} & f^{+}(v_i) \\ =& \sum_{1\leq k<i} f(v_kv_i) + f(v_iv_{i+1}) + \sum_{i+1<k\leq n} f(v_iv_k) \\ <& \sum_{1\leq k<i} f(v_kv_{i+1}) + f(v_iv_{i+1}) + \sum_{i+1<k\leq n} f(v_{i+1}v_k) \\ =& f^{+}(v_{i+1}). \end{aligned} \tag{3.1.1}$$

Hence $f^{+}(v_1) < f^{+}(v_2) < \cdots < f^{+}(v_n)$. We see that the listing of vertices $v_1, v_2, \cdots, v_n$ with the property $f^{+}(v_1) < f^{+}(v_2) < \cdots < f^{+}(v_n)$ is independent of the arbitrarily given $r_1 < r_2 < r_3 < \cdots < r_{\binom{n}{2}}$. Thus, K_n is uniformly $\mathbb{R}$-antimagic. □

3.2 Cartesian products of uniformly $\mathbb{R}$-antimagic graphs and complete graphs

We introduce the concept of $\prec$ in Section 2.3.

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Definition 3.2.1. *Let G be a graph and A be a subset of $\mathbb{R}$ with $|A| = |E(G)|$. If g is an edge labeling of G with labels in A and K, L are nonempty subsets of $E(G)$ such that $g(x) < g(y)$ for all $x \in K$, $y \in L$, then we write $K \prec L$ under g.*

The following trivial lemma will be used in the proofs of Theorems 3.2.3 and 3.3.1.

Lemma 3.2.2. *[6] Let G be an arbitrary graph and A be a subset of $\mathbb{R}$ with $|A| = |E(G)|$. Let g be an edge labeling of G with labels in A. Suppose that A_1, A_2, B_1, B_2 are pairwise disjoint nonempty subsets of the edge set $E(G)$ with $|A_1| = |B_1|$, $|A_2| = |B_2| = 1$ such that $A_1 \prec B_1 \cup B_2$ and $A_2 \prec B_1$ under g. Then*

$$\sum_{e \in A_1 \cup A_2} g(e) < \sum_{e \in B_1 \cup B_2} g(e). \tag{3.2.1}$$

Proof. Let $A_2 = \{a\}$ and b be an arbitrary edge in B_1. Since $A_2 \prec B_1$ under g, we have $g(a) < g(b)$. Since $A_1 \prec B_1 \cup B_2$ under g and $|A_1| = |B_2 \cup (B_1 - \{b\})|$, we have

$$\sum_{e \in A_1} g(e) < \sum_{e \in B_2 \cup (B_1 - \{b\})} g(e). \tag{3.2.2}$$

Note that

$$\sum_{e \in A_1 \cup A_2} g(e) = g(a) + \sum_{e \in A_1} g(e), \tag{3.2.3}$$

and

$$\sum_{e \in B_1 \cup B_2} g(e) = g(b) + \sum_{e \in B_2 \cup (B_1 - \{b\})} g(e). \tag{3.2.4}$$

Combining (3.2.2), (3.2.3), (3.2.4) and $g(a) < g(b)$, we have

$$\sum_{e \in A_1 \cup A_2} g(e) < \sum_{e \in B_1 \cup B_2} g(e). \tag{3.2.5}$$

□

We need the following notations. Let G be a graph, and A be a subset of $\mathbb{R}$ with $|A| = |E(G)|$. If f is an edge labeling of G with labels in A and D is a non-trivial connected subgraph of G which contains no isolated vertices, then we use $f_{E(D)}$ to denote the restriction of f to $E(D)$ with range $f(E(D))$. Obviously, $f_{E(D)}$ is an edge labeling of D with labels in $f(E(D))$. Moreover, for a vertex $v \in V(D)$, we use $f^+_{E(D)}(v)$ to denote $\left(f_{E(D)}\right)^+(v)$. Recall that $E_D(v)$

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is the set of all edges incident to v in D. Thus, $f^+_{E(D)}(v) = \sum_{e\in E_D(v)} f(e)$.

Let G and H be two graphs with $V(G) = \{u_1, u_2, \cdots, u_m\}$ and $V(H) = \{v_1, v_2, \cdots, v_n\}$, respectively. The *Cartesian product* of G and H, denoted by $G\Box H$, is the graph with vertex set $V(G) \times V(H)$ such that (u_i, v_j) is adjacent to (u_k, v_l) if either $u_i = u_k$ and $v_jv_l \in E(H)$ or $v_j = v_l$ and $u_iu_k \in E(G)$. For the convenience of the following discussions, we will use the following notations in the proofs of Theorems 3.2.3 and 3.3.1. In the graph $G\Box H$, the vertex $(u_i, v_j) \in V(G) \times V(H)$ is represented by $w_{i,j}$. For $j = 1, 2, \cdots, n$, we use G_j to denote the subgraph of $G\Box H$ induced by the vertices $w_{i,j}$ ($i = 1, 2, \cdots, m$).

Note 1. The graphs G, G_1, G_2, $\cdots$, G_n are isomorphic, and for each i ($i = 1, 2, \cdots, m$) the vertices $u_i \in V(G)$, $w_{i,1} \in V(G_1)$, $w_{i,2} \in V(G_2)$, $\cdots$, $w_{i,n} \in V(G_n)$ are the corresponding vertices under these isomorphisms.

Also, we use E_j to denote $E(G_j)$; that is, E_j is the set of all edges in G_j. For $1 \le j < l \le n$ and $v_jv_l \in E(H)$, we use $E_{j,l}$ to denote the set $\{w_{i,j}w_{i,l} | \ i = 1, 2, \cdots, m\}$, i.e., $E_{j,l}$ the set of all edges joining the vertices in G_j and the vertices in G_l. We see that $E(G\Box H)$ is the disjoint union of E_j ($j = 1, 2, \cdots, n$) and $E_{j,l}$ ($1 \le j < l \le n$, $v_jv_l \in E(H)$).

The notations for the vertices $w_{i,j}$, the subgraphs G_j and the edge sets $E_j, E_{j,l}$ of $G\Box H$ will be used in the proofs of Theorems 3.2.3 and 3.3.1.

Theorem 3.2.3. *[6] Let G be a regular and uniformly $\mathbb{R}$-antimagic graph. Then $G\Box K_n$ ($n \ge 2$) is also regular and uniformly $\mathbb{R}$-antimagic.*

Proof. Since both G and K_n are regular, it is trivial that $G\Box K_n$ is regular. Since G is uniformly $\mathbb{R}$-antimagic, we assume that $u_1, u_2, \cdots, u_m$ ($m \ge 3$) is the sequence of vertices of G with the uniformly $\mathbb{R}$-antimagic property. We see that the edge set $E\left(G\Box K_n\right)$ is the union of E_j ($j = 1, 2, \cdots, n$) and $E_{j,l}$ ($1 \le j < l \le n$).

We prove that $G\Box K_n$ ($n \ge 2$) is uniformly $\mathbb{R}$-antimagic. Let $A \subseteq \mathbb{R}$ with $|A| = |E(G\Box K_n)|$ be arbitrarily given. Define g to be an edge labeling of $G\Box K_n$ with labels in A by the following three rules:

Rule 3.2.1. For $j = 1, 2, \cdots, n-1$, $E_j \prec E_{j,j+1} \prec E_{j,j+2} \prec \cdots \prec E_{j,n} \prec E_{j+1}$.

Rule 3.2.2. For $1 \le j < l \le n$, and for $i = 1, 2, \cdots, m-1$, $g(w_{i,j}w_{i,l}) < g(w_{i+1,j}w_{i+1,l})$ $(i.e.,\ g(w_{1,j}w_{1,l}) < g(w_{2,j}w_{2,l}) < g(w_{3,j}w_{3,l}) < \cdots < g(w_{m,j}w_{m,l}))$.

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Rule 3.2.3. For $j = 1, 2, \cdots, n$ and for $i = 1, 2, \cdots, m-1$, $g^+_{E_j}(w_{i,j}) < g^+_{E_j}(w_{i+1,j})$ ($i.e.$, $g^{\,+}_{E_j}(w_{1,j}) < g^{\,+}_{E_j}(w_{2,j}) < g^{\,+}_{E_j}(w_{3,j}) < \cdots < g^{\,+}_{E_j}(w_{m,j})$).

That the edge labeling g with labels in A can have Rule 3.2.3 deriving from the fact that the sequence of vertices $u_1, u_2, \cdots, u_m$ has the uniformly $\mathbb{R}$-antimagic property in G and the fact stated in Note 1.

Claim 1. *For* $j = 1, 2, \cdots, n$, $g^{\,+}(w_{1,j}) < g^{\,+}(w_{2,j}) < g^{\,+}(w_{3,j}) < \cdots < g^{\,+}(w_{m,j})$.

Check of Claim 1. We need to show $g^{\,+}(w_{i,j}) < g^{\,+}(w_{i+1,j})$ for $i = 1, 2, \cdots, m-1$. Let $J = \{1, 2, \cdots, n\}$. Note that

$$g^{\,+}(w_{i,j}) = g^{\,+}_{E_j}(w_{i,j}) + \sum_{l \in J-\{j\}} g(w_{i,j}w_{i,l}), \tag{3.2.6}$$

and

$$g^{\,+}(w_{i+1,j}) = g^{\,+}_{E_j}(w_{i+1,j}) + \sum_{l \in J-\{j\}} g(w_{i+1,j}w_{i+1,l}). \tag{3.2.7}$$

By Rule 3.2.3, $g^{\,+}_{E_j}(w_{i,j}) < g^{\,+}_{E_j}(w_{i+1,j})$.
By Rule 3.2.2, for $1 \leq j < l \leq n$, $g(w_{i,j}w_{i,l}) < g(w_{i+1,j}w_{i+1,l})$, it implies

$$\sum_{l \in J-\{j\}} g(w_{i,j}w_{i,l}) < \sum_{l \in J-\{j\}} g(w_{i+1,j}w_{i+1,l}). \tag{3.2.8}$$

Thus, $g^{\,+}(w_{i,j}) < g^{\,+}(w_{i+1,j})$, which completes the check of Claim 1.

Claim 2. *For* $j = 1, 2, \cdots, n-1$, $g^{\,+}(w_{m,j}) < g^{\,+}(w_{1,j+1})$.

Check of Claim 2. Let $J = \{1, 2, \cdots, n\}$. Note that

$$\begin{aligned} g^{\,+}(w_{m,j}) &= g^{\,+}_{E_j}(w_{m,j}) + \sum_{k \in J-\{j\}} g(w_{m,j}w_{m,k}) \\ &= g^{\,+}_{E_j}(w_{m,j}) + g(w_{m,j}w_{m,j+1}) + \sum_{k \in J-\{j,j+1\}} g(w_{m,k}w_{m,j}), \end{aligned} \tag{3.2.9}$$

and

$$\begin{aligned} g^{\,+}(w_{1,j+1}) &= g^{\,+}_{E_{j+1}}(w_{1,j+1}) + \sum_{k \in J-\{j+1\}} g(w_{1,j+1}w_{1,k}) \\ &= g^{\,+}_{E_{j+1}}(w_{1,j+1}) + g(w_{1,j}w_{1,j+1}) + \sum_{k \in J-\{j,j+1\}} g(w_{1,k}w_{1,j+1}). \end{aligned} \tag{3.2.10}$$

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Let $A_1 = E_{G_j}(w_{m,j}) \subseteq E_j$, $A_2 = \{w_{m,j}w_{m,j+1}\} \subseteq E_{j,j+1}$, $B_1 = E_{G_{j+1}}(w_{1,j+1}) \subseteq E_{j+1}$, $B_2 = \{w_{1,j}w_{1,j+1}\} \subseteq E_{j,j+1}$. Thus,

$$\sum_{e \in A_1 \cup A_2} g(e) = g^{+}_{E_j}(w_{m,j}) + g(w_{m,j}w_{m,j+1}), \tag{3.2.11}$$

and

$$\sum_{e \in B_1 \cup B_2} g(e) = g^{+}_{E_{j+1}}(w_{1,j+1}) + g(w_{1,j}w_{1,j+1}). \tag{3.2.12}$$

By Rule 3.2.1, $E_j \prec E_{j,j+1} \prec E_{j+1}$. Since $A_1 \subseteq E_j$, $B_1 \subseteq E_{j+1}$, $A_2, B_2 \subseteq E_{j,j+1}$, we have $A_1 \prec B_1 \cup B_2$ and $A_2 \prec B_1$. Also, note $|A_1| = |B_1|$, $|A_2| = |B_2| = 1$. Thus, by Lemma 3.2.2, $\sum_{e \in A_1 \cup A_2} g(e) < \sum_{e \in B_1 \cup B_2} g(e)$. Hence

$$g^{+}_{E_j}(w_{m,j}) + g(w_{m,j}w_{m,j+1}) < g^{+}_{E_{j+1}}(w_{1,j+1}) + g(w_{1,j}w_{1,j+1}). \tag{3.2.13}$$

By Rule 3.2.1, $E_{k,j} \prec E_{k,j+1}$ if $k < j$, and $E_{j,k} \prec E_{j+1,k}$ if $k > j+1$, and we see that $w_{m,k}w_{m,j} \in E_{k,j}$, $w_{1,k}w_{1,j+1} \in E_{k,j+1}$. Thus, $g(w_{m,k}w_{m,j}) < g(w_{1,k}w_{1,j+1})$, which implies

$$\sum_{k \in J - \{j,j+1\}} g(w_{m,k}w_{m,j}) < \sum_{k \in J - \{j,j+1\}} g(w_{1,k}w_{1,j+1}). \tag{3.2.14}$$

Combining (3.2.13) and (3.2.14), we obtain $g^{+}(w_{m,j}) < g^{+}(w_{1,j+1})$. This completes the check of Claim 2.

From Claims 1 and 2, we obtain

$$\begin{aligned}
& g^{+}(w_{1,1}) < g^{+}(w_{2,1}) < \cdots < g^{+}(w_{m,1}) \\
<& g^{+}(w_{1,2}) < g^{+}(w_{2,2}) < \cdots < g^{+}(w_{m,2}) \\
<& g^{+}(w_{1,3}) < g^{+}(w_{2,3}) < \cdots < g^{+}(w_{m,3}) \\
<& \cdots < \qquad \cdots\cdots < \qquad \cdots \\
<& g^{+}(w_{1,n}) < g^{+}(w_{2,n}) < \cdots < g^{+}(w_{m,n}).
\end{aligned}$$

We also see that the order of the vertices $w_{1,1}$, $w_{2,1}$, $w_{3,1}$, $\cdots$, $w_{m,1}$, $w_{1,2}$, $w_{2,2}$, $w_{3,2}$, $\cdots$, $w_{m,2}$, $w_{1,3}$, $w_{2,3}$, $w_{3,3}$, $\cdots$, $w_{m,3}$, $w_{1,4}$, $\cdots$, $w_{m,n-1}$, $w_{1,n}$, $w_{2,n}$, $w_{3,n}$, $\cdots$, $w_{m,n}$ satisfying the above-mentioned strict inequalities is independent of the chosen $A \subseteq \mathbb{R}$ with $|A| = |G \square K_n|$. Thus, $G \square K_n$ $(n \geq 2)$ is uniformly $\mathbb{R}$-antimagic.

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$\square$

3.3 Cartesian products of uniformly $\mathbb{R}$-antimagic graphs and cycles

It has been proved that the Cartesian product of two or more cycles is antimagic [18]. We further propose that $G\square C_n$ is (uniformly) $\mathbb{R}$-antimagic where G is a regular and uniformly $\mathbb{R}$-antimagic graph. In $G\square C_n$, the labels we use are in each subset A of real numbers with $|A| = |E(G)|$ and the labels used in [7, 18] are in $\{1, 2, \cdots, |E(G)|\}$. Because of the difference in labels, we have to modify the order of labelings, which are different from those in [7, 18]. We use some strategies in the construction of labelings.

Theorem 3.3.1. *[6] Let G be a regular and uniformly $\mathbb{R}$-antimagic graph. Then $G\square C_n$ is also regular and uniformly $\mathbb{R}$-antimagic.*

Proof. Since both G and C_n are regular, it is trivial that $G\square C_n$ is regular. Now we show that $G\square C_n$ is uniformly $\mathbb{R}$-antimagic. By Theorem 3.2.3, $G\square K_3$ is uniformly $\mathbb{R}$-antimagic. Thus, $G\square C_3$ is uniformly $\mathbb{R}$-antimagic. Using Theorem 3.2.3 twice, we see that $(G\square K_2)\square K_2$ is uniformly $\mathbb{R}$-antimagic. Thus, $G\square C_4$ is uniformly $\mathbb{R}$-antimagic since $(G\square K_2)\square K_2$ is isomorphic to $G\square C_4$. We assume that $n \geq 5$.

Assume that the cycle C_n has vertex set $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and the edge set $E(C_n) = \{v_1v_2\} \cup \{v_iv_{i+2} |\ i = 1, 2, \cdots, n-2\} \cup \{v_{n-1}v_n\}$. We use the notations for the vertices, subgraphs and edge sets of $G\square H$ which are defined in Theorem 3.2.3 above, where H is now taken to be C_n. We see that the edge set $E(G\square C_n)$ is the union of E_j ($j = 1, 2, \cdots, n$) and $E_{1,2}$, $E_{j,j+2}$ ($j = 1, 2, \cdots, n-2$), $E_{n-1,n}$.

Now we prove that $G\square C_n$ is uniformly $\mathbb{R}$-antimagic. Since G is uniformly $\mathbb{R}$-antimagic, we assume that $u_1, u_2, \cdots, u_m$ ($m \geq 3$) is the sequence of vertices of G with the uniformly $\mathbb{R}$-antimagic property. Let $A \subseteq \mathbb{R}$ with $|A| = |E(G\square C_n)|$ be arbitrarily given. Define g to be an edge labeling of $G\square C_n$ with labels in A by the following three rules:

Rule 3.3.1. Rules of $\prec$ on $G\square C_n$.

1. $E_1 \prec E_{1,2} \prec E_2$,

2. for $j = 2, 3, \cdots, n-2$, $E_j \prec E_{j-1,j+1} \prec E_{j+1}$,

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3. $E_{n-1} \prec E_{n-2,n} \prec E_{n-1,n} \prec E_n$ (Hence $E_1 \prec E_{1,2} \prec E_2 \prec E_{1,3} \prec E_3 \prec E_{2,4} \prec E_4 \prec E_{3,5} \prec E_5 \prec \cdots \prec E_{n-3} \prec E_{n-4,n-2} \prec E_{n-2} \prec E_{n-3,n-1} \prec E_{n-1} \prec E_{n-2,n} \prec E_{n-1,n} \prec E_n$.).

Rule 3.3.2. For $v_jv_l \in E(C_n),\ g(w_{1,j}w_{1,l}) < g(w_{2,j}w_{2,l}) < g(w_{3,j}w_{3,l}) < \cdots < g(w_{m,j}w_{m,l})$.

Rule 3.3.3. For $j = 1, 2, \cdots, n$, we have $g^{\,+}_{E_j}(w_{1,j}) < g^{\,+}_{E_j}(w_{2,j}) < g^{\,+}_{E_j}(w_{3,j}) < \cdots < g^{\,+}_{E_j}(w_{m,j})$.

That the edge labeling g with labels in A can have Rule 3.3.3 deriving from the fact that the sequence of vertices $u_1, u_2, \cdots, u_m$ has the uniformly $\mathbb{R}$-antimagic property in G and the fact stated in Note 1.

Claim 1. *For* $j = 1, 2, \cdots, n$, $g^{\,+}(w_{1,j}) < g^{\,+}(w_{2,j}) < g^{\,+}(w_{3,j}) < \cdots < g^{\,+}(w_{m,j})$.

Check of Claim 1.

We need to show $g^{\,+}(w_{i,j}) < g^{\,+}(w_{i+1,j})$ for $i = 1, 2, \cdots, m-1$. Note that

$$g^{\,+}(w_{i,j}) = g^{\,+}_{E_j}(w_{i,j}) + \sum_{v_jv_l \in E(C_n)} g(w_{i,j}w_{i,l}), \tag{3.3.1}$$

and

$$g^{\,+}(w_{i+1,j}) = g^{\,+}_{E_j}(w_{i+1,j}) + \sum_{v_jv_l \in E(C_n)} g(w_{i+1,j}w_{i+1,l}). \tag{3.3.2}$$

By Rule 3.3.3, $g^{\,+}_{E_j}(w_{i,j}) < g^{\,+}_{E_j}(w_{i+1,j})$.

From Rule 3.3.2, we obtain that for fixed i, $i = 1, 2, \cdots, m-1$,

$$\sum_{v_jv_l \in E(C_n)} g(w_{i,j}w_{i,l}) < \sum_{v_jv_l \in E(C_n)} g(w_{i+1,j}w_{i+1,l}). \tag{3.3.3}$$

Thus, $g^{\,+}(w_{i,j}) < g^{\,+}(w_{i+1,j})$. This completes the check of Claim 1.

Claim 2. *For* $j = 1, 2, \cdots, n-1$, $g^{\,+}(w_{m,j}) < g^{\,+}(w_{1,j+1})$.

Check of Claim 2. We distinguish five cases: Case 1., $j = 1$; Case 2., $j = 2$; Case 3., $j = 3, 4, \cdots, n-3$; Case 4., $j = n-2$; and Case 5., $j = n-1$.

Case 1. $j = 1$.

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We need to show that $g^+(w_{m,1}) < g^+(w_{1,2})$. Let $A_1 = E_{G_1}(w_{m,1})$ and $A_2 = \{w_{m,1}w_{m,2}\}$. Then

$$g^+(w_{m,1}) = g(w_{m,1}w_{m,3}) + \sum_{e \in A_1 \cup A_2} g(e). \tag{3.3.4}$$

Let $B_1 = E_{G_2}(w_{1,2})$ and $B_2 = \{w_{1,1}w_{1,2}\}$. Then

$$g^+(w_{1,2}) = g(w_{1,2}w_{1,4}) + \sum_{e \in B_1 \cup B_2} g(e). \tag{3.3.5}$$

From Rule 3.3.1, $E_1 \prec E_{1,2} \prec E_2 \prec E_{1,3} \prec E_{2,4}$. Since $E_{1,3} \prec E_{2,4}$, we have

$$g(w_{m,1}w_{m,3}) < g(w_{1,2}w_{1,4}). \tag{3.3.6}$$

Since $E_1 \prec E_{1,2} \prec E_2$, $A_1 \subseteq E_1$, $A_2, B_2 \subseteq E_{1,2}$, $B_1 \subseteq E_2$, we have $A_1 \prec B_1 \cup B_2$, $A_2 \prec B_1$. Since G is reqular, we have $|A_1| = |B_1|$. Trivially, $|A_2| = |B_2| = 1$. Thus, by Lemma 3.2.2,

$$\sum_{e \in A_1 \cup A_2} g(e) < \sum_{e \in B_1 \cup B_2} g(e). \tag{3.3.7}$$

From the aforementioned, we obtain $g^+(w_{m,1}) < g^+(w_{1,2})$.

Case 2. $j = 2$.

We need to show that $g^+(w_{m,2}) < g^+(w_{1,3})$. Note that

$$g^+(w_{m,2}) = g^+_{E_2}(w_{m,2}) + g(w_{m,1}w_{m,2}) + g(w_{m,2}w_{m,4}), \tag{3.3.8}$$

and

$$g^+(w_{1,3}) = g^+_{E_3}(w_{1,3}) + g(w_{1,1}w_{1,3}) + g(w_{1,3}w_{1,5}). \tag{3.3.9}$$

Since $E_2 \prec E_3$, and G_2 and G_3 are regular with the same degree, we have

$$g^+_{E_2}(w_{m,2}) < g^+_{E_3}(w_{1,3}). \tag{3.3.10}$$

Since $E_{1,2} \prec E_{1,3}$, we have

$$g(w_{m,1}w_{m,2}) < g(w_{1,1}w_{1,3}). \tag{3.3.11}$$

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Since $E_{2,4} \prec E_{3,5}$, we have

$$g(w_{m,2}w_{m,4}) < g(w_{1,3}w_{1,5}). \tag{3.3.12}$$

Thus, we obtain $g^{+}(w_{m,2}) < g^{+}(w_{1,3})$.

Case 3. $j = 3, 4, \cdots, n-3$.

We need to show that $g^{+}(w_{m,j}) < g^{+}(w_{1,j+1})$. For $n = 5$, we do not need to consider this case. Assume that $n \geq 6$. Note that

$$g^{+}(w_{m,j}) = g^{+}_{E_j}(w_{m,j}) + g(w_{m,j-2}w_{m,j}) + g(w_{m,j}w_{m,j+2}), \tag{3.3.13}$$

and

$$g^{+}(w_{1,j+1}) = g^{+}_{E_{j+1}}(w_{1,j+1}) + g(w_{1,j-1}w_{1,j+1}) + g(w_{1,j+1}w_{1,j+3}). \tag{3.3.14}$$

From Rule 3.3.1(2), we have $E_j \prec E_{j-1,j+1} \prec E_{j+1} \prec E_{j,j+2}$ for $2 \leq j \leq n-3$. Since $E_j \prec E_{j+1}$, and G_j and G_{j+1} are regular with the same degree, we have

$$g^{+}_{E_j}(w_{m,j}) < g^{+}_{E_{j+1}}(w_{1,j+1}). \tag{3.3.15}$$

Since $E_{j-2,j} \prec E_{j-1,j+1}$, we have

$$g(w_{m,j-2}w_{m,j}) < g(w_{1,j-1}w_{1,j+1}). \tag{3.3.16}$$

Since $E_{j,j+2} \prec E_{j+1,j+3}$, we have

$$g(w_{m,j}w_{m,j+2}) < g(w_{1,j+1}w_{1,j+3}). \tag{3.3.17}$$

Accordingly, we obtain $g^{+}(w_{m,j}) < g^{+}(w_{1,j+1})$.

Case 4. $j = n-2$.

We need to show that $g^{+}(w_{m,n-2}) < g^{+}(w_{1,n-1})$. Note that

$$g^{+}(w_{m,n-2}) = g^{+}_{E_{n-2}}(w_{m,n-2}) + g(w_{m,n-4}w_{m,n-2}) + g(w_{m,n-2}w_{m,n}), \tag{3.3.18}$$

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and

$$g^{+}(w_{1,n-1}) = g^{+}_{E_{n-1}}(w_{1,n-1}) + g(w_{1,n-3}w_{1,n-1}) + g(w_{1,n-1}w_{1,n}). \tag{3.3.19}$$

Also note that $E_{n-4,n-2} \prec E_{n-2} \prec E_{n-3,n-1} \prec E_{n-1}$. Since $w_{m,n-4}w_{m,n-2} \in E_{n-4,n-2}$, $w_{1,n-3}w_{1,n-1} \in E_{n-3,n-1}$, we have

$$g(w_{m,n-4}w_{m,n-2}) < g(w_{1,n-3}w_{1,n-1}). \tag{3.3.20}$$

Since $E_{G_{n-2}}(w_{m,n-2}) \subseteq E_{n-2}$, $E_{G_{n-1}}(w_{1,n-1}) \subseteq E_{n-1}$, we have

$$g^{+}_{E_{n-2}}(w_{m,n-2}) < g^{+}_{E_{n-1}}(w_{1,n-1}). \tag{3.3.21}$$

Furthermore, $E_{n-2,n} \prec E_{n-1,n}$, this implies

$$g(w_{m,n-2}w_{m,n}) < g(w_{1,n-1}w_{1,n}). \tag{3.3.22}$$

Hence, we obtain $g^{+}(w_{m,n-2}) < g^{+}(w_{1,n-1})$.

Case 5. $j = n-1$.

We need to show that $g^{+}(w_{m,n-1}) < g^{+}(w_{1,n})$. Let $A_1 = E_{G_{n-1}}(w_{m,n-1})$ and $A_2 = \{w_{m,n-1}w_{m,n}\}$. Then

$$g^{+}(w_{m,n-1}) = g(w_{m,n-3}w_{m,n-1}) + \sum_{e \in A_1 \cup A_2} g(e). \tag{3.3.23}$$

Let $B_1 = E_{G_n}(w_{1,n})$ and $B_2 = \{w_{1,n-1}w_{1,n}\}$. Then

$$g^{+}(w_{1,n}) = g(w_{1,n-2}w_{1,n}) + \sum_{e \in B_1 \cup B_2} g(e). \tag{3.3.24}$$

Note that $E_{n-3,n-1} \prec E_{n-1} \prec E_{n-2,n} \prec E_{n-1,n} \prec E_n$. From $E_{n-3,n-1} \prec E_{n-2,n}$ and $w_{m,n-3}w_{m,n-1} \in E_{n-3,n-1}$, $w_{1,n-2}w_{1,n} \in E_{n-2,n}$, we have

$$g(w_{m,n-3}w_{m,n-1}) < g(w_{1,n-2}w_{1,n}). \tag{3.3.25}$$

From $E_{n-1} \prec E_{n-1,n} \prec E_n$ and $A_1 \subseteq E_{n-1}$, $A_2 \subseteq E_{n-1,n}$, $B_1 \subseteq E_n$, $B_2 \subseteq E_{n-1,n}$, we

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have $A_1 \prec B_1 \cup B_2$ and $A_2 \prec B_1$. Since G is reqular, we have $|A_1| = |B_1|$. Trivially, $|A_2| = |B_2| = 1$. Thus, by Lemma 3.2.2,

$$\sum_{e \in A_1 \cup A_2} g(e) < \sum_{e \in B_1 \cup B_2} g(e). \tag{3.3.26}$$

Therefore, we obtain $g^{+}(w_{m,n-1}) < g^{+}(w_{1,n})$.

These complete the check of Claim 2.

From Claims 1 and 2, we obtain

$$\begin{aligned} g^{+}(w_{1,1}) < \quad & g^{+}(w_{2,1}) < \cdots < \quad g^{+}(w_{m,1}) \\ < g^{+}(w_{1,2}) < \quad & g^{+}(w_{2,2}) < \cdots < \quad g^{+}(w_{m,2}) \\ < g^{+}(w_{1,3}) < \quad & g^{+}(w_{2,3}) < \cdots < \quad g^{+}(w_{m,3}) \\ < \cdots < \quad & \qquad \cdots\cdots < \qquad \cdots \\ < g^{+}(w_{1,n}) < \quad & g^{+}(w_{2,n}) < \cdots < \quad g^{+}(w_{m,n}). \end{aligned}$$

We also see that the order of the vertices $w_{1,1}$, $w_{2,1}$, $w_{3,1}$, $\cdots$, $w_{m,1}$, $w_{1,2}$, $w_{2,2}$, $w_{3,2}$, $\cdots$, $w_{m,2}$, $w_{1,3}$, $w_{2,3}$, $w_{3,3}$, $\cdots$, $w_{m,3}$, $w_{1,4}$, $\cdots$, $w_{m,n-1}$, $w_{1,n}$, $w_{2,n}$, $w_{3,n}$, $\cdots$, $w_{m,n}$ satisfying the above-mentioned strict inequalities is independent of the chosen $A \subseteq \mathbb{R}$ with $|A| = |E(G \Box C_n)|$. Thus, $G \Box C_n$ is uniformly $\mathbb{R}$-antimagic. This completes the proof of the theorem. □

The following Corollaries derive directly from Theorems 3.2.3 and 3.3.1.

Corollary 3.3.1. *[6] The graph $G_1 \Box G_2 \Box \cdots \Box G_n$ $(n \geq 2)$ is uniformly $\mathbb{R}$-antimagic, where G_1 is regular and uniformly $\mathbb{R}$-antimagic, and for $i \geq 2$ each G_i is a complete graph of order ≥ 2 or a cycle.*

Corollary 3.3.2. *[6] The graph $G_1 \Box G_2 \Box \cdots \Box G_n$ $(n \geq 2)$ is uniformly $\mathbb{R}$-antimagic, where each G_i is a complete graph of order ≥ 2 or a cycle.*

Proof. Each G_i is a complete graph of order ≥ 2 or a cycle.

Case 1. *Some $G_i \neq K_2$.*

Without loss of generality, assume $G_1 \neq K_2$. Then G_1 is a cycle or a complete graph of order ≥ 3. By Theorems 3.1.2 and 3.1.3, G_1 is uniformly $\mathbb{R}$-antimagic. Then the Corollary derives from Corollary 3.3.1.

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Case 2. $G_i = K_2$ *for* $i = 1, 2, \cdots, n$.

Since $K_2 \square K_2 \cong C_4$, by Theorem 3.1.2, $G_1 \square G_2$ is uniformly $\mathbb{R}$-antimagic. Again, the Corollary derives from Corollary 3.3.1. □

Note that the hypercube Q_n is isomorphic to $G_1 \square G_2 \square \cdots \square G_n$, where each $G_i = K_2$ for $i = 1, 2, \cdots, n$. The following corollary derives from Corollary 3.3.2.

Corollary 3.3.3. *[6] Hypercube* Q_n $(n \geq 2)$ *is uniformly* $\mathbb{R}$*-antimagic.*

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Chapter 4

Some irregular graphs

In Chapter 4, we prove that wheels, paws, and paths of order ≥ 6 are $\mathbb{R}$-antimagic.

4.1 Wheels

Let C_n denote the cycle of order n. A *wheel* W_n $(n \geq 3)$ is the graph obtained by connecting a single vertex to every vertex of the cycle C_n. In this section, we prove that wheels are $\mathbb{R}$-antimagic.

Theorem 4.1.1. *[6] Every wheel is $\mathbb{R}$-antimagic.*

Proof. Let W_n be the wheel with $V(W_n) = \{v_1, v_2, \cdots, v_n\} \cup \{v\}$ and $E(W_n) = \{v_1v_2\} \cup \{v_iv_{i+2} | \ i = 1, 2, \cdots, n-2\} \cup \{v_{n-1}v_n\} \cup \{vv_i | \ i = 1, 2, \cdots, n\}$. To prove the theorem, let $r_1 < r_2 < r_3 < \cdots < r_{2n}$ be the arbitrarily given real numbers. We distinguish two cases: Case 1, $r_{n-1} + r_n < r_{n+1} + r_{n+2} + \cdots + r_{2n-1}$; and Case 2, $r_{n+1} + r_{n+2} + \cdots + r_{2n-1} \leq r_{n-1} + r_n$.

Case 1. $r_{n-1} + r_n < r_{n+1} + r_{n+2} + \cdots + r_{2n-1}$.

We define an edge labeling f of W_n with labels in $\{r_1, r_2, r_3, \cdots, r_{2n}\}$ by $f(v_1v_2) = r_1$, $f(v_iv_{i+2}) = r_{i+1}$ for $i = 1, 2, \cdots, n-2$, $f(v_{n-1}v_n) = r_n$ and $f(vv_i) = r_{n+i}$ for $i = 1, 2, \cdots, n$ (see Figure 4.1). Then $f^+(v_1) = r_1 + r_2 + r_{n+1}$, $f^+(v_i) = r_{i-1} + r_{i+1} + r_{n+i}$ for $i = 2, \cdots, n-1$, and $f^+(v_n) = r_{n-1} + r_n + r_{2n}$. Note that

$$f^+(v_1) = r_1 + r_2 + r_{n+1} < r_1 + r_3 + r_{n+2} = f^+(v_2), \tag{4.1.1}$$

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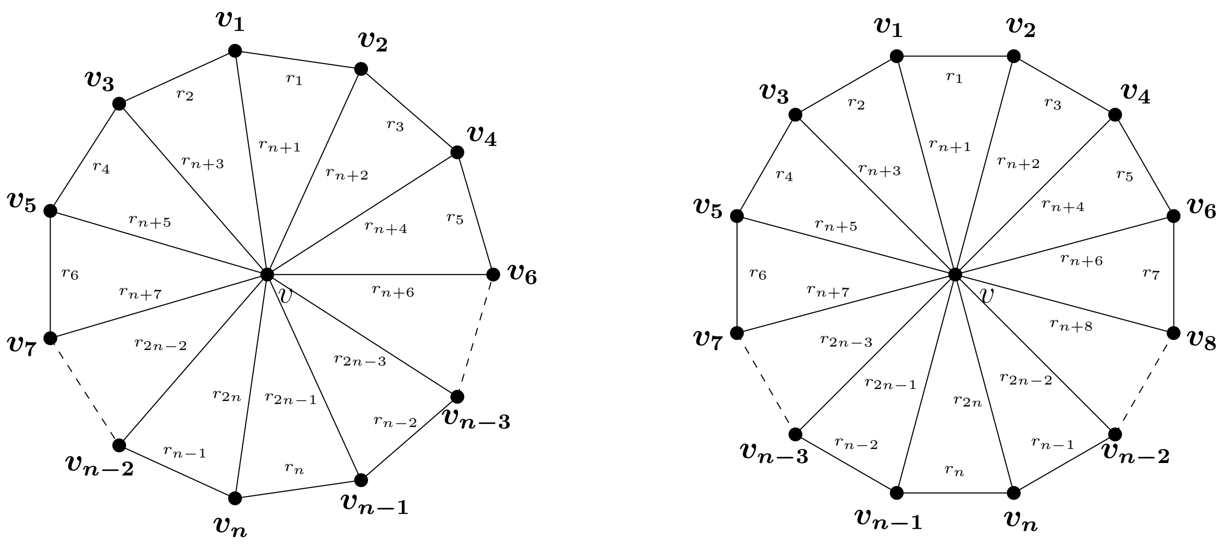

n is odd ***n is even***

Figure 4.1: Edge labeling of W_n if $r_{n-1} + r_n < r_{n+1} + r_{n+2} + \cdots + r_{2n-1}$

$$f^+(v_i) = r_{i-1} + r_{i+1} + r_{n+i} < r_i + r_{i+2} + r_{n+i+1} = f^+(v_{i+1}) \tag{4.1.2}$$

for $i = 2, \cdots, n-2$,

$$f^+(v_{n-1}) = r_{n-2} + r_n + r_{2n-1} < r_{n-1} + r_n + r_{2n} = f^+(v_n), \tag{4.1.3}$$

and

$$\begin{aligned} f^+(v_n) &= (r_{n-1} + r_n) + r_{2n} \\ &< (r_{n+1} + r_{n+2} + \cdots + r_{2n-1}) + r_{2n} \\ &= f^+(v). \end{aligned} \tag{4.1.4}$$

Hence

$$f^+(v_1) < f^+(v_2) < \cdots < f^+(v_n) < f^+(v). \tag{4.1.5}$$

Case 2. $r_{n+1} + r_{n+2} + \cdots + r_{2n-1} \le r_{n-1} + r_n$.

We define an edge labeling f of W_n with labels in $\{r_1, r_2, r_3, \cdots, r_{2n}\}$ by $f(v_1v_2) = r_{n+1}$, $f(v_iv_{i+2}) = r_{n+i+1}$ for $i = 1, 2, \cdots, n-2$, $f(v_{n-1}v_n) = r_{2n}$, and $f(vv_i) = r_i$ for

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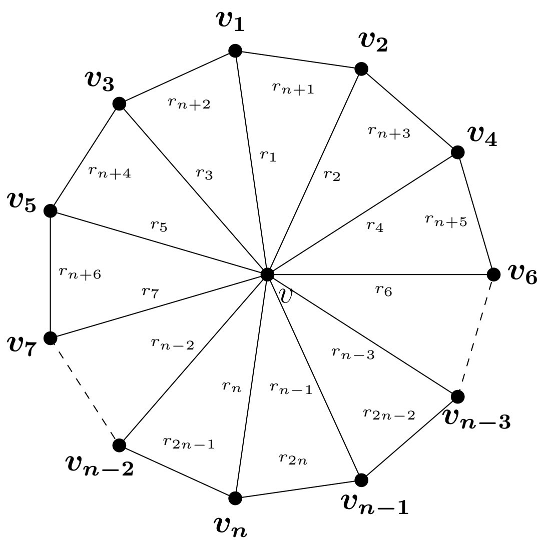

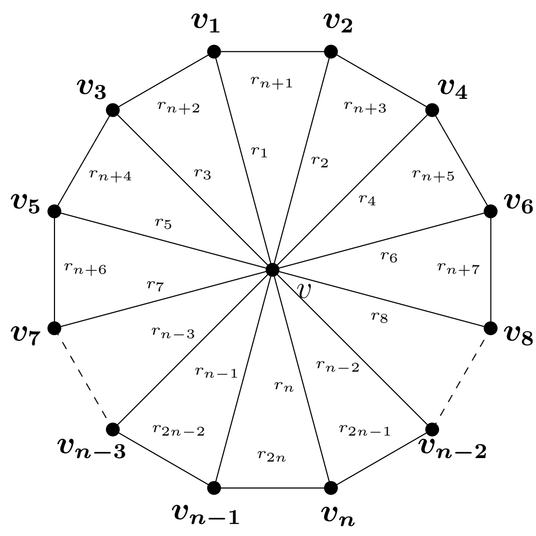

Figure 4.2: Edge labeling of W_n if $r_{n+1} + r_{n+2} + \cdots + r_{2n-1} \leq r_{n-1} + r_n$

$i = 1, 2, \cdots, n$ (see Figure 4.2). Then $f^+(v_1) = r_{n+1}+r_{n+2}+r_1$, $f^+(v_i) = r_{n+i-1}+r_{n+i+1}+r_i$ for $i = 2, \cdots, n-1$, and $f^+(v_n) = r_{2n-1} + r_{2n} + r_n$. Note that

$$f^+(v_1) = r_{n+1} + r_{n+2} + r_1 < r_{n+1} + r_{n+3} + r_2 = f^+(v_2), \tag{4.1.6}$$

$$f^+(v_i) = r_{n+i-1} + r_{n+i+1} + r_i < r_{n+i} + r_{n+i+2} + r_{i+1} = f^+(v_{i+1}) \tag{4.1.7}$$

for $i = 2, \cdots, n-2$,

$$f^+(v_{n-1}) = r_{2n-2} + r_{2n} + r_{n-1} < r_{2n-1} + r_{2n} + r_n = f^+(v_n), \tag{4.1.8}$$

and

$$\begin{aligned} f^+(v) &= r_1 + (r_2 + r_3 + \cdots + r_{n-1} + r_n) \\ &< r_1 + (r_{n+1} + r_{n+2} + \cdots + r_{2n-2} + r_{2n-1}) \\ &\leq r_1 + (r_{n-1} + r_n) < f^+(v_1). \end{aligned} \tag{4.1.9}$$

Hence

$$f^+(v) < f^+(v_1) < f^+(v_2) < \cdots < f^+(v_n). \tag{4.1.10}$$

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This completes the proof. □

4.2 Paws

A paw is a graph with a vertex set $\{v_1, v_2, v_3, v_4\}$ and an edge set $\{v_1v_2, v_2v_3, v_3v_4, v_2v_4\}$.

Theorem 4.2.1. *Every paw is $\mathbb{R}$-antimagic.*

Proof. Let G be the paw with vertex set $\{v_1, v_2, v_3, v_4\}$ and an edge set $\{v_1v_2, v_2v_3, v_3v_4, v_2v_4\}$. Let $r_1 < r_2 < r_3 < r_4$ be arbitrarily given real numbers. We distinguish five cases: Case 1., $0 \le r_1 < r_2 < r_3 < r_4$; Case 2., $r_1 < 0 \le r_2 < r_3 < r_4$; Case 3., $r_1 < r_2 < 0 \le r_3 < r_4$; Case 4., $r_1 < r_2 < r_3 < 0 < r_4$; and Case 5., $r_1 < r_2 < r_3 < r_4 \le 0$.

Case 1. $0 \le r_1 < r_2 < r_3 < r_4$.

We define an edge labeling f of G with labels in $\{r_1, r_2, r_3, r_4\}$ by $f(v_1v_2) = r_1$, $f(v_2v_3) = r_3$, $f(v_3v_4) = r_2$, and $f(v_2v_4) = r_4$ (see Figure 4.3). Then vertex sums $f^+(v_1) = r_1$, $f^+(v_2) = r_1 + r_3 + r_4$, $f^+(v_3) = r_2 + r_3$, and $f^+(v_4) = r_2 + r_4$

Note that $f^+(v_1) < f^+(v_3)$, $f^+(v_3) < f^+(v_4)$, and $f^+(v_4) < f^+(v_2)$ for $0 \le r_1$ and $r_2 < r_3$. Thus, the vertex sums are all distinct.

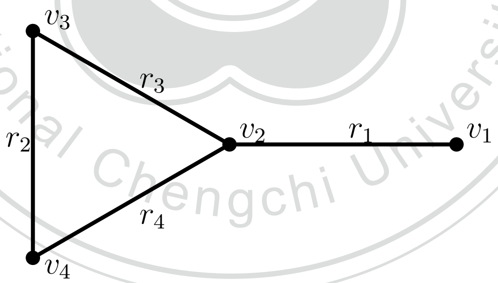

Figure 4.3: Edge labeling f of paw

Case 2. $r_1 < 0 \le r_2 < r_3 < r_4$.

We take a look at both of these situations: $r_2 < r_1 + r_3$ and $r_1 + r_3 \le r_2$.

Subcase 2.1. $r_2 < r_1 + r_3$.

We follow the same procedure as in Case 1 in terms of labeling the edges (see Figure 4.3). Then vertex sums $f^+(v_1) = r_1$, $f^+(v_2) = r_1 + r_3 + r_4$, $f^+(v_3) = r_2 + r_3$, and $f^+(v_4) = r_2 + r_4$

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Note that $f^+(v_1) < f^+(v_3)$ for $r_1 < 0$ and $0 < r_2 < r_3$, $f^+(v_3) < f^+(v_4)$, and $f^+(v_4) < f^+(v_2)$ for $r_2 < r_1 + r_3$. Thus, the vertex sums are all distinct.

Subcase 2.2. $r_1 + r_3 \le r_2$.

We define an edge labeling g of G with labels in $\{r_1, r_2, r_3, r_4\}$ by $g(v_1v_2) = r_1$, $g(v_2v_3) = r_2$, $g(v_3v_4) = r_4$, and $g(v_2v_4) = r_3$ (see Figure 4.4). Then vertex sums $g^+(v_1) = r_1$, $g^+(v_2) = r_1 + r_2 + r_3$, $g^+(v_3) = r_2 + r_4$, and $g^+(v_4) = r_3 + r_4$.

Note that $g^+(v_1) < g^+(v_2)$ for $0 \le r_2, r_3$, and $g^+(v_3) < g^+(v_4)$. Also note that

$$\begin{aligned} g^+(v_2) =& r_1 + r_2 + r_3 = r_2 + (r_1 + r_3) \\ \le& r_2 + r_2 < r_2 + r_4 = g^+(v_3). \end{aligned} \tag{4.2.1}$$

Thus, the vertex sums are all distinct.

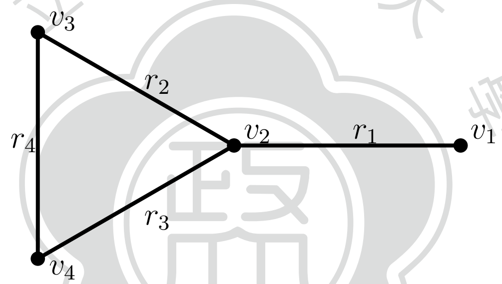

Figure 4.4: Edge labeling g of paw

Case 3. $r_1 < r_2 < 0 \le r_3 < r_4$.

We distinguish two subcases: $r_2 + r_3 \neq 0$ and $r_2 + r_3 = 0$.

Subcase 3.1. $r_2 + r_3 \neq 0$.

We use the edge labeling g of G with labels in $\{r_1, r_2, r_3, r_4\}$ by $g(v_1v_2) = r_1$, $g(v_2v_3) = r_2$, $g(v_3v_4) = r_4$, and $g(v_2v_4) = r_3$ (see Figure 4.4). Then vertex sums $g^+(v_1) = r_1$, $g^+(v_2) = r_1 + r_2 + r_3$, $g^+(v_3) = r_2 + r_4$, and $g^+(v_4) = r_3 + r_4$

Note that

$$g^+(v_2) = r_1 + r_2 + r_3 = r_2 + (r_1 + r_3) < r_2 + r_3 < r_2 + r_4 = g^+(v_3) \tag{4.2.2}$$

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for $r_1 < 0$ and $r_3 < r_4$, and $g^+(v_3) < g^+(v_4)$. Also note that $g^+(v_1) < g^+(v_3)$ for $r_1 < r_2$, $0 < r_4$, and $g^+(v_1) = r_1 \neq r_1 + r_2 + r_3 = g^+(v_2)$ under the assumption of this subcase. Thus, the vertex sums are all distinct.

Subcase 3.2. $r_2 + r_3 = 0$.

- $r_1 + r_4 < r_2$.

We use the edge labeling h of G with labels in $\{r_1, r_2, r_3, r_4\}$ by $h(v_1v_2) = r_4$, $h(v_2v_3) = r_1$, $h(v_3v_4) = r_2$, and $h(v_2v_4) = r_3$ (see Figure 4.5). Then vertex sums $h^+(v_1) = r_4$, $h^+(v_2) = r_1 + r_3 + r_4$, $h^+(v_3) = r_1 + r_2$, and $h^+(v_4) = r_2 + r_3$

Note that

$$h^+(v_3) = r_1 + r_2 < r_1 + r_3 + r_4 = h^+(v_2), \tag{4.2.3}$$

for $r_2 < 0 \leq r_3 < r_4$, and

$$h^+(v_2) = r_1 + r_3 + r_4 = (r_1 + r_4) + r_3 < r_2 + r_3 = h^+(v_4), \tag{4.2.4}$$

for $r_2 < 0 \leq r_3 < r_4$, and $h^+(v_4) = r_2 + r_3 = 0 < r_4 = h^+(v_1)$. The vertex sums are all distinct.

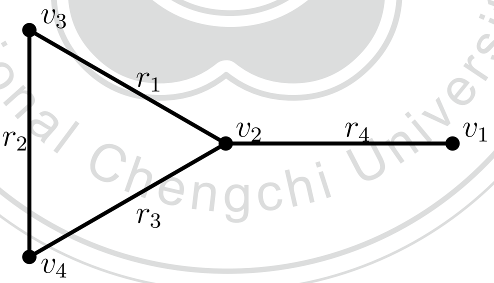

Figure 4.5: Edge labeling h of paw

- $r_2 < r_1 + r_4$.

We use the edge labeling h of G with labels in $\{r_1, r_2, r_3, r_4\}$ by $h(v_1v_2) = r_4$, $h(v_2v_3) = r_1$, $h(v_3v_4) = r_2$, and $h(v_2v_4) = r_3$ (see Figure 4.5). Then vertex sums $h^+(v_1) = r_4$, $h^+(v_2) = r_1 + r_3 + r_4$, $h^+(v_3) = r_1 + r_2$, and $h^+(v_4) = r_2 + r_3$

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Note that $h^+(v_3) = r_1 + r_2 < r_2 + r_3 = h^+(v_4)$, and

$$h^+(v_4) = r_2 + r_3 < (r_1 + r_4) + r_3 = h^+(v_2) \tag{4.2.5}$$

for $r_2 < r_1 + r_4$, and

$$h^+(v_2) = r_1 + r_3 + r_4 < (r_2 + r_3) + r_4 = r_4 = h^+(v_1), \tag{4.2.6}$$

for $r_1 < r_2$, $r_2 + r_3 = 0$. The vertex sums are all distinct.

- $r_1 + r_4 = r_2$.

We use the edge labeling s of G with labels in $\{r_1, r_2, r_3, r_4\}$ by $s(v_1v_2) = r_1$, $s(v_2v_3) = r_2$, $s(v_3v_4) = r_3$, and $s(v_2v_4) = r_4$ (see Figure 4.6). Then vertex sums $s^+(v_1) = r_1$, $s^+(v_2) = r_1 + r_2 + r_4$, $s^+(v_3) = r_2 + r_3$, and $s^+(v_4) = r_3 + r_4$.

Note that

$$s^+(v_1) = r_1 = r_1 + (r_2 + r_3) < r_1 + r_2 + r_4 = s^+(v_2) \tag{4.2.7}$$

for $r_2 + r_3 = 0$, $r_3 < r_4$, and

$$s^+(v_2) = r_1 + r_2 + r_4 = (r_1 + r_4) + r_2 = r_2 + r_2 < r_2 + r_3 = s^+(v_3), \tag{4.2.8}$$

for $r_1 + r_4 = r_2$, $r_2 < r_3$, and $s^+(v_3) = r_2 + r_3 < r_3 + r_4 = s^+(v_4)$. The vertex sums are all distinct.

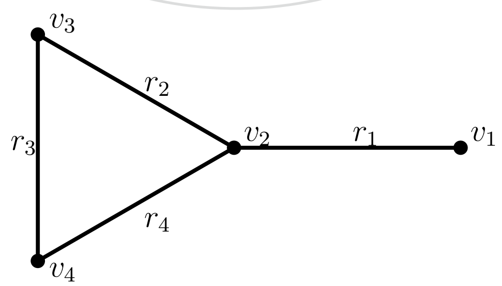

Figure 4.6: Edge labeling s of paw

Case 4. $r_1 < r_2 < r_3 < 0 < r_4$.

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We have $-r_4 < 0 < -r_3 < -r_2 < -r_1$. By Case 2, G is $\{-r_4, -r_3, -r_2, -r_1\}$-antimagic. By Lemma 2.3.1, we can have G is $\{r_1, r_2, r_3, r_4\}$-antimagic.

Case 5. $r_1 < r_2 < r_3 < r_4 \leq 0$.

In this case, $0 \leq -r_4 < -r_3 < -r_2 < -r_1$. By Case 1, G is $\{-r_4, -r_3, -r_2, -r_1\}$-antimagic. By Lemma 2.3.1, we can have G is $\{r_1, r_2, r_3, r_4\}$-antimagic. This completes the proof. □

4.3 Paths

Let P_{m+1} denote a path with m edges for $m \geq 1$. It has been shown that P_n ($n \geq 3$) is R^+-antimagic [14]. Actually, no path P_i has an antimagic labeling with labels in $\{-1, 0, \dots, i-3\}$ for $i \in \{3, 4, 5\}$ [12]. Hence, P_i ($i = 3, 4, 5$) is not $\mathbb{R}$-antimagic.

Theorem 4.3.1. *The path $P_{m+1}(m \geq 5)$ is $\mathbb{R}$-antimagic graph.*

Proof. Let P be a path with m ($m \geq 5$) edges. Let $L = \{r_1, r_2, \dots, r_m\}$ where $r_1, r_2, \dots, r_m$ be real numbers, and $r_1 < r_2 < \cdots < r_m$. To prove the theorem, we need to assign the numbers in L to the edges of P so that the vertex sums of P are all distinct. We distinguish the following cases: Case 1: L does not contain 0, Case 2: L contains 0.

Case 1. *L does not contain 0.*

A path with size m can be obtained by deleting an edge from C_{m+1}. Let C_{m+1} be the cycle with vertex set $\{v_1, v_2, \cdots, v_{m+1}\}$. From Theorem 3.1.2, C_{m+1} is $\mathbb{R}$-antimagic. There is an edge labeling f of C_{m+1} with labels in $L \cup \{0\}$ such that $f^+(v_1) < f^+(v_2) < \cdots < f^+(v_{m+1})$. In $E(C_{m+1})$, there is an edge $e = xy$ such that $f(e) = 0$ where $x, y \in V(C_{m+1})$. Let $P = C_{m+1} - e$. Then P is an x, y-path (see Figure 4.7). We define an edge labeling g of P with labels in L by $g(e') = f(e')$ where $e' \in E(C_m - e)$. Then, the vertex sum at x is $g^+(x) = f^+(x) - f(e) = f^+(x)$, and the vertex sum at y is $g^+(y) = f^+(y) - f(e) = f^+(y)$. Hence, $g^+(v_i) = f^+(v_i)$ for all $i = 1, 2, \dots, m+1$. Since the vertex sums of C_{m+1} are distinct, the vertex sums of P are distinct.

Case 2. *L contains 0.*

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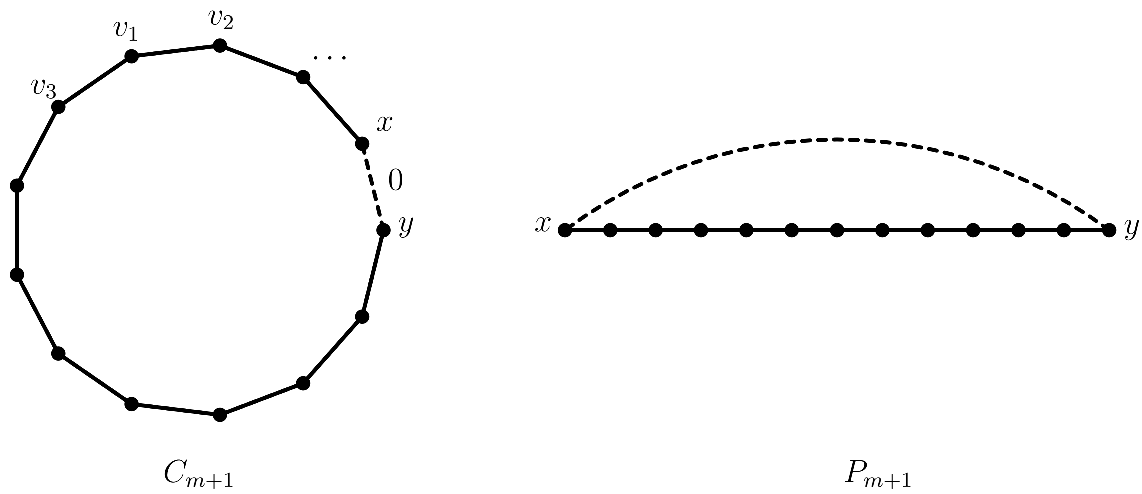

Figure 4.7: L does not contain 0

We will use the edge labeling of C_m to generate the edge labeling of P_{m+1}. Let C_m be the cycle with vertex set $\{v_1, v_2, \cdots, v_m\}$ and edge set $\{v_1v_2\} \cup \{v_iv_{i+2}| \ i = 1, 2, \cdots, m-2\} \cup \{v_{m-1}v_m\}$. As in the proof of Theorem 3.1.2, an edge labeling f of C_m with labels in L is defined by $f(v_1v_2) = r_1$, $f(v_iv_{i+2}) = r_{i+1}$ for $i = 1, 2, \cdots, m-2$, and $f(v_{m-1}v_m) = r_m$. We have $f^+(v_1) = r_1+r_2$, $f^+(v_i) = r_{i-1}+r_{i+1}$ for $i = 2, \cdots, m-1$, and $f^+(v_m) = r_{m-1}+r_m$. Since $r_1+r_2 < r_1+r_3 < r_2+r_4 < r_3+r_5 < r_4+r_6 < \cdots < r_{m-3}+r_{m-1} < r_{m-2}+r_m < r_{m-1}+r_m$, we have $f^+(v_1) < f^+(v_2) < \cdots < f^+(v_m)$.

From Lemma 2.3.1, we have that if G is L-antimagic, then G is $-L$-antimagic. Thus, we only need to consider the following cases: subcase 2.1. $r_1 = 0$, subcase 2.2. $r_2 = 0$, and subcase 2.3. $r_i = 0$ where $3 \le i \le m-2$.

Subcase 2.1. $r_1 = 0$.

We split the vertex v_1 into two new vertices x and y, obtaining an x, y-path P with the same set of edges as C_m (see Figure 4.8). We define an edge labeling g of P with labels in L by $g(e) = f(e)$ where $e \in E(C_m)$. Then, the vertex sum at x is $g^+(x) = f^+(v_1) - r_2 = r_1 = 0$, and the vertex sum at y is $g^+(y) = f^+(v_1) - r_1 = r_2$. Since $0 = r_1 < r_2 < r_1 + r_3 < r_2 + r_4 < r_3 + r_5 < r_4 + r_6 < \cdots < r_{m-3} + r_{m-1} < r_{m-2} + r_m < r_{m-1} + r_m$, we have $g^+(x) < g^+(y) < g^+(v_2) < g^+(v_3) < \cdots < g^+(v_m)$. We see that the vertex sums are all distinct.

Subcase 2.2. $r_2 = 0$.

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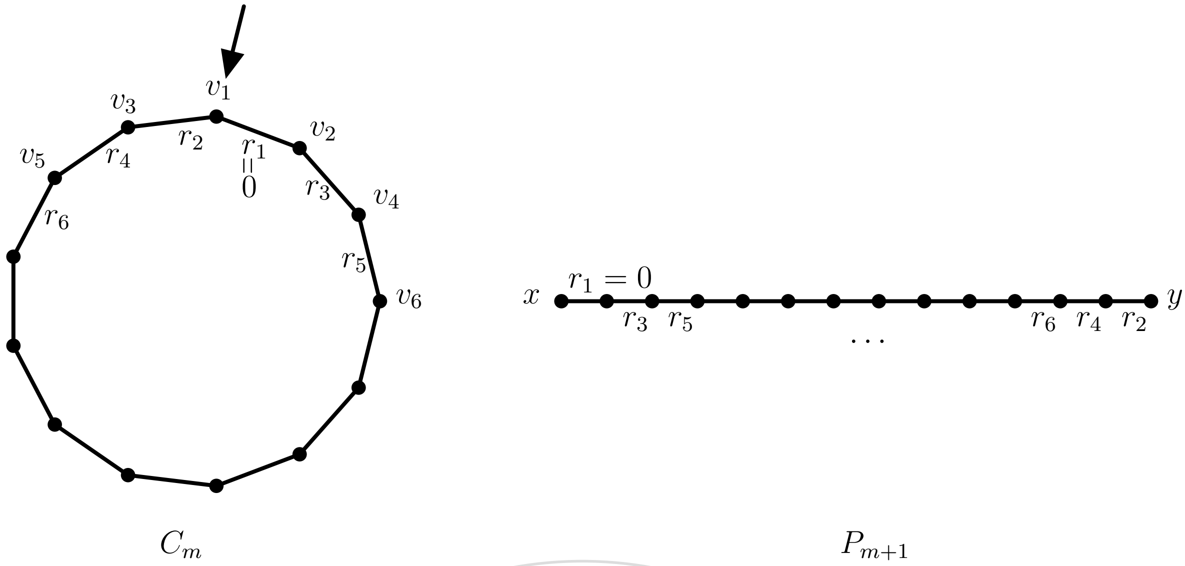

Figure 4.8: $r_1 = 0$

We split the vertex v_4 into two new vertices x and y, obtaining an x, y-path P with the same set of edges as C_m (see Figure 4.9). We define an edge labeling g of P with labels in L by $g(e) = f(e)$ where $e \in E(C_m)$. Then, the vertex sum at x is $g^+(x) = f^+(v_4) - r_5 = r_3$, and the vertex sum at y is $g^+(y) = f^+(v_4) - r_3 = r_5$. Since $r_1 < 0$, $r_1 + r_3 < r_3$. Hence, $r_1 + r_2 < r_1 + r_3 < r_3 < r_4 = r_2 + r_4 < r_5 < r_4 + r_6 < \cdots < r_{m-3} + r_{m-1} < r_{m-2} + r_m < r_{m-1} + r_m$. We have $g^+(v_1) < g^+(v_2) < g^+(x) < g^+(v_3) < g^+(y) < g^+(v_5) < \cdots < g^+(v_m)$. We see that the vertex sums are all distinct.

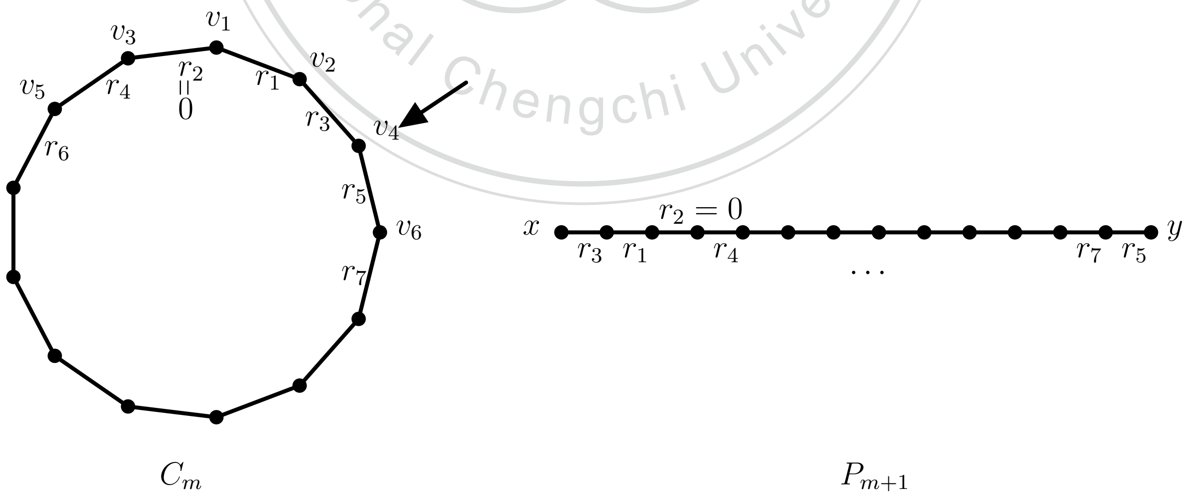

Figure 4.9: $r_2 = 0$

Subcase 2.3. $r_i = 0$ *where* $3 \le i \le m-2$.

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We split the vertex v_i into two new vertices x and y, obtaining an x, y-path P with the same set of edges as C_m (see Figure 4.10). We define an edge labeling g of P with labels in L by $g(e) = f(e)$ where $e \in E(C_m)$. Then, the vertex sum at x is $g^+(x) = f^+(v_i) - r_{i+1} = r_{i-1}$, and the vertex sum at y is $g^+(y) = f^+(v_i) - r_{i-1} = r_{i+1}$. Since $r_1 + r_2 < r_1 + r_3 < \cdots < r_{i-2} + r_i = r_{i-2} < r_{i-1} < r_{i+1} < r_{i+2} = r_i + r_{i+2} < \cdots < r_{m-3} + r_{m-1} < r_{m-2} + r_m < r_{m-1} + r_m$, we have $g^+(v_1) < g^+(v_2) < \cdots < g^+(v_{i-1}) < g^+(x) < g^+(y) < g^+(v_{i+1}) < \cdots < g^+(v_m)$. We see that the vertex sums are all distinct. $\square$

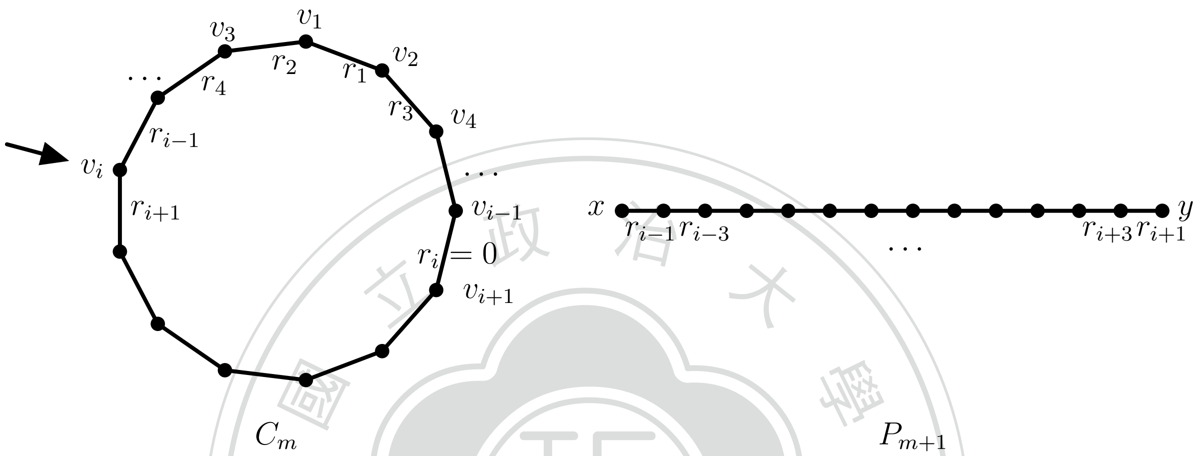

Figure 4.10: $r_i = 0$ $(3 \leq i \leq m - 2)$

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Chapter 5

Conclusions and further studies

5.1 Results

In this thesis, we propose the notion of $\mathbb{R}$-antimagic graph. This is a generalization of $\mathbb{R}^+$-antimagic graph. Every $\mathbb{R}$-antimagic graph is $\mathbb{R}^+$-antimagic, and every $\mathbb{R}^+$-antimagic is antimagic. Not all $\mathbb{R}^+$-antimagic graphs (e.g. stars and $P_n, n = 3, 4, 5$) are $\mathbb{R}$-antimagic.

In Chapter 2, We prove that every $\mathbb{R}$-antimagic graph is $\mathbb{C}$-amtimagic. We also show that every $\mathbb{R}^+$-antimagic graph is also $\mathbb{R}$-antimagic if the graph is regular. Additionally, we discover a class of regular graphs that are $\mathbb{R}$-antimagic (see Theorem 2.3.5). One of the graphs in this class is the Peterson graph.

In Chapter 3, we show that cycles, and complete graphs of order ≥ 3 are $\mathbb{R}$-antimagic. Assume that G is a complete graph (except K_1) or a cycle with $V(G) = \{u_1, u_2, \cdots, u_n\}$. We have found that all the vertices of G can be listed as $u_1, u_2, \cdots, u_n$ such that for every $A \subseteq \mathbb{R}$ with $|A| = |E(G)|$, there is an edge labeling f of G with labels in A such that $f^+(u_1) < f^+(u_2) < \cdots < f^+(u_n)$. The property we call uniformly $\mathbb{R}$-antimagic property which is independent of the choice of the subset A of $\mathbb{R}$. Clearly, every uniformly $\mathbb{R}$-antimagic is $\mathbb{R}$-antimagic. We prove that Cartesian products $G_1 \square G_2 \square \cdots \square G_n$ $(n \geq 2)$ are uniformly $\mathbb{R}$-antimagic, where each G_i is a complete graph of order ≥ 2 or a cycle.

In Chapter 4, we prove that wheels, paws, and paths of order ≥ 6 are $\mathbb{R}$-antimagic.

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5.2 Discussions

In our study, we use labelings modified from those in [7,18] and make them more systematic for Cartesian products of cycles and complete graphs. The proofs in this dissertation provide efficient algorithms for finding edge labelings of Cartesian products of cycles and complete graphs. Our contribution is to quickly find the edge labelings of Cartesian products of cycles and complete graphs through the algorithms we constructed. It has been proved that the Cartesian products $G_1 \Box G_2 \Box \cdots \Box G_n$ $(n \geq 2)$ of $G_1, G_2, \cdots, G_n$ are (uniformly) $\mathbb{R}$-antimagic if each G_i is either a complete graph (except K_1) or a cycle in Section 3.2 and 3.2.

We construct some classes of uniformly $\mathbb{R}$-antimagic graphs through Cartesian products. The *join* of simple graphs G and H is denoted by $G + H$ with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$. That some join graphs are antimagic has been proved in [2,17]. In [17], Wang et al. use the way of listing edges in [8] to show that a class of join graphs is antimagic. It makes the method of labelings in our study more plausible.

We were the first to propose uniformly $\mathbb{R}$-antimagic property [6]. Let G be a graph, and f be an edge labeling of G with labels in $\{1, 2, \ldots, |E(G)|\}$. If for any two distinct vertices u, v of G, $f^+(u) < f^+(v)$ whenever $deg(u) < deg(v)$, then G is called *strongly antimagic* [3,4]. Some results strongly antimagic graphs have been shown in [3,4]. Uniformly $\mathbb{R}$-antimagic property and strongly antimagic are similar concepts, but different. Assume that G is uniformly $\mathbb{R}$-antimagic and that for any two distinct vertices u and v of G, the vertex sum of u is less than the vertex sum of v. The degree of vertex u may not be less than the degree of vertex v.

5.3 Further studies

Generalized Petersen graphs $P(n, k)$ $(1 \leq k \leq \frac{n}{2})$, which consist of the vertex set $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and the edge set $\{u_i u_{i+1} \mid i = 0, 1, 2, \ldots, n-1\} \cup \{v_i v_{i+k} \mid i = 0, 1, 2, \ldots, n-1\} \cup \{u_i v_i \mid i = 0, 1, 2, \ldots, n-1\}$ with indices taken by modulo n, were defined by Watkins [20]. The standard Petersen graph is the instance $P(5, 2)$ (see Figure 2.5). Another example is $P(7, 3)$ (see Figure 5.1). From Theorem 2.3.5 and Corollary 2.3.1, we have the following conjecture:

Conjecture. *Generalized Petersen graphs* $P(n, k)$ *are* $\mathbb{R}$*-antimagic where* $(n, k) = 1$.

In the future, we would like to study the following issues:

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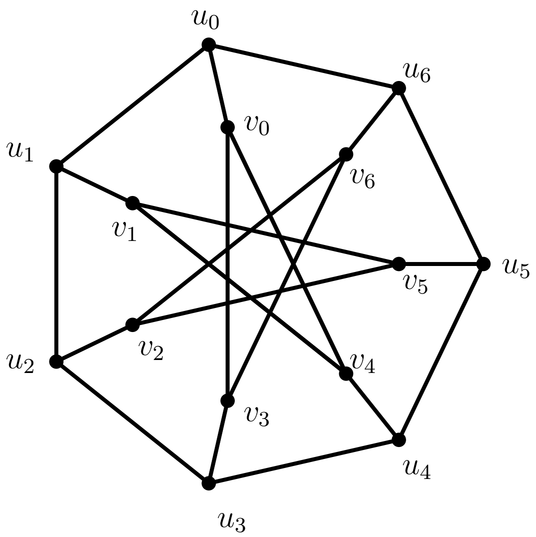

Figure 5.1: Generalized Petersen graph $P(7,3)$

1. $\mathbb{R}$-antimagicness of other graphs (e.g., generalized Petersen graph, paths, trees, caterpillars, forests etc.)

2. $\mathbb{R}$-antimagicness of some special classes of regular graphs

3. $\mathbb{R}$-antimagic labelings of some Cartesian products of some types of graphs

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