國立政治大學應用數學系

碩士學位論文

三角點陣上的簡單隨機漫步
Simple Random Walk on Triangle Lattice

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中文摘要

在本篇文章中，我們將介紹在二維三角點陣上的簡單隨機漫步。我們首先介紹位勢核函數 $a(x)$，其中 $x \in \mathbb{Z}^2$，我們求得在 $\|x\|$ 趨近於無窮下，$a(x)$ 會近似於 $\ln \|x\|$，並對其收斂速度進行討論。此外，假設 S_n 為一在三角點陣上的簡單隨機漫步，我們觀察到 $a(S_n)$ 在不通過原點的情況下是為鞅，我們設 S_n 的起始點位於大小兩圓 $B(R)$ 與 $B(r)$ 之間，利用可選停止定理，我們將 $a(\cdot)$ 與逃脫兩圓之間機率做了連結，並且我們發現在 R 趨近於無窮下先碰到大圓 $B(R)$ 的機率為 $O(1/\ln R)$。在特別情況下，我們也能求得逃脫原點的機率。再者，比較三角點陣與正方點陣，我們觀察到兩者在逃脫大小圓的機率行為是沒有差別的。最後，我們介紹了有關調和測度與容度，這些工具可以將我們的結果延伸至逃脫任意有限集合，我們也介紹些定理證明調和測度是為從無窮遠處開始到入口點的機率，並一樣討論其收斂速度。

關鍵字：隨機漫步、位勢核、振盪積分、鞅、可選停止定理、調和測度、容度

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Abstract

In this thesis, we will introduce the simple random walk on the triangular lattice. We first introduce the potential kernel function $a(x)$ for $x \in \mathbb{Z}^2$. We conclude that $a(x) \approx \ln \|x\|$ as $\|x\| \to \infty$. Moreover, the rate of convergence is discussed too. Besides, let S_n be the simple random walk on the triangular lattice. We observe that $a(S_n)$ is a martingale without visiting the origin. We set our S_n starting at the point between two circle, $B(r)$ and $B(R)$ with $r < R$. Using the optional stopping theorem, we make the connection between $a(\cdot)$ and escaping probability from two circle. Moreover, as $R \to \infty$, we find that the probability that visiting $B(R)$ first is $O(1/\ln R)$. In the specific case, we can also find the probability that escaping from the origin. Futhermore, compare triangular lattice with the square lattice, we observe that there is no difference between them in the behavior of escaping from circle. Finally, we introduce the concept of harmonic measure and capacity. These can extend our results to calculate the probability of escaping from any finite set. We also introduce some theorem to prove that the harmonic measure is the probability of entrance point starting at infinity and also discuss the rate of convergence.

Keywords: random walk, potential kernel , oscillatory integral, martingale, optional stopping theorem, harmonic measure, capacity

DOI:10.6814/NCCU202200379

Contents

DOI:10.6814/NCCU202200379

DOI:10.6814/NCCU202200379

List of Tables

DOI:10.6814/NCCU202200379

List of Figures

DOI:10.6814/NCCU202200379

Chapter 1

Introduction

1.1 Random Walk

In 1905, the problem of random walk was first introduced by Karl Pearson [8]. He asked that a man starts from a point O and walks randomly in a straight line and what's the probability that he is at a distance between r and $r + \delta r$ from his starting point O. Nowadays, these topics are still one part of the most interesting problems for the researchers in probability, physics and finance. Random walks have been widely used to engineering and many scientific fields. It explains the observed behaviors of many processes in these fields. For example, the simulation of the path of a molecule as it travels in a liquid or gas or prediction of the price of a fluctuating stock.

A well-known random walk model is a random walk on an integer lattice, where at each step one jumps to another site according to some probability distribution. In a simple random walk, one can only jump to neighboring sites of the lattice and the probabilities of jumping to each one of its neighboring sites are the same. Besides, a simple random walk in one-dimensional converges to a Brownian motion. This result was introduced by Monroe D. Donsker [2] in 1951. Brownian motion is the random motion of particles suspended in a medium. This motion is named after Robert Brown, who first described the phenomenon in 1827 [1]. In 1905, the theoretical physicist Albert Einstein published a paper [3] where he modeled the motion of the pollen particles.

To discuss the proposition of random walk, we will use the well-known method, martingale. The concept of martingale in probability theory was introduced by Paul Lévy in 1934 [7].

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Martingale referred to a class of betting strategies that was popular in 18th-century France. And in probability theory, a martingale is a sequence of random variables for which the conditional expectation of the next value is equal to the present value. It's easy to see that a simple random walk is a martingale. And we will use the theory of martingale to deal with the model.

As a general fact, a random walk may be recurrent (i.e., it returns to its starting point for infinitely many times almost surely) or transient (i.e., it has a positive probability that one will never return to its starting point). In 1912, a fundamental result about the simple random walk is introduced by Geogre Pólya [9].

Theorem 1.1.1. *The simple random walk in dimension d is recurrent for $d = 1, 2$ and transient for $d \geq 3$.*

Moreover, the recurrent case can be classified further as "positive recurrent" in which case the expected time of revisiting its starting point is finite and "null recurrent" in which case the corresponding expected time is infinite. And both one-dimensional and two-dimensional simple random walks are null recurrent.

We will focus on the recurrent random walks. It is easy to see that for the one-dimensional simple random walk, the probability that one gets more than distance n away from its starting point without revisiting it is $\frac{1}{n}$. And for the two-dimensional simple random walk, the probability is approximately $(1.0293737 + \frac{2}{\pi} \ln n)^{-1}$. The result above is not quiet easy to get, and it is introduced by Serguei Popov [10]. Note that although both of these probabilities converge to zero as $n \to \infty$, the two-dimensional case is much slower than the one-dimensional case. That means that 2-dimensional case is not quite "recurrent" as we think.

In this thesis, we will focus on two-dimensional cases. Let $G = (E, V)$ where $V = \mathbb{Z}^2$ be a graph in two-dimensional. Let $X_1, X_2, \cdots$ be a sequence of G-valued i.i.d random variables. And let $S_n = X_0 + X_1 + \cdots + X_n$ be the sum of them with $X_0 = 0$. Then S_n can be seen as a random walk on two-dimensional integer lattice. We define "potential kernel" as follow:

$$a(x) = \sum_{n=0}^{\infty} P(S_n = 0) - P(S_n = x)$$

The concept of this function is our knowledge that two-dimensional random walk is recurrent. Hence, $\sum_{n=0}^{\infty} P(S_n = 0)$ will be infinite. We want to observe how the difference of two infinite numbers. There are many references analysing the potential kernel. Frank

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Spitzer [11] has shown that if the random walk satisfies some conditions, then $a(x) \approx \ln \|x\|$ as $\|x\| \to \infty$. Moreover, Fukai, Yasunari, and Kôhei Uchiyama [4] introduced more general cases with the error terms. Gregory F. Lawler [6] applied the local central limit theorem to show the same result. For the simple random walk case, $a(x) = \frac{2}{\pi} \ln \|x\| + \frac{2\gamma + \ln 8}{\pi} + O(\|x\|^{-2})$ where γ is Euler's constant.

1.2 Triangular Lattice and Spread-out Model

In this thesis, we introduce the random walk on the triangular lattice. There are many kinds of triangular lattices. The following figures show some usual kinds of triangular lattices. Note that each of them can be transformed into each other. And we will focus on the equilateral triangle case i.e. the model at the left side of the Fig(1.1).

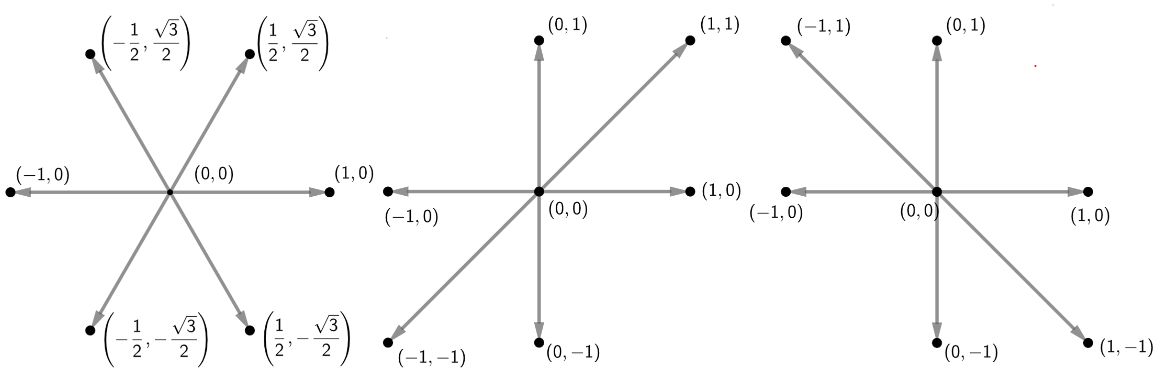

Figure 1.1: Different kinds of triangular lattice

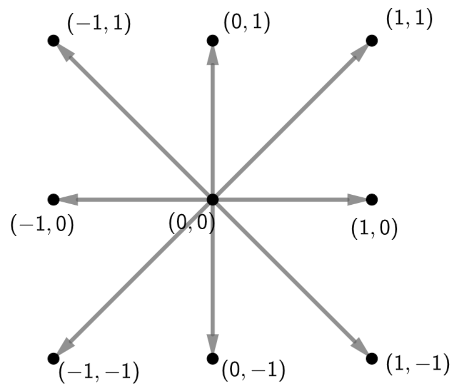

Figure 1.2: Spread-out model

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In this model, we will assign the distribution by giving the same probability for going to each way. We define it as the simple random walk on the triangualr lattice. And in this thesis we discuss the potential kernel on it (see Theorem 2.1.1 in the next chapter). Moreover, our method for dealing with potential kernel can be applied to any setting in which the distribution that is symmetric with finite values and its fourth moment can be expressed as its quadratic form, such as spread-out model which is shown in the Fig(1.2).

The rest of this thesis is organized as follows. The main results are presentd in Chapter 2. We give the proof of our main theorm of potential kenrel in Section 2.1, and apply our main theorem to the triangualr lattice in Section 2.2. In Chapter 3, we describe the proposition of potential kernel on triangualr lattice and discuss the escaping probability in Section 3.1. Moreover, we discuss the restricted Green function in Section 3.2. In Chapter 4, we introduce the concept of harmonic measure and capacity.

1.3 Notation

Here we list the notation recurrently used in this thesis

- $\mathbb{L}_T$ is the set of triangualr lattice generated by $\{(1,0),(\frac{1}{2},\frac{\sqrt{3}}{2}),(\frac{1}{2},-\frac{\sqrt{3}}{2})\}$;
- $B(x,r)=\{y:\|y-x\|\leq r\}$ is the ball in $\mathbb{Z}^2$ or $\mathbb{L}_T$; $B(r)$ stands for $B(0,r)$;
- we write $x\sim y$ if x and y are neighbours;
- $\mathbf{1}_{\{\text{event}\}}$ is the indicator function of event;
- $\mathbb{P}_x$ and $\mathbb{E}_x$ are probability and expectation for a process starting from x;
- $\tau_\Lambda\geq 0$ and $\tau_\Lambda^+\geq 1$ are entrance and hitting times of Λ;

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Chapter 2

Main Result

2.1 Potential Kernel on Integer Lattice

Let $G = (E, V)$ where $V = \mathbb{Z}^2$ be a graph in two-dimensional space. Let $X_1, X_2, \cdots$ be a sequence of G-valued i.i.d random variables. And let $S_n = X_0 + X_1 + \cdots + X_n$ be the recurrent random walk starting at the orgin on G, i.e. $X_0 = (0, 0)$. We define the potential kernel $a(\cdot)$ by

$$a(x) = \sum_{n=0}^{\infty} \big(P(S_n = (0,0)) - P(S_n = x)\big) \tag{2.1.1}$$

Let $\phi_{X_1}(\theta)$ be the characteristic function of X_1. Since $\{X_n\}_{n=0}^{\infty}$ are i.i.d , $\phi_{S_n}(\theta) = \phi_{X_1+\cdots+X_n}(\theta) = (\phi_{X_1}(\theta))^n$. Also our random variables are on $\mathbb{Z}^2$, we can use Inverse formula and get

$$P(S_n = x) = \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} (\phi_{X_1}(\theta))^n e^{-i\theta\cdot x}\, d^2\theta$$

Hence,

$$\begin{aligned} a(x) &= \sum_{n=0}^{\infty} \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} (\phi_{X_1}(\theta))^n (1 - e^{-i\theta\cdot x})\, d^2\theta \\ &= \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \sum_{n=0}^{\infty} (\phi_{X_1}(\theta))^n (1 - e^{-i\theta\cdot x})\, d^2\theta \\ &= \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{1 - e^{-i\theta\cdot x}}{1 - \phi_{X_1}(\theta)}\, d^2\theta \end{aligned} \tag{2.1.2}$$

To evaluate $a(\cdot)$, we consider a simpler case in which the random variable is symmetric with finite values. That means that our random walk is on the integer lattice $\mathbb{Z}^2$ and all of the even

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moments of X_1 exist, and all of the odd moments are 0. Hence, the characteristic function of X_1 can be written as

$$\phi_{X_1}(\theta) = E[1 + i\langle\theta, X_1\rangle - \frac{\langle\theta, X_1\rangle^2}{2} - i\frac{\langle\theta, X_1\rangle^3}{6} + O(\|\theta\|^4)] = 1 - \frac{E\langle\theta, X_1\rangle^2}{2} + O(\|\theta\|^4).$$

Note that $E\langle\theta, X_1\rangle^2 = \theta^T\sum\theta$, where $\sum$ is the covariance matrix of X_1. Since the covariance matrix is symmetric and positive definite, i.e. there exists a matrix Λ so that $\sum = \Lambda\Lambda^T$, and $E\langle\theta, X_1\rangle^2 = \left\|\Lambda^T\theta\right\|^2$. Hence,

$$1 - \phi_{X_1}(\theta) = \frac{\left\|\Lambda^T\theta\right\|^2}{2} + O(\|\theta\|^4). \tag{2.1.3}$$

To estimate $a(x)$, we consider the following formula and decompose it into five parts :

$$\begin{aligned} a(x) &= \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2}\frac{1-e^{-i\theta\cdot x}}{1-\phi_{X_1}(\theta)}\,d^2\theta \\ &= \frac{1}{(2\pi)^2}\int_0^\infty dt\int_{[-\pi,\pi]^2} e^{-t(1-\phi_{X_1}(\theta))}(1-e^{-i\theta\cdot x})\,d^2\theta \\ &= I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned}$$

where,

$$\begin{aligned} I_1 &= \frac{1}{(2\pi)^2}\int_0^1 dt\int_{[-\pi,\pi]^2} e^{-t(1-\phi_{X_1}(\theta))}(1-e^{-i\theta\cdot x})\,d^2\theta \\ &\quad - \frac{1}{(2\pi)^2}\int_1^U dt\int_{[-\pi,\pi]^2} e^{-t(1-\phi_{X_1}(\theta))}e^{-i\theta\cdot x}\,d^2\theta \\ I_2 &= \frac{1}{(2\pi)^2}\int_T^\infty dt\int_{[-\pi,\pi]^2}[e^{-t(1-\phi_{X_1}(\theta))} - e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}](1-e^{-i\theta\cdot x})\,d^2\theta \\ I_3 &= \frac{1}{(2\pi)^2}\int_T^\infty dt\int_{[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}(1-e^{-i\theta\cdot x})\,d^2\theta \\ &\quad + \frac{1}{(2\pi)^2}\int_1^T dt\int_{[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}\,d^2\theta \end{aligned}$$

DOI:10.6814/NCCU202200379

$$I_4 = \frac{1}{(2\pi)^2}\int_1^T dt \int_{[-\pi,\pi]^2} [e^{-t(1-\phi_{X_1}(\theta))} - e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}]\, d^2\theta$$

$$I_5 = -\frac{1}{(2\pi)^2}\int_U^T dt \int_{[-\pi,\pi]^2} [e^{-t(1-\phi_{X_1}(\theta))} - e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}] e^{-i\theta\cdot x}\, d^2\theta$$
$$-\frac{1}{(2\pi)^2}\int_U^T dt \int_{[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}} e^{-i\theta\cdot x}\, d^2\theta$$

where $T \approx \|x^2\|$,$U \approx \|x\|$ whose values will be chosen later. First, we consider I_3

$$I_3 = \frac{1}{(2\pi)^2}\int_T^\infty dt \int_{\mathbb{R}^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}} (1-e^{-i\theta\cdot x})\, d^2\theta$$
$$-\frac{1}{(2\pi)^2}\int_T^\infty dt \int_{\mathbb{R}^2\setminus[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}} (1-e^{-i\theta\cdot x})\, d^2\theta$$
$$+\frac{1}{(2\pi)^2}\int_1^T dt \int_{\mathbb{R}^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}\, d^2\theta$$
$$-\frac{1}{(2\pi)^2}\int_1^T dt \int_{\mathbb{R}^2\setminus[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}\, d^2\theta$$

Here, we let $\theta' = \Lambda^T\theta$, then $\theta = (\Lambda^T)^{-1}\theta'$, and $\left|\det(\frac{\partial\theta}{\partial\theta'})\right| = \left|(\det\Lambda^T)^{-1}\right| = \frac{1}{\sqrt{\det\sum}}$. Also, $i\theta\cdot x = i\left\langle(\Lambda^T)^{-1}\theta', x\right\rangle = i\left\langle\theta', ((\Lambda^T)^{-1})^T x\right\rangle = i\left\langle\theta', \Lambda^{-1}x\right\rangle$. We can get

$$\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}} (e^{-i\theta\cdot x})\, d^2\theta = \frac{1}{2\pi}\frac{1}{t}\int_{\mathbb{R}^2} \overbrace{(\frac{1}{\sqrt{2\pi\frac{1}{t}}})^2 e^{-\frac{\|\theta'\|^2}{2\frac{1}{t}}} (e^{-i\theta'\cdot\Lambda^{-1}x})}^{cf\ of\ Gaussian\ distribution} \left|\det(\frac{\partial\theta}{\partial\theta'})\right|\, d^2\theta'$$
$$= \frac{1}{t}\frac{1}{2\pi\sqrt{\det\sum}} e^{-\frac{\left\|\Lambda^{-1}x\right\|^2}{2t}} \tag{2.1.4}$$

Also, since Λ is symmetry and positive definite, there exists a $\delta > 0$ so that $\frac{\left\|\Lambda^T\theta\right\|^2}{2} > \delta\left\|\theta\right\|^2$ for all $\theta \in \mathbb{R}^2$. Hence,

$$\left|\int_T^\infty dt \int_{\mathbb{R}^2\setminus[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}} (1-e^{-i\theta\cdot x})\, d^2\theta\right| \le 2\int_T^\infty dt \int_{\mathbb{R}^2\setminus[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}\, d^2\theta$$
$$\le 2\int_T^\infty dt \int_{\mathbb{R}^2\setminus[-\pi,\pi]^2} e^{-t\delta\|\theta\|^2}\, d^2\theta$$
$$\le O(1)\int_T^\infty \frac{e^{-\pi^2 t}}{t} \le \frac{O(1)}{T} = O(\|x\|^{-2}) \tag{2.1.5}$$

DOI:10.6814/NCCU202200379

By the same way as (2.1.5), we can get

$$\frac{1}{(2\pi)^2}\int_1^T dt\int_{\mathbb{R}^2\setminus[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}\,d^2\theta$$
$$=\frac{1}{(2\pi)^2}\int_1^\infty dt\int_{\mathbb{R}^2\setminus[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}\,d^2\theta-\frac{1}{(2\pi)^2}\int_T^\infty dt\int_{\mathbb{R}^2\setminus[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}\,d^2\theta$$
$$=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2\setminus[-\pi,\pi]^2}\frac{e^{-\frac{\left\|\Lambda^T\theta\right\|^2}{2}}}{\frac{\|\Lambda^T\theta\|^2}{2}}\,d^2\theta+O(\|x\|^{-2})=c_1+O(\|x\|^{-2})$$

Therefor, by (2.1.4) and (2.1.5)

$$I_3=\int_T^\infty\frac{1}{2\pi\sqrt{\det\sum}}\frac{1}{t}[1-e^{-\frac{\left\|\Lambda^{-1}x\right\|^2}{2t}}]\,dt+\int_1^T\frac{1}{2\pi\sqrt{\det\sum}}\frac{1}{t}\,dt-c_1+O(\|x\|^{-2})$$
$$=\frac{1}{2\pi\sqrt{\det\sum}}[\int_0^{\frac{\left\|\Lambda^{-1}x\right\|^2}{2T}}(\frac{1-e^{-u}}{u})\,du+\ln T]-c_1+O(\|x\|^{-2})$$

Let $T=\frac{\left\|\Lambda^{-1}x\right\|^2}{2}\approx\|x\|^2$, we have

$$I_3=\frac{1}{2\pi\sqrt{\det\sum}}\ln\left\|\Lambda^{-1}x\right\|^2-c_1+c_2+O(\|x\|^{-2})\tag{2.1.6}$$

where $c_1=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2\setminus[-\pi,\pi]^2}\frac{e^{-\frac{\left\|\Lambda^T\theta\right\|^2}{2}}}{\frac{\|\Lambda^T\theta\|^2}{2}}\,d^2\theta$ and $c_2=\frac{1}{2\pi\sqrt{\det\sum}}[-\ln 2+\int_0^1\frac{1-e^{-u}}{u}\,du]$

Next, we consider I_2, and by using Taylor's formula we get

$$\begin{aligned}e^{-t(1-\phi_{X_1}(\theta))}-e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}&=e^{-t(\frac{\left\|\Lambda^T\theta\right\|^2}{2}+O(1)\left\|\Lambda^T\theta\right\|^4)}-e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}\\&=e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}[e^{-t(O(1)\left\|\Lambda^T\theta\right\|^4)}-1]\\&=O(1)te^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}\left\|\Lambda^T\theta\right\|^4\end{aligned}\tag{2.1.7}$$

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Hence,

$$
\begin{aligned}
I_2 &\le \frac{2}{(2\pi)^2}\int_T^\infty dt \int_{[-\pi,\pi]^2} \left| e^{-t(1-\phi_{X_1}(\theta))} - e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}} \right| d^2\theta \\
&\le O(1)\int_T^\infty dt \int_0^{\sqrt{2}\pi} tr^5 e^{-\frac{tr^2}{2}}\, dr \\
&\le O(1)\int_T^\infty \frac{1}{t^2}\, dt = \frac{O(1)}{T} = O(\|x\|^{-2})
\end{aligned} \tag{2.1.8}
$$

Similarly,

$$
\begin{aligned}
I_4 &= \frac{1}{(2\pi)^2}\int_1^\infty dt \int_{[-\pi,\pi]^2} [e^{-t(1-\phi_{X_1}(\theta))} - e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}]\, d^2\theta \\
&- \frac{1}{(2\pi)^2}\int_T^\infty dt \int_{[-\pi,\pi]^2} [e^{-t(1-\phi_{X_1}(\theta))} - e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}]\, d^2\theta \\
&= c_3 + O(\|x\|^{-2})
\end{aligned} \tag{2.1.9}
$$

where $c_3 = \dfrac{1}{(2\pi)^2}\displaystyle\int_{[-\pi,\pi]^2} [\frac{e^{-(1-\phi_{X_1}(\theta))}}{1-\phi_{X_1}(\theta)} - \frac{e^{-\frac{\left\|\Lambda^T\theta\right\|^2}{2}}}{\frac{\left\|\Lambda^T\theta\right\|^2}{2}}]\, d^2\theta$

To estimate the rest of terms, we need to find the fourth moment of the characteristic function of X_1,

$$
\phi_{X_1}(\theta) = 1 - \frac{E\left\langle \theta, X_1\right\rangle^2}{2} + \frac{E\left\langle \theta, X_1\right\rangle^4}{4!} + O(\|\theta\|^6)
$$

In the triangualr lattice case, we have

$$
\begin{aligned}
E\left\langle \theta, X_1\right\rangle^2 &= \left\|\Lambda^T\theta\right\|^2 = \frac{1}{3}[\theta_1^2+\theta_2^2+(\theta_1+\theta_2)^2] = \frac{2}{3}[\theta_1^2+\theta_2^2+\theta_1\theta_2] \\
E\left\langle \theta, X_1\right\rangle^4 &= \frac{1}{3}[\theta_1^4+\theta_2^4+(\theta_1+\theta_2)^4] = \frac{2}{3}[\theta_1^4+\theta_2^4+2\theta_1^3\theta_2+3\theta_1^2\theta_2^2+2\theta_1\theta_2^3] \\
(E\left\langle \theta, X_1\right\rangle^2)^2 &= \frac{4}{9}[\theta_1^4+\theta_2^4+2\theta_1^3\theta_2+3\theta_1^2\theta_2^2+2\theta_1\theta_2^3] = \frac{2}{3}E\left\langle \theta, X_1\right\rangle^4
\end{aligned}
$$

where $\theta = (\theta_1, \theta_2)$. Hence,

$$
E\left\langle \theta, X_1\right\rangle^4 = \frac{3}{2}(E\left\langle \theta, X_1\right\rangle^2)^2 = \frac{3}{2}\left\|\Lambda^T\theta\right\|^4 \tag{2.1.10}
$$

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Note that the following method can also be adapted for the the random walk whose fourth moment can be expressed as the quadratic forms. For convenience, we let $E\langle\theta, X_1\rangle^4 = \frac{3}{2}\left\|\Lambda^T\theta\right\|^4$ and this yields,

$$1-\phi_{X_1}(\theta)=\frac{1}{2}\left\|\Lambda^T\theta\right\|^2-\frac{1}{16}\left\|\Lambda^T\theta\right\|^4+O(\left\|\Lambda^T\theta\right\|^6)$$

Similarly as (2.1.7), we can get

$$e^{-t(1-\phi_{X_1}(\theta))}-e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}=te^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}[\frac{1}{16}\left\|\Lambda^T\theta\right\|^4+O(\left\|\Lambda^T\theta\right\|^6)] \tag{2.1.11}$$

Now, we consider I_5 and let $I_5=-I_{5,1}-I_{5,2}$, where

$$I_{5,1}=\frac{1}{(2\pi)^2}\int_U^T dt\int_{[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}e^{-i\theta\cdot x}\,d^2\theta$$

$$I_{5,2}=\frac{1}{(2\pi)^2}\int_U^T dt\int_{[-\pi,\pi]^2}[e^{-t(1-\phi_{X_1}(\theta))}-e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}]e^{-i\theta\cdot x}\,d^2\theta$$

Consider $I_{5,1}$, we have

$$\begin{aligned}I_{5,1}=&\frac{1}{(2\pi)^2}\int_U^T dt\int_{\mathbb{R}^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}e^{-i\theta\cdot x}\,d^2\theta\\&-\frac{1}{(2\pi)^2}\int_U^T dt\int_{\mathbb{R}^2\setminus[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}e^{-i\theta\cdot x}\,d^2\theta\end{aligned}$$

Using the same argument of (2.1.5), we get

$$\begin{aligned}\left|\int_U^T dt\int_{\mathbb{R}^2\setminus[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}e^{-i\theta\cdot x}\,d^2\theta\right|&\leq\int_U^T dt\int_{\mathbb{R}^2\setminus[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}\,d^2\theta\\&\leq\int_U^T\frac{e^{-\pi^2t}}{t}\,dt\leq\frac{e^{-\pi^2U}}{U}\leq O(\|x\|^{-2})\end{aligned}$$

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and use the way of (2.1.4)

$$\frac{1}{(2\pi)^2}\int_U^T dt\int_{\mathbb{R}^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}} e^{-i\theta\cdot x}\, d^2\theta = \int_U^T \frac{1}{t}\frac{1}{2\pi\sqrt{\det\sum}} e^{-\frac{\left\|\Lambda^{-1}x\right\|^2}{2t}}\, dt$$
$$= \frac{1}{2\pi\sqrt{\det\sum}}\int_1^{\frac{T}{2U}} \frac{e^{-u}}{u}\, du$$
$$= \frac{1}{2\pi\sqrt{\det\sum}}\int_1^{\infty} \frac{e^{-u}}{u}\, du - \frac{1}{2\pi\sqrt{\det\sum}}\int_{\frac{T}{2U}}^{\infty} \frac{e^{-u}}{u}\, du$$
$$= c_4 + O(\|x\|^{-2})$$

Therefore,

$$I_{5,1} = c_4 + O(\|x\|^{-2}) \tag{2.1.12}$$

Consider $I_{5,2}$, by using (2.1.11) we have

$$I_{5,2} = \frac{1}{(2\pi)^2}\int_U^T \frac{t}{16}\, dt\int_{\mathbb{R}^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}} \left\|\Lambda^T\theta\right\|^4 e^{-i\theta\cdot x}\, d^2\theta$$
$$- \frac{1}{(2\pi)^2}\int_U^T \frac{t}{16}\, dt\int_{\mathbb{R}^2-[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}} \left\|\Lambda^T\theta\right\|^4 e^{-i\theta\cdot x}\, d^2\theta$$
$$+ O(1)\frac{1}{(2\pi)^2}\int_U^T t\, dt\int_{[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}} \left\|\Lambda^T\theta\right\|^6 e^{-i\theta\cdot x}\, d^2\theta$$

and then by the same argument of (2.1.5), we get

$$\left|\int_U^T \frac{t}{16}\, dt\int_{\mathbb{R}^2-[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}} \left\|\Lambda^T\theta\right\|^4 e^{-i\theta\cdot x}\, d^2\theta\right| \le O(1)\int_U^T t\, dt\int_{\sqrt{2}\pi}^{\infty} e^{-t\frac{r^2}{2}} r^5\, dr$$
$$\le O(1)\int_U^T \frac{e^{-\pi^2 t}}{t}\, dt \le O(\|x\|^{-2})$$

and by the same argument of (2.1.8), we get

$$\left|\int_U^T \frac{t}{16}\, dt\int_{[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}} \left\|\Lambda^T\theta\right\|^6 e^{-i\theta\cdot x}\, d^2\theta\right| \le O(1)\int_U^T t\, dt\int_0^{\sqrt{2}\pi} e^{-t\frac{r^2}{2}} r^7\, dr$$
$$\le O(1)\int_U^T \frac{1}{t^3}\, dt = \frac{O(1)}{U^2} = O(\|x\|^{-2})$$

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Now, consider

$$\int_{\mathbb{R}^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}} \left\|\Lambda^T\theta\right\|^4 e^{-i\theta\cdot x}\, d^2\theta = 4\frac{d^2}{dt^2}\big(\int_{\mathbb{R}^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}} e^{-i\theta\cdot x}\, d^2\theta\big)$$
$$= 4\frac{d^2}{dt^2}\big(\frac{1}{2\pi\sqrt{\det\sum}}\frac{1}{t}e^{-\frac{\left\|\Lambda^{-1}x\right\|^2}{2t}}\big)$$
$$= \frac{1}{\pi\sqrt{\det\sum}}[\frac{4}{t^3} - \frac{4\left\|\Lambda^{-1}x\right\|^2}{t^4} + \frac{\left\|\Lambda^{-1}x\right\|^4}{t^5}]e^{-\frac{\left\|\Lambda^{-1}x\right\|^2}{2t}}$$

Therefore,

$$I_{5,2} = \int_U^T \frac{1}{16\pi\sqrt{\det\sum}}[\frac{4}{t^2} - \frac{4\left\|\Lambda^{-1}x\right\|^2}{t^3} + \frac{\left\|\Lambda^{-1}x\right\|^4}{t^4}]e^{-\frac{\left\|\Lambda^{-1}x\right\|^2}{2t}}\, dt + O(\|x\|^{-2})$$
$$= \frac{1}{16\pi\sqrt{\det\sum}}\int_1^{\frac{T}{U}}[4(\frac{2u}{\left\|\Lambda^{-1}x\right\|^2})^2 - 4\left\|\Lambda^{-1}x\right\|^2(\frac{2u}{\left\|\Lambda^{-1}x\right\|^2})^3$$
$$+ \left\|\Lambda^{-1}x\right\|^4(\frac{2u}{\left\|\Lambda^{-1}x\right\|^2}^4)\frac{\left\|\Lambda^{-1}x\right\|^2}{2u^2}e^{-u}\, du + O(\|x\|^{-2})$$
$$= \frac{1}{16\pi\sqrt{\det\sum}\left\|\Lambda^{-1}x\right\|^2}\int_1^{\frac{T}{U}}(8 - 16u + 8u^2)e^{-u}\, du + O(\|x\|^{-2})$$
$$= O(\|x\|^{-2}) \tag{2.1.13}$$

Note that for the square lattice or the spread-out models whose covariance matrice are diagonal, it's much easier to estimate $I_{5,2}$. In these cases, we let $\Lambda^T = [\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}]$ and $x = (x_1, x_2), \theta = (\theta_1, \theta_2)$. It's not necessary to use (2.1.11), and we can just use (2.1.7). Then the part of $I_{5,2}$ will be

$$\int_{[-\pi,\pi]^2}[e^{-t(1-\phi_{X_1}(\theta))} - e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}]e^{-i\theta\cdot x}\, d^2\theta$$
$$= O(1)t\int_{[-\pi,\pi]^2} e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}\left\|\Lambda^T\theta\right\|^4\, d^2\theta$$
$$= O(1)t\big(\int_{-\pi}^{\pi} e^{-t\frac{k^2\theta_1^2}{2}}\theta_1^4 cos(\theta_1 x_1)\, d\theta_1\big)\big(\int_{-\pi}^{\pi} e^{-t\frac{k^2\theta_2^2}{2}}\theta_2^4 cos(\theta_2 x_2)\, d\theta_2\big)$$

Using integration by parts, we have

$$\int_{-\pi}^{\pi} e^{-t\frac{k^2\theta_1^2}{2}}\theta_1^4 cos(\theta_1 x_1)\, d\theta_1$$
$$= \frac{1}{x_1}e^{-t\frac{k^2\theta_1^2}{2}}\theta_1^4 sin(\theta_1 x_1)\Big|_{-\pi}^{\pi} + \frac{1}{x_1}k^2 t\int_{-\pi}^{\pi} e^{-t\frac{k^2\theta_1^2}{2}}\theta_1^5 sin(\theta_1 x_1)\, d\theta_1$$

DOI:10.6814/NCCU202200379

Note that the first part above is 0, and using integration by parts again, we can get that the integral above is $O(|x_1|^{-2})$. Therefore, we find that for the case in which the covariance matrix is diagonal,

$$\int_{[-\pi,\pi]^2}[e^{-t(1-\phi_{X_1}(\theta))} - e^{-t\frac{\left\|\Lambda^T\theta\right\|^2}{2}}]e^{-i\theta\cdot x}\, d^2\theta = O(|x_1|^{-2}\,|x_2|^{-2}) = O(\|x\|^{-2}) \tag{2.1.14}$$

Hence, by $I_{5,1}$ and $I_{5,2}$ we get

$$I_5 = -c_4 + O(\|x\|^{-2}) \tag{2.1.15}$$

where $c_4 = \dfrac{1}{2\pi\sqrt{\det\sum}}\displaystyle\int_1^\infty \frac{e^{-u}}{u}\, du$

Finally, to estimate I_1, we let $I_1 = I_{1,1} - I_{1,2}$ where

$$I_{1,1} = \frac{1}{(2\pi)^2}\int_0^1 dt\int_{[-\pi,\pi]^2} e^{-t(1-\phi_{X_1}(\theta))}(1 - e^{-i\theta\cdot x})\, d^2\theta$$

$$I_{1,2} = \frac{1}{(2\pi)^2}\int_1^U dt\int_{[-\pi,\pi]^2} e^{-t(1-\phi_{X_1}(\theta))}e^{-i\theta\cdot x}\, d^2\theta$$

Consider

$$\begin{aligned}
I_{1,1} &= \frac{1}{(2\pi)^2}\int_0^1 dt\int_{[-\pi,\pi]^2} e^{-t(1-\phi_{X_1}(\theta))}\, d^2\theta - \frac{1}{(2\pi)^2}\int_0^1 dt\int_{[-\pi,\pi]^2} e^{-t(1-\phi_{X_1}(\theta))}e^{-i\theta\cdot x}\, d^2\theta \\
&= c_5 - \frac{1}{(2\pi)^2}\int_0^1 e^{-t}\, dt\int_{[-\pi,\pi]^2}\sum_{n=0}^\infty \frac{\phi_{X_1}(\theta)^n}{n!}t^n e^{-i\theta\cdot x}\, d^2\theta \\
&= c_5 - \int_0^1 e^{-t}\sum_{n=0}^\infty \frac{t^n}{n!}P(S_n = x)\, dt
\end{aligned} \tag{2.1.16}$$

And since

$$\sum_{n=0}^\infty \frac{t^n}{n!}P(S_n = x) = \sum_{n=\lfloor\frac{\|x\|}{\sqrt{2}}\rfloor}^\infty \frac{t^n}{n!}P(S_n = x) \le \sum_{n=\lfloor\frac{\|x\|}{\sqrt{2}}\rfloor}^\infty \frac{t^n}{n!}$$

and using $\displaystyle\sum_{n=M}^\infty \frac{t^n}{n!} \le \frac{t^M}{M!}e^t \le \frac{1}{\sqrt{2\pi M}}(\frac{te}{M})^M)e^t$,

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we have

$$\int_0^1 e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}P(S_n=x)\,dt \le O(1)\int_0^1 (\frac{e}{\|x\|})^{\|x\|}\frac{1}{\sqrt{2\pi\,\|x\|}}t^{\|x\|}\,dt$$
$$= O(1)(\frac{e}{\|x\|})^{\|x\|}\frac{1}{\sqrt{2\pi\,\|x\|}}\frac{1}{(\|x\|+1)} \le O(\|x\|^{-2})$$

Then

$$I_{1,1} = c_5 + O(\|x\|^{-2}) \tag{2.1.17}$$

To estimate $I_{1,2}$, we use the same argument of (2.1.16)

$$I_{1,2} = \int_1^U I_t(x)\,dt \text{ where } I_t(x) = \sum_{n=0}^{\infty}\frac{t^n}{n!}P(S_n=x)$$

And we use the following Lemma which is referrd to [5]

Lemma 2.1.1. *For any $t \ge 1$,$x \in \mathbb{Z}^2$ we have*

$$I_t(x) \le e^{-\|x\|_\infty + t\frac{\sigma^2}{2}}$$

where $\sigma^2 = \sum_y \|y\|^2 P(X_1 = y)$.

Using the lemma above, we have

$$I_{1,2} \le \int_1^U e^{-\|x\|_\infty + t\frac{\sigma^2}{2}} \le \left(e^{-\|x\|_\infty + U\frac{\sigma^2}{2}}\right)U$$

Choose $U = \frac{\|x\|_\infty}{\sigma^2} \approx \|x\|$, we have

$$I_{1,2} \le e^{-\frac{\|x\|_\infty}{2}}\frac{\|x\|_\infty}{\sigma^2} \le O(\|x\|^{-2}) \tag{2.1.18}$$

Hence, by $I_{1,1}$ and $I_{1,2}$ we get

$$I_1 = c_5 + O(\|x\|^{-2}) \tag{2.1.19}$$

where $c_5 = \frac{1}{(2\pi)^2}\int_0^1 dt\int_{[-\pi,\pi]^2} e^{-t(1-\phi_{X_1}(\theta))}\,d^2\theta$

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So it suffices to show the lemma above

Proof. W.L.O.G. let $\|x\|_\infty = |x_1|$ and define

$$\begin{aligned}\varphi_t(s) &= \sum_{x\in\mathbb{Z}^2} e^{sx_1} I_t(x) = e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}\big(\sum_{x\in\mathbb{Z}^2} e^{sx_1}P(X_1=x)\big)^n \\ &= e^{-t}\exp[t\sum_y e^{sy_1}P(X_1=y)] \\ &= e^{-t}\exp[t\sum_y P(X_1=y)\cosh(sy_1)] \\ &= \exp[t\big(\sum_y P(X_1=y)(\cosh(sy_1)-1)\big)]\end{aligned}$$

Also, since $\varphi_t(s) \geq e^{sx_1} I_t(x)$ for all $s > 0$. Hence

$$I_t(x) \leq \exp[-s\|x\|_\infty + t\big(\sum_y P(X_1=y)(\cosh(sy_1)-1)\big)]$$

Besides, $\cosh u \leq 1 + u^2$ if $|u| \leq 1$. We have

$$I_t(x) \leq \exp[-s\|x\|_\infty + \frac{ts^2\sigma^2}{2}]$$

where $\sigma^2 = \sum_y \|y\|^2 P(X_1 = y)$ if $|s| \leq 1$. Take $s = 1$ we get

$$I_t(x) \leq e^{-\|x\|_\infty + t\frac{\sigma^2}{2}}$$

Finally, from all above, we have the following theorem

Theorem 2.1.2. *Let $X_1, X_2,$...be a sequence of i.i.d. $\mathbb{Z}^2$-valued symmetric random variables with finite values, and its fourth moment can be expressed as its quadratic form, then as $\|x\| \to \infty$ its potential kernel will be*

$$a(x) = \frac{1}{2\pi\sqrt{\det\sum}}\ln\left\|\Lambda^{-1}x\right\|^2 + C + O(\|x\|^{-2}), \qquad x\in\mathbb{Z}^2,$$

where C is a constant that depends on the distribution of X_1.

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Remark 2.1.3. *For the square lattice or spread-out models, the above theorem also applies and can be proved in a much easier way as (2.1.14) has shown.*

2.2 Potential Kernel on Triangular Lattice

Next, we consider the triangualr lattice $\mathbb{L}_T$ generated by$\{(1,0),(\frac{1}{2},\frac{\sqrt{3}}{2}),(\frac{1}{2},-\frac{\sqrt{3}}{2})\}$. Let $X_1, X_2, \dots$ be a sequence of i.i.d random variables that gives probability 1/6 to the following vectors $\{\pm(1,0),\pm(\frac{1}{2},\frac{\sqrt{3}}{2}),\pm(\frac{1}{2},-\frac{\sqrt{3}}{2})\}$, and let $S_n = X_0 + X_1 + \cdots + X_n$ with $X_0 = 0$, then S_n is the simple random walk on $\mathbb{L}_T$ starting from the origin. To estimate the potential kernel on triangular lattice, we will use the Theorem 2.1.2. But note that the triangular lattice $\mathbb{L}_T$ is not on the integer lattice, so we cannot apply Theorem 2.1.2 directly on it. Hence, we consider the matrix

$$A = \begin{bmatrix} 1 & -1/\sqrt{3} \\ 0 & 2/\sqrt{3} \end{bmatrix}$$

The matrix A maps the triangular lattice $\mathbb{L}_T$ to $\mathbb{Z}^2$ by sending $\{(1,0),(\frac{1}{2},\frac{\sqrt{3}}{2}),(\frac{1}{2},-\frac{\sqrt{3}}{2})\}$ to $\{(1,0),(0,1),(1,-1)\}$, as figure **??** shows. In this new integer lattice, we will give

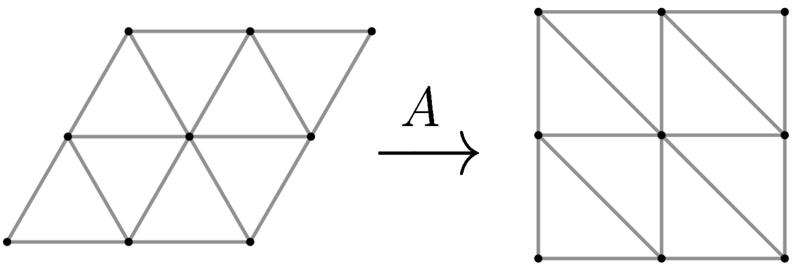

Figure 2.1: Triangular lattice to integer lattice

it the distribution by giving probability 1/6 to the movement to each of the following vectors: $\{\pm(1,0),\pm(0,1),\pm(1,-1)\}$. We will call this new random walk on new integer lattice $A\mathbb{L}_T$. It's easy to see that $A\mathbb{L}_T$ has the same potential kernel as simple random walk on triangular lattice. And since it's on integer lattice, we can apply Theorem 2.1.2 on $A\mathbb{L}_T$ and help us to estimate the potential kernel on triangular lattice. In $A\mathbb{L}_T$, its covariance matrix $\sum$ and Λ^{-1} are

$$\sum = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}, \Lambda^{-1} = \begin{bmatrix} (\sqrt{3}+1)/2 & (\sqrt{3}-1)/2 \\ (\sqrt{3}-1)/2 & (\sqrt{3}+1)/2 \end{bmatrix}$$

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Hence, by Theorem 2.1.2 as $\|x\| \to \infty$, the potential kernel $a(\cdot)$ on $A\mathbb{L}_T$ is

$$a(x) = \frac{\sqrt{3}}{2\pi} \ln(2x_1^2 + 2x_1x_2 + 2x_2^2) + C + O(\|x\|^{-2}),\ x = (x_1, x_2) \in A\mathbb{L}_T \tag{2.2.1}$$

Remark 2.2.1. *By computer calculation, C = 0.8121600594636381 ...*

Therefore, for $x \in \mathbb{L}_T$, to compute the potential kernel, we need to transfer it to $A\mathbb{L}_T$ first, and then obtain 2.2.1. Note that $\|\Lambda^{-1}Ax\|^2 = 2\|x\|^2$. We have the following result,

Remark 2.2.2. *The potential kernel of simple random walk on triangular lattice as $\|x\| \to \infty$ will be*

$$a(x) = \frac{\sqrt{3}}{\pi} \ln \|x\| + \frac{\sqrt{3}}{2\pi} \ln 2 + C + O(\|x\|^{-2}),\ x \in \mathbb{L}_T$$

The following table is the approximate values for $a(\cdot)$ on $A\mathbb{L}_T$ as $0 \leq x = (x_1, x_2) \leq 6$ which

Table 2.1: Approximate values for $a(\cdot)$ on $A\mathbb{L}_T$

	x_1 = 0	1	2	3	4	5	6
x_2 = 0	- - -	1.003236	1.385388	1.608932	1.767540	1.890565	1.991084
1		1.306084	1.539654	1.710301	1.842502	1.949864	2.040065
2			1.688236	1.814913	1.921806	2.013149	2.092453
3				1.911781	1.998637	2.076072	2.145351
4					2.070388	2.136457	2.197065

is compared to the table of real $a(\cdot)$.

Table 2.2: Values for $a(\cdot)$ on $A\mathbb{L}_T$

	x_1 = 0	1	2	3	4	5	6
x_2 = 0	0	1	1.384053	1.608639	1.767455	1.890533	1.991069
1		1.307973	1.539866	1.710295	1.842485	1.949852	2.040058
2			1.688372	1.814955	1.921817	2.013151	2.092452
3				1.911807	1.998650	2.076077	2.145353
4					2.070396	2.136462	2.197067

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Chapter 3

Proposition on Triangular Lattice

3.1 Escaping Probability

At first, we introduce the optional stopping theorem [10] which will be often used in the following content.

Theorem 3.1.1. *(Optional stopping theorem) Let X_n be a submartingale and τ be a finite stopping time. For a constant $c > 0$, suppose that at least one of the following conditions holds :*

$(i).\tau \leq c\, a.s.$;

$(ii).\,|X_{n\vee\tau}| \leq c\, a.s.\ for\ all\ n \geq 0$;

$(iii).\mathbb{E}\tau < \infty\ and\ \mathbb{E}[|X_{n+1} - X_n|\,|\mathcal{F}_n] \leq c\, a.s.\ for\ all\ n \geq 0.$

Then $\mathbb{E}_\tau \geq \mathbb{E}_0$. If X_n is a martingale and at least one of the conditions (i) - (iii) holds, then $\mathbb{E}_\tau = \mathbb{E}_0$.

For the following chapter, we will focus on the simple random walk on the triangualr lattice. Let $S_n = X_0 + X_1 + \cdots + X_n$ be the simple random walk starting at X_0 on the triangualr lattice. Denote $a(\cdot)$ be the potential kernel of it, and we have the following proposition.

Proposition 3.1.2. *The process $a(S_{n\wedge\tau_0})$, where τ_0 is a stopping time to 0, is a martingale.*

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Proof. First, we show that $a(\cdot)$ is harmonic outside the origin, i.e.

$$a(x) = \frac{1}{6}\sum_{y\sim x} a(y),\ x \in \mathbb{L}_T \neq 0 \tag{3.1.1}$$

Since, for $x \neq 0$, by (2.1.1)

$$\begin{aligned}
a(x) &= \lim_{N\to\infty} \Big(\sum_{n=0}^{N} P(S_n = (0,0)) - \sum_{n=1}^{N} P(S_n = x) \Big) \\
&= \lim_{N\to\infty} \Big(\sum_{n=0}^{N} \sum_{y\sim x} \frac{1}{6} P(S_n = (0,0)) - \sum_{n=1}^{N} \sum_{y\sim x} \frac{1}{6} P(S_{n-1} = y) \Big) \\
&= \lim_{N\to\infty} P(S_N = 0) + \frac{1}{6} \sum_{y\sim x} \lim_{N\to\infty} \sum_{n=0}^{N-1} \big(P(S_n = (0,0)) - P(S_n = y) \big) \\
&= \frac{1}{6} \sum_{y\sim x} a(y)
\end{aligned} \tag{3.1.2}$$

The harmonicity of $a(\cdot)$ outside the orgin implies the result.

Also, let $\mathcal{N} = \pm\{(1,0), (\frac{1}{2}, \frac{\sqrt{3}}{2}), (\frac{1}{2}, -\frac{\sqrt{3}}{2})\}$ be the six neighbors of the origin. We can get that

$$a(x) = 1 \text{ for all } x \in \mathcal{N} \tag{3.1.3}$$

To see this, first, note that by symmetry of the triangualr lattice, $P(S_n = x)$ takes the same value for all $x \in \mathcal{N}$ for any n, then similar to (2.1.2)

$$\begin{aligned}
a((0,0)) &= \lim_{N\to\infty} \Big(\sum_{n=0}^{N} P(S_n = (0,0)) - \sum_{n=1}^{N} P(S_n = (0,0)) - 1 \Big) \\
&= \lim_{N\to\infty} \Big(\sum_{n=0}^{N} P(S_n = (0,0)) - \sum_{n=1}^{N} \sum_{x\sim 0} \frac{1}{6} P(S_{n-1} = x) \Big) \\
&= \lim_{N\to\infty} \Big(\sum_{n=0}^{N} P(S_n = (0,0)) - \sum_{n=1}^{N} P(S_{n-1} = (1,0)) - 1 \Big) \\
&= \lim_{N\to\infty} \Big(\sum_{n=0}^{N} P(S_n = (0,0)) - P(S_n = (1,0)) \Big) - \lim_{N\to\infty} \big(P(S_N = (1,0)) \big) - 1 \\
&= a((1,0)) - 1
\end{aligned}$$

Hence, $a((1,0)) - 1 = a((0,0)) = 0$, and we can obtain (3.1.3).

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In the forthcoming section, we will discuss $\sum_{y\in\partial B(x,r)} v(y)a(y)$ for any probability measure v on $\partial B(x,r)$ with fixed x as $r \gg \|x\|$. To simplify calculation, we consider for all $x, y \in \mathbb{L}_T$ with $\|x\| \gg \|y\|$, due to Remark 2.2.2, we have

$$a(x+y) - a(x) = \ln(1+\frac{\|y\|}{\|x\|}) + O(\|x+y\|^{-2} + \|x\|^{-2}) = O(\frac{\|y\|}{\|x\|}) \tag{3.1.4}$$

Futhermore, we also define the function

$$a(r) = \frac{\sqrt{3}}{\pi}\ln r + \frac{\sqrt{3}}{2\pi}\ln 2 + C,\ r \in \mathbb{R} \tag{3.1.5}$$

to express the potential kernel for those points which have the same norm.
Therefor, by (3.1.4) and (3.1.5), we can get

$$\begin{aligned}\sum_{y\in\partial B(x,r)} v(y)a(y) &= \sum_{y\in\partial B(x,r)} v(y)a(y-x+x) = \sum_{y\in\partial B(x,r)} v(y)\big(a(y-x) + O(\frac{\|x\|\vee 1}{\|y-x\|})\big) \\ &= a(r) + O(\frac{\|x\|\vee 1}{r})\big) \quad as \quad r\to\infty \end{aligned} \tag{3.1.6}$$

We can obtain the following result by (3.1.6), Remark 2.2.2, Proposition 3.1.2, and the optional stopping theorem :

Proposition 3.1.3. *For all $x \in \mathbb{L}_T$ and R >r >0 with $x \in B(y,R) - B(r)$, then*

$$\mathbb{P}_x[\tau_{\partial B(y,R)} > \tau_{\partial B(r)}] = \frac{\ln\frac{\|x\|}{R} + O(\frac{\|y\|\vee 1}{R} + \frac{1}{\|x\|^2})}{\ln\frac{r}{R} + O(\frac{1}{r} - \frac{\|y\|\vee 1}{R})} \ \textit{as}\ r, R\to\infty$$

Proof. Consider $p = \mathbb{P}_x[\tau_{\partial B(y,R)} > \tau_{\partial B(r)}]$.
Let $a\!\uparrow = \mathbb{E}_x(a(S_{\tau_{\partial B(y,R)}})|\tau_{\partial B(y,R)} < \tau_{\partial B(r)}), a\!\downarrow = \mathbb{E}_x(a(S_{\tau_{\partial B(r)}})|\tau_{\partial B(y,R)} > \tau_{\partial B(r)})$.
Since $S_0 = x$ and $a(S_{n\wedge\tau_0})$ is a martingale.
Also $\left|a(S_{n\wedge\tau_B(r)\wedge\tau_B(y,R)})\right| \le M a.s.$ for all $n \ge 0$, which satisfy the condition (ii) of Optional stopping theorem. Hence, we can get

$$a(x) = a(S_0) = \mathbb{E}_x a(S_{\tau_B(r)\wedge\tau_B(y,R)}) = pa\!\downarrow + (1-p)a\!\uparrow,$$

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It shows that

$$p = \frac{a(x) - a\uparrow}{a\downarrow - a\uparrow}$$

Note that by (3.1.6), we can simplify the $a\uparrow$ and $a\downarrow$ to

$$a\uparrow = a(R) + O(\frac{\|y\| \vee 1}{R}), a\downarrow = a(r) + O(\frac{1}{r})$$

By above and Remark 2.2.2, we finally get

$$p = \frac{a(x) - a(R) + O(\frac{\|y\|\vee 1}{R})}{a(r) - a(R) + O(\frac{1}{r} - \frac{\|y\|\vee 1}{R})} = \frac{ln\frac{\|x\|}{R} + O(\frac{\|y\|\vee 1}{R} + \frac{1}{\|x\|^2})}{ln\frac{r}{R} + O(\frac{1}{r} - \frac{\|y\|\vee 1}{R})}$$

For the specific case, we have the following lemma :

Lemma 3.1.4. *For $x \in B(y, R)$ and $x \neq 0$, then*

$$\mathbb{P}_x[\tau_{\partial B(y,R)} > \tau_0] = \frac{a(R) - a(x) + O(\frac{\|y\|\vee 1}{R})}{a(R) + O(\frac{\|y\|\vee 1}{R})} \text{ as } R \to \infty$$

Proof. Let $p = \mathbb{P}_x[\tau_{\partial B(y,R)} > \tau_0]$. Similarly to Proposition 3.1.3, again by the optional stopping theorem,

$$a(x) = (1 - p)\mathbb{E}_x(a(S_{\tau_{\partial B(y,R)}})|\tau_{\partial B(y,R)} < \tau_0)$$

and by (3.1.6) we can get the lemma.

Note that Lemma 3.1.4 implies that :

$$\mathbb{P}_0[\tau_{\partial B(y,R)} < \tau_0^+] = \frac{1}{a(R) + O(\frac{\|y\|\vee 1}{R})} \tag{3.1.7}$$

which means the probability of escaping from circle before returning to origin.

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To get the next proposition, we need to first show a lemma here :

Lemma 3.1.5. *For any finite set* $\Lambda \subset \mathbb{L}_T$*, it holds that* $\mathbb{E}_x \tau_{\Lambda^c} < \infty$ *for all* x*.*

Proof. For any $x \in \Lambda, \exists\, n_x \in \mathbb{N}$ so that $\mathbb{P}_x(\tau_{\Lambda^c} \geq n_x) < 1$. Let $n_0 = \max_{x\in\Lambda}\{n_x\}$ and $p_0 = \max_{x\in\Lambda} \mathbb{P}_x(\tau_{\Lambda^c} \geq n_0)$. Hence,

$$\mathbb{E}_x \tau_{\Lambda^c} = \sum_{n=1}^{\infty} \mathbb{P}_x(\tau_{\Lambda^c} \geq n) = \sum_{k=0}^{\infty} \sum_{n=kn_0+1}^{(k+1)n_0} \mathbb{P}_x(\tau_{\Lambda^c} \geq n) \leq \sum_{k=0}^{\infty} n_0 p_0^k < \infty$$

Proposition 3.1.6. *For all* $x \in \mathbb{L}_T$ *and R >r >0 with* $x \in B(R) - B(r)$*.*
Let $T = \min\{\tau_{\partial B(R)}, \tau_{\partial B(r)}\}$*, then*

$$\mathbb{E}_x(T) = r^2 O(1 + \frac{1}{r})p + R^2 O(1 + \frac{1}{R})(1 - p) - \left\|x^2\right\|$$

where $p = \mathbb{P}_x[\tau_{\partial B(R)} > \tau_{\partial B(r)}]$ *that can be obtained by Proposition 3.1.3.*

Proof. Note that $\|S_n\|^2 - n$ is a martingale. Also since $B(R) - B(r)$ is finite, by lemma 3.1.5, it holds that $\mathbb{E}_x \tau_{B(R)-B(r)} < \infty$ which satisfies the the condition (iii) of Optional stopping theorem. Hence, we can get

$$\begin{aligned}
\|S_0\|^2 = \|x\|^2 &= \mathbb{E}_x(\|S_T\|^2 - T) \\
&= \mathbb{E}_x(\|S_T\|^2 | \tau_{\partial B(R)} > \tau_{\partial B(r)})p + \mathbb{E}_x(\|S_T\|^2 | \tau_{\partial B(R)} < \tau_{\partial B(r)})(1 - p) - \mathbb{E}_x(T) \\
&= r^2 O(1 + \frac{1}{r})p + R^2 O(1 + \frac{1}{R})(1 - p) - \mathbb{E}_x(T)
\end{aligned}$$

and so

$$E[T] = r^2 O(1 + \frac{1}{r})p + R^2 O(1 + \frac{1}{R})(1 - p) - \left\|x^2\right\|$$

Note that if we fix r and let $R \to \infty$, then $\mathbb{E}_x(T) \to \infty$.
Although $(1 - p) \to 0$ as $R \to \infty$, R grows faster to infinity than that $(1 - p)$ decays to 0.

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The following figures are the theoretical results compared to simulation. We simulate the situation that the simple random walk on triangualr lattice starting at $\|x\| \approx 100$. Observe the probability of visiting small circle first and the escaping expectation times. We fix the raidus of a small circle $r = 50$ and change the raidus of a big circle to $R = 150, 215, 500$.

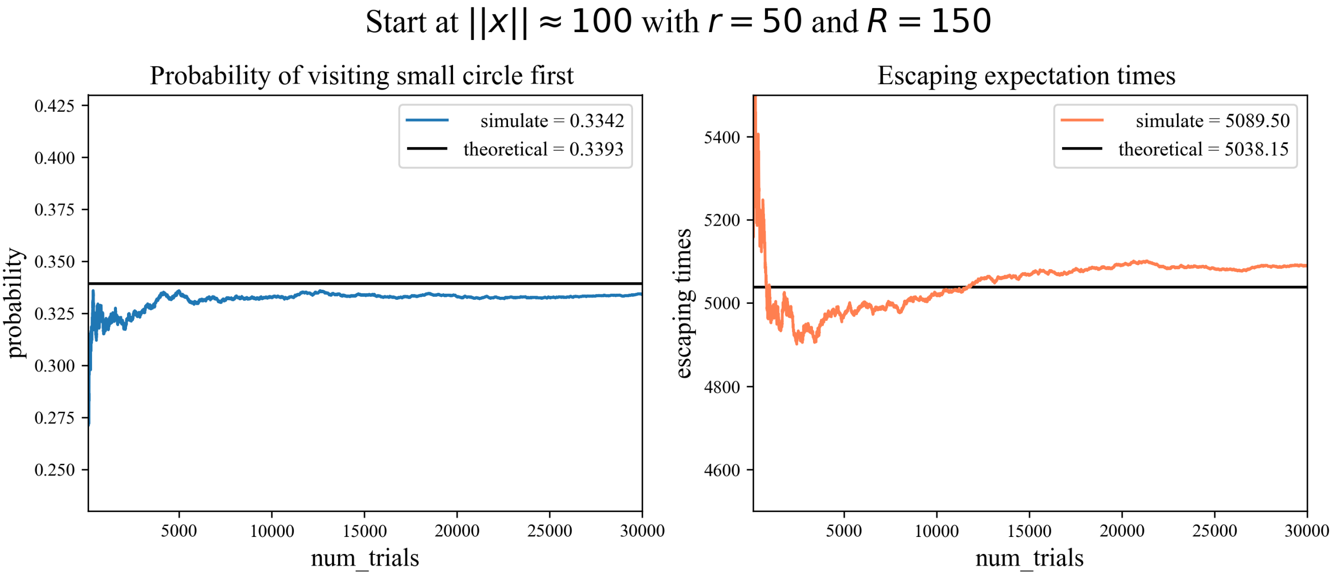

Figure 3.1: Start at $\|x\| \approx 100$ with $r = 50$ and $R = 150$

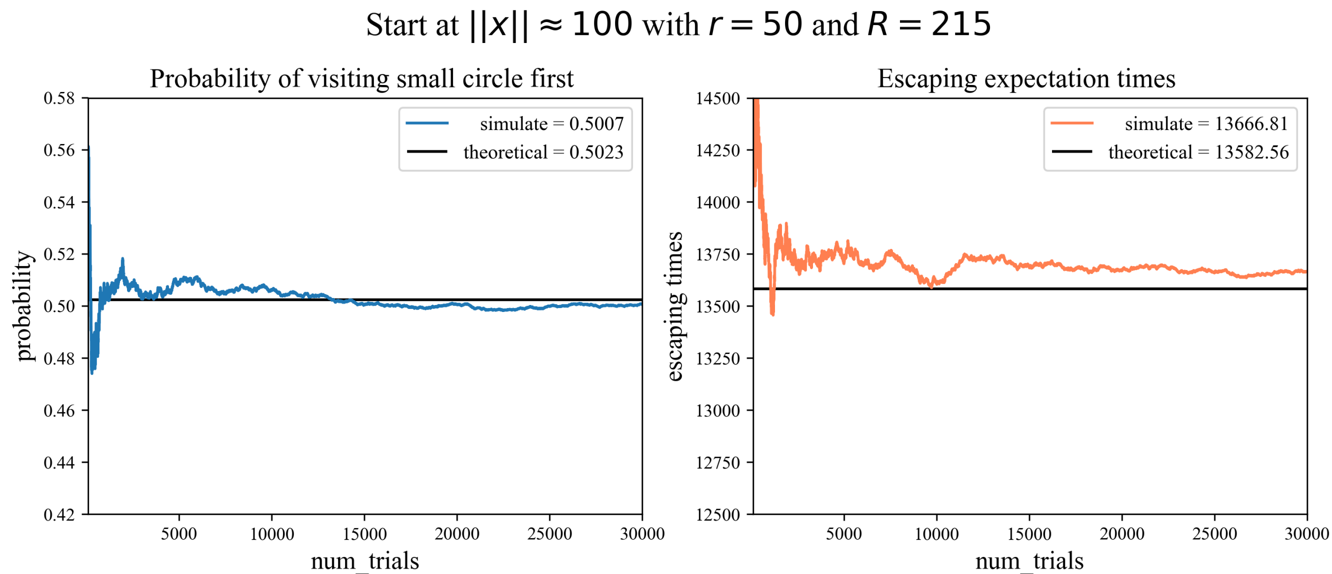

Figure 3.2: Start at $\|x\| \approx 100$ with $r = 50$ and $R = 215$

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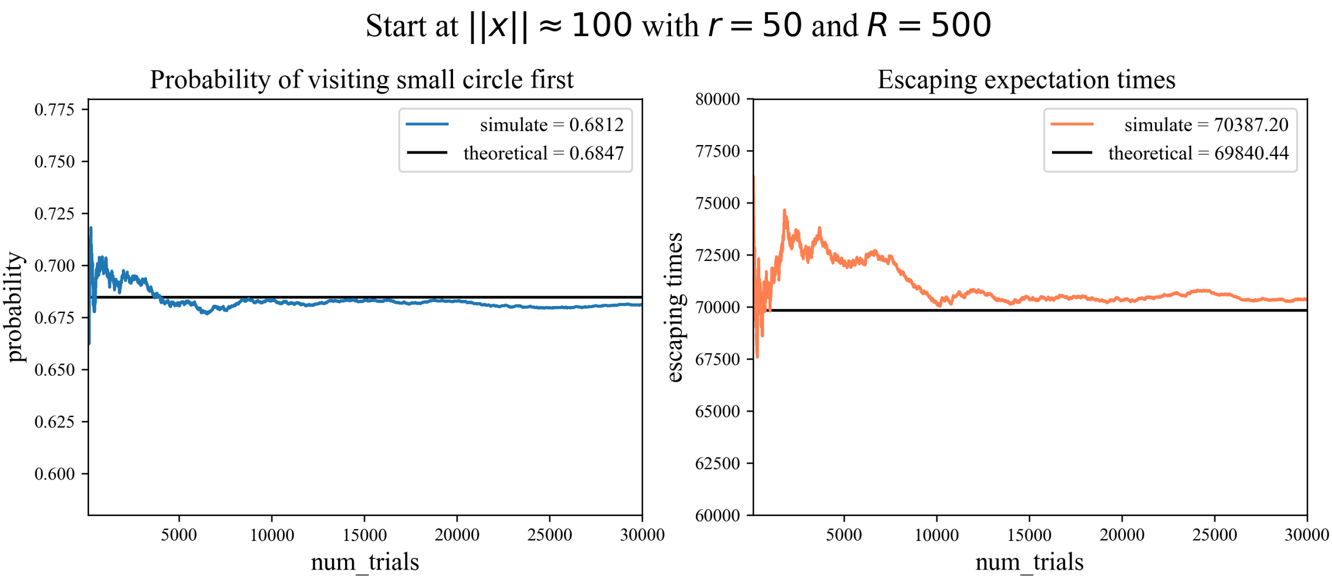

Figure 3.3: Start at $\|x\| \approx 100$ with $r = 50$ and $R = 500$

Moreover, note that in our proposition 3.1.3 which is the probability of escaping from a circle on triangualr lattice. There is no difference between triangualr lattice and simple random walk on integer lattice(see [10]). The following table is the comparison of them. Again, we start at $\|x\| = 53$ and fix the raidus of small circle $r = 50$ and change the raidus of big circle to $R = 60, 100, 500$.

	Triangualr		SRW	
	Theoretical	Simulate	Theoretical	Simulate
R=60	0.6804	0.6503	0.6804	0.6268
R=100	0.9159	0.9054	0.9159	0.8954
R=500	0.9746	0.9714	0.9746	0.9682

Table 3.1: Comparison between triangular lattice and simple random walk

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3.2 Green's Function

Next, let Λ be a finite subset of $\mathbb{L}_T$. For $x, y \in \Lambda$ in $\mathbb{L}_T$, we define the restricted Green's function

$$G_\Lambda(x,y) = \mathbb{E}_x \sum_{k=0}^{\tau_{\Lambda^c}-1} \mathbf{1}_{\{S_k=y\}}$$

to be the mean times of visits to y starting from x before escaping out of Λ. Note that

$$G_\Lambda(x,y) = G_\Lambda(y,x) \text{ for any } x, y \in \Lambda. \tag{3.2.1}$$

To prove this, since our random walk is symmetry, we have

$$\begin{aligned} G_\Lambda(x,y) &= \mathbb{E}_x \sum_{k=0}^{\tau_{\Lambda^c}-1} \mathbf{1}_{\{S_k=y\}} = \mathbb{E}_x \sum_{k=0}^{\infty} \mathbf{1}_{\{S_k=y,\tau_{\Lambda^c}>k\}} \\ &= \sum_{k=0}^{\infty} \mathbb{P}_x\{S_k = y, \tau_{\Lambda^c} > k\} = \sum_{k=0}^{\infty} \mathbb{P}_y\{S_k = x, \tau_{\Lambda^c} > k\} = G_\Lambda(y,x) \end{aligned}$$

Moreover, We have the following theorem connecting the restricted Green's function to the potential kernel :

Theorem 3.2.1. *Assume that Λ is finite, then we have*

$$G_\Lambda(x,y) = \mathbb{E}_x a(S_{\tau_{\Lambda^c}} - y) - a(x-y)$$

Proof. First, Proposition 3.1.2 and (3.1.3) imply that the process $a(S_n - y)$ is a submartingale. Since $a(S_n - y)$ is a martingale when $S_n \neq y$. However, for $S_n = y$, $E[a(S_{n+1} - y) \mid a(0)] = 1 > a(0) = 0$. Hence, by Doob's decomposition, we know that there exits an unique decompostion of $a(S_n - y)$ as the sum of a martingale and a predictable process. Here, we let

$$N_y^{(n-1)} = \sum_{k=0}^{n-1} \mathbf{1}_{\{S_k=y\}}$$

to be the number of visiting to y up to time $n-1$. Note that for $N_y^{(-1)} = 0$. Also let $Y_n = a(S_n - y) - N_y^{(n-1)}$ will be a martingale. Again since for $S_n \neq y$,

$$E[a(S_{n+1} - y) - N_y^{(n)} \mid \mathcal{F}_n] = a(S_n - y) - N_y^{(n-1)}$$

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And for $S_n = y$,

$$E[a(S_{n+1} - y) - N_y^{(n)} \mid \mathcal{F}_n] = a(S_n - y) + 1 - N_y^{(n-1)} - 1 = a(S_n - y) - N_y^{(n-1)}$$

Moreover, because Λ is finite, by Lemma 3.1.5 we know that $\mathbb{E}_x\tau_{\Lambda^c} < \infty$, which satisfing the condition (iii) of Optional stopping theorem. Hence, by Optional stopping theorem we can get

$$\begin{aligned} a(x-y) &= \mathbb{E}_x Y_0 = \mathbb{E}_x Y_{\tau_{\Lambda^c}} = \mathbb{E}_x\big(a(S_{\tau_{\Lambda^c}} - y) - N_y^{\tau_{\Lambda^c}-1}\big) \\ &= \mathbb{E}_x a(S_{\tau_{\Lambda^c}} - y) - G_\Lambda(x, y) \end{aligned}$$

Thus, we prove the theorem.

Next, we will use Theorem 3.2.1 for the special case when $\Lambda = B(R)$ with a large R. Let $x, y \in B(R)$, and for any $z \in \partial B(R)$, it satisfies that

$$a(z-y) = a(R) + O(\frac{\|y\|+1}{R})$$

Since the triangle inequailty gives that $\|z-y\| \le \|y\| + R + 1$ and $\ln\|z-y\| = \ln\big(R(1 + \frac{\|z-y\|-R}{R})\big) = \ln R + ln(1 + \frac{\|z-y\|-R}{R}) = \ln R + O(\frac{\|y\|+1}{R})$. , we apply Theorem 3.2.1 for $x, y \in B(R)$ and get

$$\begin{aligned} G_{B(R)}(x,y) &= \mathbb{E}_x\big(a(z-y)\big) - a(x-y) \\ &= \mathbb{E}_x\big(a(R) + O(\frac{\|y\| \wedge \|x\| + 1}{R})\big) - a(x-y) \\ &= \frac{\sqrt{3}}{\pi}\ln\frac{R}{\|x-y\|} + O(\frac{\|y\| \wedge \|x\| + 1}{R} + \frac{1}{\|x-y\|^2}) \end{aligned} \tag{3.2.2}$$

Note that due to the symmetry property (3.2.1), we can choose the smaller one of $\|x\|$ and $\|y\|$ to prove the error term.

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Chapter 4

Harmonic Measure and Capacity

4.1 Harmonic Measure

In the previous chapter, we have discussed the probability for a simple random walk on triangular lattice to escape from a circle. In this chapter, we will introduce the concept of harmonic measure and capacity. First, we need to define some tools and notations. For a finite set $\Lambda \subset \mathbb{L}_T$ and $x \in \mathbb{L}_T$, we define

$$q_\Lambda(x) = a(x - y_0) - \mathbb{E}_x a(S_{\tau_\Lambda} - y_0) \tag{4.1.1}$$

which y_0 is some dot of Λ. Later we will prove that the value of $q_\Lambda(\cdot)$ does not depend on y_0. Note that $q_\Lambda(x) = 0$ when $x \in \Lambda$ and $q_\Lambda(x)$ is invariant under translations, i.e., $q_{\Lambda+z}(x + z) = q_\Lambda(x)$ for all $z \in \mathbb{L}_T$. We have the following proposition to describe what the $q_\Lambda(x)$ is.

Proposition 4.1.1. *For any finite set $\Lambda \subset \mathbb{L}_T$ and any $x, y \in \mathbb{L}_T$*

$$q_\Lambda(x) = \lim_{R\to\infty} a(R)\mathbb{P}_x[\tau_{B(y,R)^C} < \tau_\Lambda] = \frac{\sqrt{3}}{\pi} \lim_{R\to\infty} \ln R\, \mathbb{P}_x[\tau_{B(y,R)^C} < \tau_\Lambda] \tag{4.1.2}$$

Proof. Since our random walk is recurrent, $\mathbb{P}_x[\tau_{B(y,R)^C} < \tau_\Lambda]$ will decay to 0 as $R \to \infty$. But note that by Proposition 3.1.3, we can observe that the decay rate of $\mathbb{P}_x[\tau_{B(y,R)^C} < \tau_\Lambda]$ is $O(1/\ln R)$. Hence, the equation above makes sense.

Without loss of generality, we can assume that $y_0 = 0$. Again, since Λ is finite, we apply the

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optional stopping theorem to the martingale $a(S_{n\wedge\tau_0})$ with the stopping time $\tau_\Lambda \wedge \tau_{B(y,R)}$.

$$
\begin{aligned}
a(x) &= \mathbb{E}_x a(S_0) = \mathbb{E}_x a(S_{\tau_\Lambda \wedge \tau_{B(y,R)}}) \\
&= \mathbb{E}_x\Big(a(S_{\tau_{B(y,R)}})\mathbf{1}_{\{\tau_{B(y,R)}<\tau_\Lambda\}}\Big) + \mathbb{E}_x\Big(a(S_{\tau_\Lambda})\mathbf{1}_{\{\tau_\Lambda<\tau_{B(y,R)}\}}\Big) \\
&= \mathbb{E}_x\Big(a(S_{\tau_{B(y,R)}})\mathbf{1}_{\{\tau_{B(y,R)}<\tau_\Lambda\}}\Big) + \mathbb{E}_x\Big(a(S_{\tau_\Lambda})(1-\mathbf{1}_{\{\tau_{B(y,R)}<\tau_\Lambda\}})\Big) \\
&= \mathbb{P}_x[\tau_{B(y,R)} < \tau_\Lambda]\mathbb{E}_x\Big(a(S_{\tau_{B(y,R)}}) \mid \tau_{B(y,R)} < \tau_\Lambda\Big) + \mathbb{E}_x a(S_{\tau_\Lambda}) \\
&\quad - \mathbb{P}_x[\tau_{B(y,R)} < \tau_\Lambda]\mathbb{E}_x\Big(a(S_{\tau_\Lambda}) \mid \tau_{B(y,R)} < \tau_\Lambda\Big)
\end{aligned}
$$

Hence, let $b = 1 + \max_{x\in\Lambda} \|x\|$, and we can get

$$
\begin{aligned}
\mathbb{P}_x[\tau_{B(y,R)} < \tau_\Lambda] &= \frac{a(x) - \mathbb{E}_x a(S_{\tau_\Lambda})}{\mathbb{E}_x\Big(a(S_{\tau_{B(y,R)}}) - a(S_{\tau_\Lambda}) \mid \tau_{B(y,R)} < \tau_\Lambda\Big)} \\
&\quad (\text{using}(3.1.6)) \\
&= \frac{q_\Lambda(x)}{a(R) - O(\ln b) + O(\frac{\|y\|+1}{R})}
\end{aligned}
\tag{4.1.3}
$$

and take $R \to \infty$ we can get (4.1.2).

Since the limit in (4.1.2) does not depend on y, it shows that the definition (4.1.1) does not depend on what y_0 is chosen to be. Later, We move y_0 to 0 for convenience.

Next, we define the harmonic measure in $\mathbb{L}_T$:

Definition 4.1.2. *For a finite set $\Lambda \subset \mathbb{L}_T$ and $y \in \Lambda$, the harmonic measure is defined as follow:*

$$
hm_\Lambda(y) = \frac{1}{6}\sum_{z\sim y} q_\Lambda(z) = \frac{1}{6}\sum_{\substack{z\sim y \\ z\notin\Lambda}} (a(z) - \mathbb{E}_z a(S_{\tau_\Lambda})) \tag{4.1.4}
$$

We can observe that the harmonic measure is proportional to the escape probability. To see this, by proposition 4.1.1 and following equation

$$
\mathbb{P}_y[\tau_{B(R)^C} < \tau_\Lambda^+] = \frac{1}{6}\sum_{\substack{z\sim y \\ z\notin\Lambda}} \mathbb{P}_z[\tau_{B(R)^C} < \tau_\Lambda]
$$

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then multiply both sides by $a(R)$ and take limit. We can get

$$hm_\Lambda(y) = \lim_{R\to\infty} a(R)\mathbb{P}_y[\tau_{B(R)^C} < \tau_\Lambda^+] \tag{4.1.5}$$

At this point, it's easy to obvious that $hm_\Lambda(\cdot)$ should be nonnegative (because of Proposition 4.1.1). But why does it sum to 1 on $\partial\Lambda$. We will discuss it in next theorem.

Theorem 4.1.3. *For all finite $\Lambda \subset \mathbb{L}_T$ and $y \in \Lambda$, we have*

$$hm_\Lambda(y) = \lim_{x\to\infty} \mathbb{P}_x[S_{\tau_\Lambda} = y] \tag{4.1.6}$$

Proof. First, without loss of generality, we can assume that $(0,0) \in \Lambda$. Next, we define the following notations:

$$N_x^\flat = \sum_{k=0}^{\tau_\Lambda^+-1} \mathbf{1}_{\{S_{\tau_\Lambda}=x\}}, \qquad N_{x,R}^\flat = \sum_{k=0}^{(\tau_\Lambda^+-1)\wedge\tau_{B(R)^C}} \mathbf{1}_{\{S_{\tau_\Lambda}=x\}}$$

$$N_{x,R}^\sharp = \sum_{k=\tau_\Lambda^+}^{\tau_{B(R)^C}} \mathbf{1}_{\{S_{\tau_\Lambda}=x\}}, \quad N_{x,R} := N_{x,R}^\flat + N_{x,R}^\sharp = \sum_{k=0}^{\tau_{B(R)^C}} \mathbf{1}_{\{S_{\tau_\Lambda}=x\}}$$

And we can write

$$\begin{aligned}
\mathbb{P}_x[S_{\tau_\Lambda} = y] &= \sum_{k=0}^{\tau_\Lambda^+-1} \mathbb{P}_y[S_k = x] = \mathbb{E}_y N_x^\flat \\
&\text{(by Monotone Convergence Theorem)} \\
&= \lim_{R\to\infty} \mathbb{E}_y N_{x,R}^\flat = \lim_{R\to\infty} (\mathbb{E}_y N_{x,R} - \mathbb{E}_y N_{x,R}^\sharp) \\
&= \lim_{R\to\infty} \big(G_{B(R)}(y,x) - \sum_{z\in\Lambda} \mathbb{P}_y[\tau_\Lambda^+ < \tau_{B(R)^C}, S_{\tau_\Lambda^+} = z] G_{B(R)}(z,x)\big) \\
&\text{(using(3.2.2))} \\
&= \lim_{R\to\infty} \big(a(R) - a(y-x) + O(\frac{\|y\|+1}{R}) \\
&\quad - \sum_{z\in\Lambda} \mathbb{P}_y[\tau_\Lambda^+ < \tau_{B(R)^C}, S_{\tau_\Lambda^+} = z](a(R) - a(z-x) + O(\frac{\|z\|+1}{R}))\big) \\
&= \lim_{R\to\infty} a(R)\big(1 - \sum_{z\in\Lambda} \mathbb{P}_y[\tau_\Lambda^+ < \tau_{B(R)^C}, S_{\tau_\Lambda^+} = z]\big) \\
&\quad - a(y-x) + \sum_{z\in\Lambda} \mathbb{P}_y[S_{\tau_\Lambda^+} = z] a(z-x)
\end{aligned}$$

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(Note that the first part above is $\mathbb{P}_y[\tau_\Lambda^+ > \tau_{B(R)^C}]$, then use$(4.1.5)$)

$$= hm_\Lambda(y) - a(y-x) + \sum_{z\in\Lambda} \mathbb{P}_y[S_{\tau_\Lambda^+} = z]a(z-x) \tag{4.1.7}$$

Moreover, by (3.1.4) we can obtain that

$$\begin{aligned} a(y-x) - \sum_{z\in\Lambda} \mathbb{P}_y[S_{\tau_\Lambda^+} = z]a(z-x) &= \sum_{z\in\Lambda} \mathbb{P}_y[S_{\tau_\Lambda^+} = z](a(y-x) - a(z-x)) \\ &= O(\frac{\text{diam}(\Lambda)}{\|x\|}) \end{aligned}$$

which converges to 0 as $x \to \infty$, and the proof is concluded.

Hence, Theorem 4.1.3 shows that $hm_\Lambda(\cdot)$ is indeed a probability measure. Next, for the following theorem, we will give a much better estimate and show how fast the convergence is.

Theorem 4.1.4. *Let Λ be a finite subset of $\mathbb{L}_T$ and assume that dist(x, Λ) $\geq$ 3diam(Λ) + 1. Then it holds that*

$$\mathbb{P}_x[S_{\tau_\Lambda} = y] = hm_\Lambda(y)\big(1 + O(\frac{diam(\Lambda)}{dist(x,\Lambda)})\big) \tag{4.1.8}$$

Proof. Again, without loss of generality, we can assume that $(0,0) \in \Lambda$ and $|\Lambda| \geq 2$, so diam(Λ) $\geq$1. Recall that in the above proof (4.1.7), we obtain that

$$\mathbb{P}_x[S_{\tau_\Lambda} = y] - hm_\Lambda(y) = -a(y-x) + \sum_{z\in\Lambda} \mathbb{P}_y[S_{\tau_\Lambda^+} = z]a(z-x) \tag{4.1.9}$$

The goal is to estimate the right-hand side more precisely. Now, let $V = \partial B(2\text{diam}(\Lambda))$ be the boundary ponit of a ball that contains Λ. Since they are all finite, there exists $k > 0$ so that

$$a(v) - a(z) \geq k \qquad \text{for all } v \in V \text{and } z \in \Lambda \tag{4.1.10}$$

Next, we apply the optional stopping theorem to the martingale $a(S_{n\wedge\tau_\Lambda^+\wedge\tau_V} - x)$ with the stopping time $\tau_\Lambda^+ \wedge \tau_V$ for the walk that starts at $y \in \partial\Lambda$. We have

$$\begin{aligned} a(y-x) &= \mathbb{E}_y a(S_{\tau_\Lambda^+\wedge\tau_V} - x) = \mathbb{E}_y(a(S_{\tau_\Lambda^+} - x)\mathbf{1}_{\{\tau_\Lambda^+<\tau_V\}}) + \mathbb{E}_y(a(S_{\tau_V} - x)\mathbf{1}_{\{\tau_V<\tau_\Lambda^+\}}) \\ &= \mathbb{E}_y a(S_{\tau_\Lambda^+} - x) + \mathbb{E}_y((a(S_{\tau_V} - x) - a(S_{\tau_\Lambda^+} - x))\mathbf{1}_{\{\tau_V<\tau_\Lambda^+\}}) \\ &= \sum_{z\in\Lambda} \mathbb{P}_y[S_{\tau_\Lambda^+} = z]a(z-x) + \mathbb{E}_y((a(S_{\tau_V} - x) - a(S_{\tau_\Lambda^+} - x))|\tau_V < \tau_\Lambda^+)\mathbb{P}_y[\tau_V < \tau_\Lambda^+] \end{aligned}$$

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Hence, the equation above with (4.1.9) implies that

$$\begin{aligned}&hm_\Lambda(y) - \mathbb{P}_x[S_{\tau_\Lambda} = y]\\&= \mathbb{E}_y((a(S_{\tau_V} - x) - a(S_{\tau_\Lambda^+} - x))|\tau_V < \tau_\Lambda^+)\mathbb{P}_y[\tau_V < \tau_\Lambda^+]\end{aligned} \tag{4.1.11}$$

Again, recall the expression (4.1.3) from the proof of Proposition 4.1.1, we have

$$\begin{aligned}\mathbb{P}_y[\tau_V < \tau_\Lambda^+] &= \frac{1}{6}\sum_{\substack{z\sim y\\ z\notin\Lambda}} \mathbb{P}_z[\tau_V < \tau_\Lambda]\\&= \frac{1}{6}\sum_{\substack{z\sim y\\ z\notin\Lambda}} \frac{q_\Lambda(z)}{\mathbb{E}_z\big((a(S_{\tau_V}) - a(S_{\tau_\Lambda}))|\tau_V < \tau_\Lambda\big)}\\&\text{(using(4.1.10))}\\&\leq \frac{hm_\Lambda(y)}{k}\end{aligned}$$

We also obtain that for any $v \in V$ and $z \in \Lambda$, we can get $a(x-v) - a(x-z) = O(\frac{\text{diam}(\Lambda)}{\text{dist}(x,\Lambda)})$. Hence, the right hand side of (4.1.11) is indeed $O(\frac{\text{diam}(\Lambda)}{\text{dist}(x,\Lambda)}) \times hm_\Lambda(y)$. Therefore, we complete the proof.

4.2 Capacity

Recall that the calculations in the Proposition 4.1.2, for a finite set $\Lambda \subset \mathbb{L}_T$ and $y_0 \in \Lambda$ we defined $q_\Lambda(x)$ in (4.1.1) as

$$q_\Lambda(x) = a(x - y_0) - \mathbb{E}_x a(S_{\tau_\Lambda} - y_0), \qquad x \in \mathbb{L}_T \tag{4.2.1}$$

and proved that $q_\Lambda(x)$ does not depend on the choice of y_0. Note that for the second term on the right hand side. If x is far away from Λ, Theorem 4.1.4 implies that

$$\begin{aligned}\mathbb{E}_x a(S_{\tau_\Lambda} - y_0) &= \sum_{z\in\Lambda} \mathbb{P}_x[S_{\tau_\Lambda} = z]a(z - y_0)\\&= \sum_{z\in\Lambda} hm_\Lambda(z)a(z-y_0)\big(1 + O(\frac{\text{diam}(\Lambda)}{\text{dist}(x,\Lambda)})\big)\end{aligned} \tag{4.2.2}$$

We can find that the main term in (4.2.2) does not depend on x. Therefore, we define

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capacity as follow:

Definition 4.2.1. *For a finite set $\Lambda \subset \mathbb{L}_T$ with $y_0 \in \Lambda$, we define its capacity by*

$$cap(\Lambda) = \sum_{x \in \Lambda} a(x - y_0) hm_\Lambda(x) \tag{4.2.3}$$

Again, we need to show that Definition 4.2.1 above does not depend on the choice of y_0. To show that, we also let $y_1 \in \Lambda$, and we have $q_\Lambda(x) = a(x - y_1) - \mathbb{E}_x a(S_{\tau_\Lambda} - y_1)$. Then by (4.2.2) we get

$$a(x - y_0) - a(x - y_1) = \big(cap(\Lambda) - \sum_{z \in \Lambda} hm_\Lambda(z) a(z - y_1)\big)(1 + O(\frac{\text{diam}(\Lambda)}{\text{dist}(x, \Lambda)}))$$

Since the left hand side converges to 0 as $x \to \infty$, the expression in the right hand side also equal to 0.

Now, recall the calculation in (4.1.3), we can rewrite as

$$\mathbb{P}_x[\tau_{B(y,R)} < \tau_\Lambda] = \frac{a(x) - cap(\Lambda)(1 + O(\frac{\text{diam}(\Lambda)}{\text{dist}(x,\Lambda)}))}{a(R) + O(R^{-1}) - cap(\Lambda)(1 + O(\frac{\text{diam}(\Lambda)}{\text{dist}(x,\Lambda)}))} \tag{4.2.4}$$

That means that if we know the capacity of Λ, we are able to compute the escape probability with higher precision. Next, we discuss the simplest cases where the capacity can be calculated. For the one-point sets, since $a((0, 0)) = 0$, it holds that $cap(\{x\}) = 0$ for any $x \in \mathbb{L}_T$. And for two-point sets, by symmetry we have

$$cap(\{x, y\}) = \frac{a(y - x)}{2} \tag{4.2.5}$$

for any $x, y \in \mathbb{L}_T$, $x \neq y$. Moreover, for the capacity of a ball $B(r)$, (3.1.6) implies that

$$cap(B(r)) = a(r) + O(r^{-1}) \tag{4.2.6}$$

It's remarkable to notice that the capacities of a two-point set $\{(0, 0), x\}$ with $\|x\| = r$ and the whole ball $B(r)$ only differ by a factor of 2.

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