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Optimal Processes Control for a Failure Mechanism

Su-Fen Yang, Ph.D.*

Department of Statistics, National Chengchi University, Taipei, Taiwan, R.O.C.

ABSTRACT

This article considers the economic statistical process control for two dependent processes with a failure mechanism which obeys Weibull shock model and has an increasing failure rate. We construct individual economic statistical X control chart to monitor the quality variable produced by the first process, and use the cause-selecting control chart to monitor the specific quality produced by the second process with minimal cost and required statistical properties. By using the proposed control charts, we can effectively and economically distinguish which process is out of control. The renewal theorem approach is extended to construct the cost model for our proposed control charts, and optimization method is used to determine the optimal design parameters of the proposed control charts. Finally,

*Correspondence: Professor Su-Fen Yang, Department of Statistics, National Chengchi University, No. 64, Chih-Nan Road, Sec. 2, Taipei, 116, Taiwan, R.O.C.; Fax: 886-2-29398024; E-mail: yang@nccu.edu.tw.

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we give an example to show how to construct and apply the proposed control charts. Sensitivity analysis that illustrates the effects of the cost and process parameters on the optimal design parameters and the minimal expected cost per unit time for the proposed control charts is also presented.

Key Words: Processes; Weibull distribution; Control charts; Renewal theorem.

1. INTRODUCTION

Control charts were first proposed by Shewhart (1931), and have become important tools for statistical quality control. An appropriate control chart indicates whether a production process is in the statistical control state or not.

Using a control chart requires the determination of three design parameters: the sample size n, the length of the sampling interval h, and the control limit coefficient k. Duncan (1956) determines the optimal design parameters of the $\overline{X}$ control charts from an economic viewpoint. The cost function of the $\overline{X}$ chart is derived under the assumption that there is only a single assignable cause and that the occurrence time of the assignable cause has an exponential distribution. Following Duncan's approach, joint economic design of $\overline{X}$ and R charts has been proposed by Jones and Case (1981), Rahim et al. (1988), Rahim (1989), Saniga (1977,1989), Saniga and Montgomery (1981), and Yang (1993). Rahim et al. (1988) discuss the use of joint economic $\overline{X}$and S^2 control charts when the process has only a single assignable cause. Collani and Sheil (1989) propose the economic design of S control chart under the assumption that the single assignable cause affects only the process variance. Literature surveys of related work is presented in Gibra (1975), Ho and Case (1994), Montgomery (1980), and Vance (1983).

The above economic control chart articles have the difficulty in estimating the quality cost (see Elsayed and Chen (1994)). Taguchi (1984) defines quality loss as ''the loss to society caused by the product after it is shipped out''. Taguchi et al. (1989) indicate that a quadratic approximation function sufficiently represents economic losses due to the deviation of the quality characteristic from its target. Kacker (1986) indicates that the concept of quadratic loss emphasizes the importance of continuously reducing performance variation. Different quality evaluation systems using the loss function are presented by Chen and

Kapur (1989). Taguchi (1984) and Taguchi et al. (1989) provide an economic design to determine the diagnosis interval and control limits for an on-line production process applying the loss function approach. The loss function as a rational approach for minimizing process variation has been widely accepted. Koo and Lin (1992) modify Duncan's cost model using Taguchi's loss function. Elsayed and Chen (1994) present an economic design of $\overline{X}$ control chart including Taguchi's quality loss function under the assumptions of a single assignable cause and continuous process operation.

Woodall (1986, 1987) comments that in many economic designs of control charts the probability of Type I error is much higher than that in a statistical design, and that this will result in more false alarms than expected. A higher Type I error probability can also cause process over-adjustment, which leads to an increase in the variance of the distribution of the interested quality characteristic. Saniga (1989) presents a method to improve economic control charts by bounding Type I and Type II error probabilities and the average time to signal (ATS) an expected shift and are, therefore, in accordance with industry's demand for low process variability and long-term quality. The design maintains a very small probability of a false alarm and a possibly incorrect adjustment. He calls the design the economic statistical design. Yang (1998) presents the economic statistical design of S control charts including Taguchi loss function.

The above articles assume that the occurrence time of assignable cause is described by exponential distribution having constant hazard rates and used constant sampling interval. However, this assumption is not always appropriate for some processes which deteriorate with time. Hu (1984) presents an economic design of $\overline{X}$ chart under Weibull shock model assuming a constant sampling interval. Banerjee and Rahim (1988) modify Hu's approach, and propose a cost model in which the length of the sampling interval varies with time. They indicate that the model under nonuniform sampling scheme provides a lower cost than that of Hu's model.

The above articles assume that there is only a single process. Today, many products are produced in several dependent processes. Consequently, it is not appropriate to monitor these processes by utilizing a control chart for each individual process. Zhang (1984) proposes the simple cause-selecting control chart to control the last step of the two dependent processes. Wade and Woodall (1993) review the basic principles of the cause-selecting chart for the case of two dependent processes and modify Zhang's approach. They give an example to illustrate the use of the individual X chart and the simple cause-selecting chart. It is shown that their approach is better than that of Zhang for the dependent processes control. They also examine the difference between the simple cause-selecting chart and the multivariate T^2 control chart. Yang (1997) proposes the economic design of an individual X control chart and a simple cause-selecting control chart under the assumption that there is only a single assignable cause and that the occurrence time of the assignable cause follows an exponential distribution. However, the economic process control for the two dependent processes under Weibull shock model has not been addressed. In this article, we include asymmetric quadratic loss function in the economic model of the two dependent processes to avoid the difficulty of estimating cost parameters, and assume that the occurrence time of a single assignable cause follows a Weibull shock model. In Sec. 2, we describe the process model and the assumptions. In Sec. 3, we derive the cost model by extending the renewal theorem approach. In Sec. 4, we give a numerical example and sensitivity analysis to show the construction and application of the proposed control charts. Finally, we provide conclusions.

2. PROCESS DESCRIPTION

In this article, we assume that the process consists of two dependent steps, and that only a single assignable cause may occur in any one of the processes. The production process model is described under the following assumptions.

2.1. Assumptions and Notation

(1) The production process consists of two dependent steps. The first step is called the previous process and the second step is called the current process. The previous process and the current process are dependent. Therefore, the quality variable X produced by the previous process will affect the quality variable Y produced by the current process.

(2) There is a single assignable cause, say AC, which may occur in one of the two processes. Once it occurs in one of the processes, it is investigated and removed from the process. We let the probability that the assignable cause occurs in the previous process be q, and the probability that the assignable cause occurs in the current process be $1-q$, where $0 \leq q < 1$ (see Fig. 1).

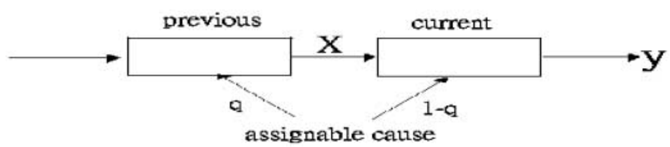

Figure 1. Two processes.

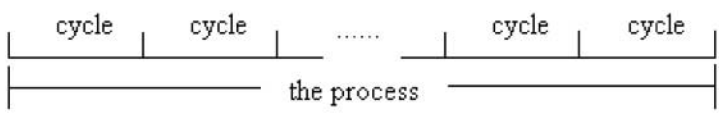

Figure 2. The processes denoted by cycles.

(3) The time before the assignable cause (T_{AC}) occurs in any one of the two processes follows a Weibull distribution, whose probability density function given by

$$f(t) = \lambda\theta t^{(\theta-1)} \exp\{-\lambda t^{\theta}\}, \quad t > 0,\ \theta \geq 1,\ \lambda > 0 \tag{1}$$

(4) The assignable cause affects only the mean of the distribution of X or Y, and their process variance is unchanged.

(5) The process with two steps is monitored by drawing a random sample (X, Y) with size one from the current process at times h_1, $(h_1 + h_2)$, $(h_1 + h_2 + h_3), \ldots$ and so on, and the points are plotted on the proposed control charts, where h_1 is the first sampling time from the start of the process, and

$$h_i = \left[i^{1/\theta} - (i-1)^{1/\theta}\right]h_1,\ i = 2, 3, \ldots \tag{2}$$

(see Banerjee and Rahim (1988))

(6) Production ceases during the search for the assignable cause and the adjustment of the process.

(7) We assume that the process is composed of independent and identically distributed cycles (Fig. 2) (Duncan, 1956).
A cycle is composed of in-control time, out-of-control time and search and adjustment time until the next starting cycle (Fig. 3).

The notation used is described as follows.

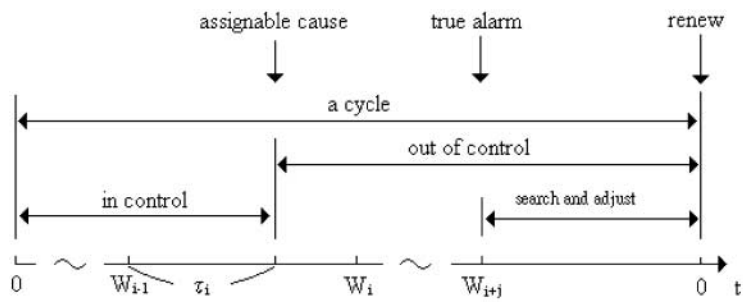

Figure 3. A complete cycle.

(1) Process Parameters

δ_{10} = Magnitude of shifts in the mean of quality variable X following the AC and while the previous process is not in control.
$\sigma^2_{y|x}$ = Variance of Y given X.
δ_{01} = Magnitude of shifts in the mean of quality variable Y given quality variable X following the AC and while the current process is not in control.
X_{00i} = Observation of the quality variable X when the previous process is in control, where $i = 1, 2, \ldots$.
X_{10i} = Observation of the quality variable X when the AC occurs in the previous process, where $i = 1, 2, \ldots$.
X_{01i} = Observation of the quality variable X when the AC occurs in the current process, where $i = 1, 2, \ldots$.
Y_{00i} = Observation of the quality variable Y when the previous process is in control, where $i = 1, 2, \ldots$.
Y_{10i} = Observation of the quality variable Y when the AC occurs in the previous process, where $i = 1, 2, \ldots$.
Y_{01i} = Observation of the quality variable Y when the AC occurs in the current process, where $i = 1, 2, \ldots$.
Ta = Target value for quality variable Y.
M = The quantity of output per unit production time.

(2) Time Parameters

W_j = The time until we take the jth sample, $W_0 = 0$, $W_j = \sum_{i=1}^{j} h_i$, $i = 1,2,3, \ldots$.
$E(T)$ = The expected cycle length.

$E(T_j)$ = The expected residual length in the cycle beyond time W_j given that the process is in control at time W_j, $j = 1, 2, \ldots$.
T_f = The expected search time when there is at least one false alarm.
T_{sr} = The expected time to search for the reason for the AC and to adjust the process to eliminate it.
τ_j = The expected time of occurrence of the AC, given that it occurred between time W_{j-1} and W_j, that is,

$$\tau_j = E(T_{AC} - W_{j-1} | W_{j-1} < T_{AC} < W_j) \tag{3}$$

(3) Cost Parameters

$E(C)$ = The expected cycle cost.
$E(C_j)$ = The expected residual cost in the cycle beyond time W_j given that the process is in control at time W_j, $j = 1, 2, \ldots$.
b = The sample cost.
D_0 = The expected loss per unit product while the process is in control.
D_1 = The expected loss per unit product following the AC and while the previous process is not in control.
D_2 = The expected loss per unit product following the AC and while the current process is not in control.
C_f = The expected search cost per false alarm.
C_{sr} = The expected cost to search for the reason for the AC and to adjust the process to eliminate it.
A_j = The coefficient of loss function, where $j = 0$ when quality variable Y is smaller than target Ta, and $j = 1$ when Y is larger than target.

(4) Probability

P_j = The probability that the AC occurs between time W_{j-1} and W_j, given that the process is still in control before time W_{j-1}, that is,

$$P_j = P(T_{\mathrm{AC}} < W_j | T_{\mathrm{AC}} > W_{j-1}) = 1 - \exp(-\lambda h_1^{\theta}), \quad j = 1, 2, \ldots, \tag{4}$$

where T_{AC} is time of AC occurrence.
$P = P_j$, since P_j is independent of time W_j.

2.2. The Possible Distribution of X and Y

When random samples of size one are taken from the current process at sampling times $W_j\,(j = 1, 2, \ldots)$, we get pairs of observations (X_{mni}, Y_{mni}) $(m, n = 1$ or 0, but cannot be all 1 at the same time, i is the number of sample). The model relating the two variables can take many forms. One of the most useful models is the simple linear regression model. We let

$$E[Y_{mni}|X_{mni}] = a_0 + a_1 X_{mni}, \tag{5}$$

where a_0 and a_1 are constants.

However, the model does not need to be linear for constructing the simple cause-selecting chart. The simple cause-selecting technique can also be applied to a nonlinear model. The possible distributions of X_{mni}, $Y_{mni}|X_{mni}$ and Y_{mni} for the in-control or out-of-control processes are illustrated as follows (see Table 1).

2.3. The Individual X Control Chart and Cause-Selecting Control Chart

The individual X chart is constructed to monitor the process state of the quality variable X, and the cause-selecting control chart is constructed to monitor the specific quality in the current process by adjusting the effect of X on Y. Since the in-control distribution of $X_{00i} \sim N(\mu_{00x}, \sigma_x^2)$, the individual X chart has upper control limit $(\mathrm{UCL}) = (\mu_{00\mathrm{x}} + \mathrm{k}_1\sigma_\mathrm{x})$, central line $(\mathrm{CL}) = \mu_{00x}$, and lower control limit $(\mathrm{LCL}) = (\mu_{00\mathrm{x}} - \mathrm{k}_1\sigma_\mathrm{x})$. Since the quality variable Y is dependent on the quality variable X, the specific quality of the current process can be specified by adjusting the effect of X on Y; that is the specific quality is presented by the cause-selecting value, $Z_{mni} = ((Y_{mni}\big|X_{mni}) - \mu_{mni})/(\sigma_{y|x})$, $m, n = 0$, or 1. Based on the in-control distribution of Z_{00i}, the cause-selecting chart can be constructed. Since the in-control distribution of the cause-selecting random variable is $Z_{mni} \sim N(0, 1)$, the cause-selecting control chart has $\mathrm{UCL} = k_2$, $\mathrm{CL} = 0$, and $\mathrm{LCL} = -k_2$.

Once the optimal design parameters (h_1, k_1, and k_2) of the individual X chart and the cause-selecting chart are determined, the proposed control charts can be effectively used to monitor the two processes with minimal cost. After each sampling, we chart the value of X on the individual X chart and plot the value of Z on the cause-selecting chart. When the value of X falls outside the control limits of the individual X chart, it indicates that the AC has occurred in the previous process. When the value of Z falls outside the control limits of the cause-selecting

Table 1. The possible distributions of quality variable X, and quality variable Y.

Process state	Distribution of X	Conditional distribution $Y\|X$	Distribution of Y
In control	$X_{00i} \sim N(\mu_{00x},\sigma_x^2)$	$Y_{00i}\|X_{00i} \sim N(a_0 + a_1 X_{00i},\sigma_{y\|x}^2) = N(\mu_{00i},\sigma_{y\|x}^2)$	$Y_{00i} \sim N(a_0 + a_1\mu_{00x},a_1^2\sigma_x^2 + \sigma_{y\|x}^2) = N(\mu_{00y},\sigma_y^2)$
Previous process out-of-control	$X_{10i} \sim N(\mu_{00x} + \delta_{10}\sigma_x,\sigma_x^2) = N(\mu_{10x},\sigma_x^2)$	$Y_{10i}\|X_{10i} \sim N(a_0 + a_1 X_{10i},\sigma_{y\|x}^2) = N(\mu_{10i},\sigma_{y\|x}^2)$	$Y_{10i} \sim N(a_0 + a_1\mu_{10x},a_1^2\sigma_x^2 + \sigma_{y\|x}^2) = N(\mu_{10y},\sigma_y^2)$
Current process out-of-control	$X_{01i} = X_{00i} \sim N(\mu_{00x},\sigma_x^2)$	$Y_{01i}\|X_{01i} \sim N(\mu_{00i} + \delta_{01}\sigma_{y\|x},\sigma_{y\|x}^2) = N(\mu_{01i},\sigma_{y\|x}^2)$	$Y_{01i} \sim N(a_0 + a_1\mu_{00x} + \delta_{01}\sigma_{y\|x},a_1^2\sigma_x^2 + \sigma_{y\|x}^2) = N(\mu_{01}y,\sigma_y^2)$

chart, it indicates that the AC has occurred in the current process. The process engineer should initiate the action to search and adjust the AC. The process would back to in-control state after it is adjusted.

2.4. Type I and Type II Error Probabilities

In this section, we define type I and type II error probabilities for the two proposed control charts. Define:

α: The probability that there is at least one false alarm indicated by the two control charts, given the previous and current processes are all in control,

$$\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2, \tag{6}$$

where α_1 is the probability that the individual X chart gives a false alarm, given the previous process is in control,

$$\begin{aligned}\alpha_1 &= P(X_{00i} > \mu_{00x} + k_1\sigma_x) + P(X_{00i} < \mu_{00x} - k_1\sigma_x) \\ &= 2(1 - \Phi(k_1)),\end{aligned} \tag{7}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of the standardized normal distribution.

α_2 is the probability that the simple cause-selecting chart gives a false alarm given the current process is in control,

$$\begin{aligned}\alpha_2 &= P((Y_{00i}|X_{00i}) > \mu_{00i} + k_2\sigma_{y|x}) + P((Y_{00i}|X_{00i}) < \mu_{00i} - k_2\sigma_{y|x}) \\ &= 2(1 - \Phi(k_2)),\end{aligned} \tag{8}$$

β_{10}: The probability that there are no alarms indicated by the two charts, given that the AC occurs in the previous process, that is

$$\beta_{10} = (1 - \alpha_2)\beta_1, \tag{9}$$

where β_1 is the probability that the individual X chart does not indicate a true alarm, given the previous process is out of control,

$$\begin{aligned}\beta_1 &= P(\mu_{00x} - k_1\sigma_x < X_{10i} < \mu_{00x} + k_1\sigma_x) \\ &= \Phi(-\delta_{10} + k_1) - \Phi(-\delta_{10} - k_1),\end{aligned} \tag{10}$$

β_{01}: The probability that there are no alarms indicated by the two charts, given that AC occurs in the current process, that is

$$\beta_{01} = (1 - \alpha_1)\beta_2, \tag{11}$$

where β_2 is the probability that the cause-selecting chart does not indicate a true alarm, given that the previous process is in control but the current process is out of control, that is

$$\beta_2 = P(-k_2 < Z_{01i} < k_2) = \Phi(k_2 - \delta_{01}) - \Phi(-k_2 - \delta_{01}). \tag{12}$$

2.5. The Asymmetric Loss Function

We consider the asymmetric loss function $L(\cdot)$ based on the distribution of quality variable Y (see Fig. 4 and Eq. (13)).

$$L(Y_{mni}) = \begin{cases} A_1[Y_{mni} - E(Y_{00i})]^2, & \text{where} \quad Y_{mni} \geq Ta \\ A_0[Y_{mni} - E(Y_{00i})]^2, & \text{where} \quad Y_{mni} < Ta, \\ m = 0, 1;\ n = 0, 1;\ i = 1, 2, \ldots \end{cases} \tag{13}$$

The expected loss per unit product when the process is in control or out of control is illustrated below.

(1) The process is in control:

$$\begin{aligned} D_0 &= E(L(Y_{00i})) = E(A_j(Y_{00i} - E(Y_{00i}))^2) \\ &= P(Y_{00i} < Ta) \int_{-\infty}^{T} A_0(y_{00i} - Ta)^2 f(y_{00i})\, dy_{00i} \\ &\quad + P(Y_{00i} > Ta) \int_{T}^{\infty} A_1(y_{00i} - Ta)^2 f(y_{00i})\, dy_{00i} \\ &= \left(\frac{A_0 + A_1}{2}\right)\sigma_y^2, \end{aligned} \tag{14}$$

where $f(\bullet)$ is probability density function.

(2) The AC occurs in the previous process:

$$\begin{aligned} D_1 &= E(L(Y_{10i})) = E(A_j(Y_{10i} - E(Y_{00i}))^2) \\ &= P(Y_{10i} < Ta) \int_{-\infty}^{T} A_0(y_{10i} - Ta)^2 f(y_{10i})\, dy_{10i} \\ &\quad + P(Y_{10i} > Ta) \int_{T}^{\infty} A_1(y_{10i} - Ta)^2 f(y_{10i})\, d_{10i} \\ &= \left\{A_0\Phi\left(\frac{-a_1\delta_{10}\sigma_x}{\sigma_y}\right) + A_1\left[1 - \Phi\left(\frac{-a_1\delta_{10}\sigma_x}{\sigma_y}\right)\right]\right\} \\ &\quad \times \left[\sigma_y^2 + (a_1\delta_{10}\sigma_x)^2\right] \end{aligned} \tag{15}$$

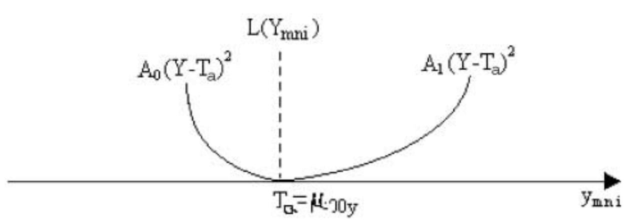

Figure 4. The asymmetric loss function.

(3) The AC occurs in the current process:

$$\begin{aligned} D_2 &= E(L(Y_{01i})) = E(A_j(Y_{01i} - E(Y_{00i}))^2) \\ &= P(Y_{01i} < Ta)\int_{-\infty}^{T} A_0(y_{01i} - Ta)^2 f(y_{01i})\,dy_{01i} \\ &\quad + P(Y_{01i} > Ta)\int_{T}^{\infty} A_1(y_{01i} - Ta)^2 f(y_{01i})\,d_{01i} \\ &= \left\{A_0\Phi\left(\frac{-\delta_{01}\sigma_{y|x}}{\sigma_y}\right) + A_1\left[1 - \Phi\left(\frac{-\delta_{01}\sigma_{y|x}}{\sigma_y}\right)\right]\right\} \\ &\quad \times \left[\sigma_y^2 + (\delta_{01}\sigma_{y|x})^2\right] \end{aligned} \tag{16}$$

3. THE COST MODEL

In this section, we extend the renewal theorem approach (Banerjee and Rahim, 1987) to obtain the cost model, which is the ratio of the expected cycle cost and the expected cycle length for the proposed individual X chart and cause-selecting chart. The design parameters of the proposed control charts can then be obtained by minimizing the cost model.

3.1. Expected Cycle Length

To extend the renewal theorem approach to derive the expected cycle length and the expected cycle cost, we study the possible states at the end of the first sampling and inspection time. Depending on the state of the system, we can compute the expected residual cycle length (the expected length in the cycle beyond the first sampling and inspection time) and the expected residual cycle cost (the expected cost in the cycle beyond the first sampling and inspection time) for each state. These values together with the associated probabilities lead us to formulate the renewal equation. The possible states are defined as follows (see Table 2).

State 1. The production is in control, and there is no false alarms indicated by the two charts.

State 2. The production is in control, but there is at least one false alarm indicated by the two charts.

Table 2. Definition of the six states.

State	Previous process in control	Current process in control	At least one alarm for the two charts
1	Yes	Yes	No
2	Yes	Yes	Yes
3	Yes	No	No
4	Yes	No	Yes
5	No	Yes	No
6	No	Yes	Yes

State 3. The AC occurs in the previous process, but there are no alarms indicated by the two charts.

State 4. The AC occurs in the previous process, and there is at least one alarm indicated by the two charts.

State 5. The AC occurs in the current process, but there are no alarms indicated by the two charts.

State 6. The AC occurs in the current process, and there is at least one alarm indicated by the two charts.

Table 3 displays the expected residual cycle length (R_i) and the associated probability (P_{ri}) of being in each respective state at the end of the first sampling and inspection time (W_1), $i = 1,2,\ldots,6$. Hence, the renewal equation is

$$\begin{aligned} E(T) &= h_1 + \mathrm{Pr}_1 E(T_1) + \mathrm{Pr}_2[T_f + E(T_1)] + \sum_{i=3}^{6} \mathrm{Pr}_i R_i \\ &= h_1 + \alpha T_f (1 - P) + T_{sr} P + qP\left(\sum_{i=1}^{\infty} h_{i+1}\beta_{10}^{i}\right) \\ &\quad + (1-q)P\left(\sum_{i=1}^{\infty} h_{i+1}\beta_{01}^{i}\right) + (1-P)E(T_1) \end{aligned} \tag{17}$$

Table 4 displays the expected residual cycle length and the associated probability of being in each respective state at the end of the sampling and inspection time W_j, $j = 1, 2, \ldots$.

Table 3. Probability and expected residual cycle length for each state at time W_1.

State	Probability	Expected residual cycle length
1	$\Pr_1 = (1-P)\ (1-\alpha)$	$R_1 = E(T_1)$
2	$\Pr_2 = (1-P)\alpha$	$R_2 = T_f + E(T_1)$
3	$\Pr_3 = q\ P\beta_{10}$	$R_3 = (1-\beta_{10})\sum_{i=1}^{\infty}(w_{i+1}-h_1)\beta_{10}^{i-1} + T_{sr}$
4	$\Pr_4 = q\ P(1-\beta_{10})$	$R_4 = T_{sr}$
5	$\Pr_5 = (1-q)\ P\beta_{01}$	$R_5 = (1-\beta_{01})\sum_{i=1}^{\infty}(w_{i+1}-h_1)\beta_{01}^{i-1} + T_{sr}$
6	$\Pr_6 = (1-q)\ P(1-\beta_{01})$	$R_6 = T_{sr}$

Table 4. Probability and expected residual cycle length for each state at time W_j.

State	Probability	Expected residual cycle length
1	$\Pr_1 = (1-P)\ (1-\alpha)$	$R_1 = E(T_j)$
2	$\Pr_2 = (1-P)\alpha$	$R_2 = T_f + E(T_j)$
3	$\Pr_3 = q\ P\beta_{10}$	$R_3 = (1-\beta_{10})\sum_{i=1}^{\infty}(w_{i+j}-w_j)\beta_{10}^{i-1} + T_{sr}$
4	$\Pr_4 = q\ P(1-\beta_{10})$	$R_4 = T_{sr}$
5	$\Pr_5 = (1-q)\ P\beta_{01}$	$R_5 = (1-\beta_{01})\sum_{i=1}^{\infty}(w_{i+j}-w_j)\beta_{01}^{i-1} + T_{sr}$
6	$\Pr_6 = (1-q)\ P(1-\beta_{01})$	$R_6 = T_{sr}$

Hence,

$$\begin{aligned} E(T_{j-1}) &= h_j + \Pr_1 E(T_j) + \Pr_2[T_f + E(T_j)] + \sum_{i=3}^{6} \Pr_i R_i \\ &= h_j + \alpha T_f\,(1-P) + T_{sr}\,P + qP\left(\sum_{i=1}^{\infty} h_{i+j}\beta_{10}^{i}\right) \\ &\quad + (1-q)P\left(\sum_{i=1}^{\infty} h_{i+j}\beta_{01}^{i}\right) + (1-P)E(T_j), j = 2, 3, \ldots. \end{aligned} \tag{18}$$

We provide a set of recursive systems for $E(T)$, $E(T_1)$, $E(T_2)$, ..., and so on. The system can be solved to obtain an expression for $E(T)$. Consequently,

$$\begin{aligned} E(T) &= h_1 \sum_{i=1}^{\infty}\left[i^{1/\theta} - (i-1)^{1/\theta}\right](1-P)^{i-1} + \alpha T_f \frac{(1-P)}{p} + T_{sr} \\ &\quad + qPh_1\left\{\sum_{j=1}^{\infty}(1-P)^{j-1}\sum_{i=1}^{\infty}\left[(i+j)^{1/\theta} - (i+j-1)^{1/\theta}\right]\beta_{10}^{i}\right\} \\ &\quad + (1-q)Ph_1\left\{\sum_{j=1}^{\infty}(1-P)^{j-1}\sum_{i=1}^{\infty}\left[(i+j)^{1/\theta} - (i+j-1)^{1/\theta}\right]\beta_{01}^{i}\right\} \end{aligned} \tag{19}$$

(see Appendix 1).

There are three infinite series

$$\sum_{i=1}^{\infty}\left[i^{1/\theta}-(i-1)^{1/\theta}\right](1-P)^{i-1},$$

$$\left\{\sum_{j=1}^{\infty}(1-P)^{j-1}\sum_{i=1}^{\infty}\left[(i+j)^{1/\theta}-(i+j-1)^{1/\theta}\right]\beta_{10}^{i}\right\},\quad \text{and}$$

$$\left\{\sum_{j=1}^{\infty}(1-P)^{j-1}\sum_{i=1}^{\infty}\left[(i+j)^{1/\theta}-(i+j-1)^{1/\theta}\right]\beta_{01}^{i}\right\}$$

in Eq. (19), for which we can calculate approximate values. Using the infinite series,

$$\left\{\sum_{j=1}^{\infty}(1-P)^{j-1}\sum_{i=1}^{\infty}[(i+j)^{1/\theta}-(i+j-1)^{1/\theta}]\beta_{10}^{i}\right\},$$

to illustrate, we describe the procedure in Appendix 2. The approximate values of other infinite series can be obtained using a similar approach.

3.2. The Expected Cycle Cost

The approach of deriving the expected cycle cost is similar to that of the expected cycle length. We present the possible states of the system, the sum (R_i') of the costs occurred in the first sampling and inspection time interval and the expected residual costs at time W_1 as shown in Table 5. Hence, the renewal equation for $E(C)$ is

$$\begin{aligned}E(C) &= \Pr{}_1[(b+D_0Mh_1)+E(C_1)] \\ &\quad + \Pr{}_2[(b+D_0Mh_1)+C_f+E(C_1)]+\sum_{i=3}^{6}\Pr{}_iR_i' \\ &= b+(h_1-\tau_1)PM[qD_1+(1-q)D_2] \\ &\quad + C_{sr}P+\frac{bPq\beta_{10}}{1-\beta_{10}}+\frac{bP(1-q)\beta_{01}}{1-\beta_{01}} \\ &\quad + D_1MqP\sum_{i=1}^{\infty}h_{i+1}\beta_{10}^{i}+D_2M(1-q)P\sum_{i=1}^{\infty}h_{i+1}\beta_{01}^{i} \\ &\quad + D_0Mh_1(1-P)+D_0MP\tau_1 \\ &\quad + (1-P)\alpha C_f+(1-P)E(C_1)\end{aligned}\tag{20}$$

Table 5. The cost in the first sampling and inspection interval and expected residual cost beyond time W_1.

State	Cost in the first sampling cost and inspection time	+	Expected residual
1	$R'_1 = b + D_0Mh_1$	+	$E(C_1)$
2	$R'_2 = b + D_0Mh_1$	+	$C_f + E(C_1)$
3	$R'_3 = b + D_0M\tau_1 + D_1M(h_1 - \tau_1)$	+	$b/(1-\beta_{10}) + D_1M(1-\beta_{10})\sum_{i=1}^{\infty}(w_{i+1} - h_1)\beta_{10}^{i-1} + C_{sr}$
4	$R'_4 = b + D_0M\tau_1 + D_1M(h_1 - \tau_1)$	+	C_{sr}
5	$R'_5 = b + D_0M\tau_1 + D_2M(h_1 - \tau_1)$	+	$b/(1-\beta_{01}) + D_2M(1-\beta_{01})\sum_{i=1}^{\infty}(w_{i+1} - h_1)\beta_{01}^{i-1} + C_{sr}$
6	$R'_6 = b + D_0M\tau_1 + D_2M(h_1 - \tau_1)$	+	C_{sr}

Proceeding in a similar fashion, we see that for $j = 2, 3, \ldots$ (see Table 6),

$$\begin{aligned}
E(C_{j-1}) &= \Pr_1[(b + D_0 M h_j) + E(C_j)] \\
&\quad + \Pr_2[(b + D_0 M h_j) + C_f + E(C_j)] + \sum_{i=3}^{6} \Pr_i R_i^{'} \\
&= b + (h_j - \tau_j) PM[qD_1 + (1-q)D_2] + C_{sr} P + \frac{bPq\beta_{10}}{1-\beta_{10}} \\
&\quad + \frac{bP(1-q)\beta_{01}}{1-\beta_{01}} + D_1 MqP \sum_{i=1}^{\infty} h_{i+j}\beta_{10}^{i} + D_2 M(1-q) \\
&\quad \times P \sum_{i=1}^{\infty} h_{i+j}\beta_{01}^{i} + D_0 M h_j (1-P) \\
&\quad + D_0 MP\tau_j + (1-P)\alpha C_f \\
&\quad + (1-P)E(C_j), \quad j = 2, 3, \ldots
\end{aligned} \tag{21}$$

We provide a set of recursive systems in the form of $E(C)$, $E(C_1)$, $E(C_2), \ldots,$ and so on. The system can then be solved to obtain an expression for $E(C)$. Consequently,

$$\begin{aligned}
E(C) &= \frac{b}{p} + [qD_1 + (1-q)D_2] MP \sum_{i=1}^{\infty} \{h_1[i^{1/\theta} - (i-1)] - \tau_i\} \\
&\quad \times (1-P)^{i-1} + C_{sr} + \frac{bq\beta_{10}}{1-\beta_{10}} \\
&\quad + \frac{b(1-q)\beta_{01}}{1-\beta_{01}} + D_1 MqP \left[\sum_{j=1}^{\infty} (1-P)^{j-1} \sum_{i=1}^{\infty} h_{i+j}\beta_{10}^{i} \right] \\
&\quad + D_2 M(1-q)P \left[\sum_{j=1}^{\infty} (1-P)^{j-1} \sum_{i=1}^{\infty} h_{i+j}\beta_{01}^{i} \right] \\
&\quad + D_0 M \sum_{i=1}^{\infty} h_i (1-P)^i \\
&\quad + D_0 MP \sum_{i=1}^{\infty} \tau_i (1-P)^{i-1} + \alpha C_f \frac{(1-P)}{P}
\end{aligned} \tag{22}$$

(see Appendix 3).

In the expression of $E(C)$, there are four infinite series. We can obtain the approximate values of these infinite series by using an algorithm similar to the one described in Appendix 2.

Table 6. Cost in the jth sampling and inspection interval and the expected residual cost beyond time W_j.

State	Cost in the jth sampling and inspection time	+	Expected residual cost
1	$R_1^{'} = b + D_0 M h_j$	+	$E(C_j)$
2	$R_2^{'} = b + D_0 M h_j$	+	$C_f + E(C_j)$
3	$R_3^{'} = b + D_0 M \tau_j + D_1 M (h_j - \tau_j)$	+	$b/(1 - \beta_{10}) + D_1 M (1 - \beta_{10}) \sum_{i=1}^{\infty} (w_{i+j} - w_j) \beta_{10}^{i-1} + C_{sr}$
4	$R_4^{'} = b + D_0 M \tau_j + D_1 M (h_j - \tau_j)$	+	C_{sr}
5	$R_5^{'} = b + D_0 M \tau_j + D_2 M (h_j - \tau_j)$	+	$b/(1 - \beta_{01}) + D_2 M (1 - \beta_{01}) \sum_{i=1}^{\infty} (w_{i+j} - w_j) \beta_{01}^{i-1} + C_{sr}$
6	$R_6^{'} = b + D_0 M \tau_j + D_2 M (h_j - \tau_j)$	+	C_{sr}

3.3. Determination of the Optimal Design Parameters for the Individual X Control Chart and the Cause-Selecting Control Chart

The approximate expected cost per unit time in the long run can be expressed as the ratio of the expected cycle cost and the expected cycle length (see Ross, 1993). The expected cost per unit time is a function of the design parameters h_1, k_1, and k_2. The optimal design parameters can be determined by minimizing the objective function. That is

$$\text{the objective function} = \frac{E(C)}{E(T)} = \frac{\text{right side of Eq. (22)}}{\text{right side of Eq. (19)}} \tag{23}$$

The proposed control charts have statistical properties as described in Saniga (1989). We constrain α and β by their upper bounds α_U, β_{10U} and β_{01U}, respectively. The upper bounds of h_1, k_1, and k_2 are h_{1U} k_{1U}, and k_{2U}, respectively. Hence the optimization model is expressed as follows (Eq. (24)).

$$\text{Minimize the objective function} = \frac{E(C)}{E(T)}$$

$$\begin{aligned} \text{Subject to } & 0 < h_1 \le h_{1U},\ 0 < k_1 \le k_{1U},\ 0 < k_2 \le k_{2U}, \\ & 0 < \alpha \le \alpha_U,\ 0 < \beta_{10} \le \beta_{10U},\ 0 < \beta_{01} \le \beta_{01U}. \end{aligned} \tag{24}$$

4. NUMERICAL EXAMPLE AND SENSITIVITY ANALYSIS

In this section, we give an example to illustrate how the proposed method is used to solve a real process control problem. Furthermore, we perform sensitivity analysis on the scale parameter θ and the shape parameter λ of the Weibull distribution, and the cost parameters to study the effects of cost and process parameters on the optimal design and the minimal cost.

4.1. Numerical Example

Assume that a cotton yarn factory produces cotton yarn in two dependent processes. The skein strength of the cotton yarn is denoted by the quality variable Y, which is produced in the current process. Yarn strength is the most important single index of spinning quality. Good yarn strength not only increases the range of usefulness of a given cotton but it indicates good spinning and weaving performance. The fiber length of the cotton yarn is denoted by the quality variable X, which is produced in the previous process. The skein strength can be obtained from knowledge of fiber length, so their relationship can be found by analysis history data. When the process is in control, the average skein strength given fiber length is expressed as model $E[Y_{00i}|X_{00i}] = 11 + 1.1X_{00i}$. The distributions of X_{00i}, $Y_{00i}|X_{00i}$, and Y_{00i} are illustrated as follows, when the previous and current processes are all in control.

$$X_{00i} \sim N(77.05, 5^2) \tag{25}$$

$$Y_{00i}|X_{00i} \sim N(11 + 1.1X_{00i},\ 8.35^2) \tag{26}$$

$$Y_{00i} \sim N(95.755,\ 10^2) \tag{27}$$

In the production process, a machine could be out-of-control in either the previous process or the current process. Since the machines do tend to deteriorate with time. It is of prime concern in process control to be able to distinguish in which one of the processes the out-of-control situation occurs. An out-of-control situation occurring in the previous process would cause only the mean of the X distribution to change. Because X influences Y, the mean of Y would then be changed. An out-of-control situation in the current process would cause only the mean of the Y distribution to change.

The individual X chart and cause-selecting chart are constructed to monitor the process effectively. To determine the optimal design parameters of the individual X chart and cause-selecting chart, the process and cost parameters are estimated as follows.

$$\lambda = 0.002, \quad \theta = 3, \quad q = 0.5, \quad \delta_{10} = 3, \quad \delta_{01} = 3, \quad b = \$20,$$
$$C_f = \$250, \quad C_{sr} = \$1000, \quad A_0 = 1, \quad A_1 = 1.2, \quad M = 40 \text{ unit},$$
$$T_f = 0.1\,\text{h}, \quad T_{sr} = 0.4\,\text{h}.$$

The optimal design parameters are determined by using a Fortran program with upper bounds $\alpha_U = 0.1$, $\beta_{10U} = 0.3$, and $\beta_{01U} = 0.3$. Hence $(h_1, k_1, k_2) = (2.92, 2.06, 1.86)$, $\alpha = 0.1$, $1 - \beta_{10} = 0.837$, $1 - \beta_{01} = 0.878$, and the expected cost per unit time is \$2730.88. Consequently, the individual X chart has $UCL = (77.05 + 10.6) = 87.65$, $CL = 77.05$, and $LCL = (77.05 - 10.6) = 66.45$; the cause-selecting chart has $UCL = 1.86$, $CL = 0$, and $LCL = -1.86$.

To monitor the process states, a sample $(X_{mni}, Y_{mni})$with size one is taken and inspected 2.92 h after the process starts. If the plotted statistic $X_{mni} > 87.65$ or $X_{mni} < 66.45$, then we conclude that the previous process is out of control and AC needs to be adjusted. If the plotted statistic $Z_{mni} > 1.86$ or $Z_{mni} < -1.86$, then we conclude that the current process is out of control and AC needs to be adjusted. If the plotted statistics $66.45 < X_{mni} < 87.65$, and/or $-1.86 < Z_{mni} < 1.86$, it indicates that the previous process is in control and/or the current process is in control and no action is taken; then the process continues and the second sample will be taken after 0.76 h, and so on. The joint power of the two proposed control charts is 0.837 when the previous process is out of control; and the joint power of the two proposed control charts is 0.878 when the current process is out of control.

4.2. Sensitivity Analysis

The effect of λ and θ on the optimal design parameters is shown in Appendix 4, where the process and cost parameters are estimated as:

$$q = 0.5,\quad \delta_{10} = 3,\quad \delta_{01} = 3,\quad b = \$20,\quad T_f = 0.1\,\text{h},$$
$$T_{sr} = 0.4\,\text{h},\quad C_f = \$250,\quad C_{sr} = \$1000,\quad A_0 = 1,$$
$$A_1 = 1.2,\quad M = 40\text{ unit},\quad \alpha_U = 0.1,\quad \beta_{10U} = 0.3,\quad \beta_{01U} = 0.3.$$

The numerical results of the sensitivity analysis provide some useful managerial insights that are valid at least over the range of the values of the parameters.

From Appendix 4, h_1, k_1, and k_2 decrease, but $E(C)/E(T)$ increases when λ or θ increases. The hazard function of the Weibull distribution is $h(t) = \lambda\theta t^{\theta-1}$. As θ or λ increase, $h(t)$ (the hazard rate) increases, leading to decreases in h_1(i.e., more frequent sampling), decreases in k_1 and k_2 (i.e., earlier detection of the out-of-control states), and increases in $E(C)/E(T)$ (increased cost).

Next, we fix the two parameters $\lambda = 0.002$, $\theta = 3$, and change the cost parameters (Appendix 5) one at a time to observe the effect of each cost parameter on the optimal design parameters.

The results of the sensitivity analysis are also summarized in Table 7 in the following manner: the parameter that is varied is always increasing; the resulting change in the parameter of interest is denoted as ↑ (for an increase), ↓ (for a decrease), or — (for no significant change).

Table 7. Changes of the design parameters when each of the cost parameters increases.

Cost parameter ↑	h_1	k_1	k_2	$E(C)/E(T)$
T_f ↑	↓	—	—	↓
T_{sr} ↑	—	—	—	↓
b ↑	↑	—	—	↑
C_f ↑	—	↑	↑	↑
C_{sr} ↑	—	—	—	↑
A_0 ↑	↑	—	—	↑
A_1 ↑	↓	—	—	↑
δ_{10} ↑	—	↑	↓	↑
δ_{01} ↑	↓	↓	↑	↑
M ↑	↓	—	—	↑↓[a]

[a]Represents increase first and then decrease.

The results of this study lead to some intuitive implications. For example, in Table 7, we see that

1. As T_f (the expected search time when there is at least one false alarm) increases, the expected cost per unit time decreases; the detection ability in this case requires a decrease in h_1 (more frequent sampling);
2. As b (the cost per unit sampled) increases, the expected cost per unit time increases; therefore, minimizing the cost requires increased h_1 (less frequent sampling);
3. As C_f (the expected search time when there is at least one true alarm) increases, the expected cost per unit time increases; therefore minimizing the cost requires increased k_1 and k_2 (larger control limit coefficient);
4. As M (the quantity of output per unit production time) increases the expected cost per unit time increases; therefore, minimizing the cost requires increased h_1 (less frequent sampling);
5. As δ_{10} (the shift parameter of the process mean for process variable X) increases, the expected cost per unit time increases; therefore, minimizing the cost requires increased h_1 (less frequent sampling), leading to increases in K_1 but decreases in k_2;
6. As δ_{01} (the shift parameter of the process mean for process variable Z) increases, the expected cost per unit time increases; therefore, minimizing the cost requires increased h_1 (less frequent sampling), leading to decreases in k_1 but increases in k_2.

5. CONCLUSIONS

In this study, we propose an approach for the design of the individual X control chart and the cause-selecting control chart having economic and statistical properties for a process with two steps in which the failure mechanism obeys a Weibull shock model. The approximate value for the expected cost per unit time involving some complicated infinite series was obtained using Fortran and IMSL subroutines. By using the proposed charts, we can effectively distinguish whether the previous process or/and the current process is/are in control or not. The asymmetric quadratic loss function is considered in the cost model, it overcomes the difficulty in cost estimation. An example was given to illustrate the application of the proposed individual X control chart and the cause-selecting control chart. The effects of the cost and process parameters on the optimal design parameters and minimum cost were investigated. The results show that λ and θ have significant effects on h_1 and $E(C)/E(T)$. This approach can easily be extended to derive the cost model for two dependent process steps with multiple Weibull shock models.

APPENDICES

Appendix 1: Derivation of $E(T)$

By Eqs. (17) and (18),

$$E(T) = h_1 + \alpha T_f(1-P) + T_{sr}P + qP\left(\sum_{i=1}^{\infty} h_{i+1}\beta_{10}^{i}\right) + (1-q)P\left(\sum_{i=1}^{\infty} h_{i+1}\beta_{01}^{i}\right) + (1-P)E(T_1)$$

and

$$E(T_{j-1}) = h_j + \alpha T_f(1-P) + T_{sr}P + qP\left(\sum_{i=1}^{\infty} h_{i+j}\beta_{10}^{i}\right) + (1-q)P\left(\sum_{i=1}^{\infty} h_{i+j}\beta_{01}^{i}\right) + (1-P)E(T_j), j = 2, 3, \ldots.$$

By solving the recursive system, we may obtain the expected cycle length, that is

$$\begin{aligned}
E(T) &= h_1 + (1-P)h_2 + (1-P)^2 h_3 + \cdots\cdots \\
&\quad + \alpha T_f(1-P) + \alpha T_f(1-P)^2 + \alpha T_f(1-P)^3 + \cdots\cdots \\
&\quad + T_{sr}P + T_{sr}P(1-P) + T_{sr}P(1-P)^2 + \cdots\cdots \\
&\quad + qP\left(\sum_{i=1}^{\infty} h_{i+1}\beta_{10}^{i}\right) + qP(1-P)\left(\sum_{i=1}^{\infty} h_{i+2}\beta_{10}^{i}\right) \\
&\quad + qP(1-P)^2\left(\sum_{i=1}^{\infty} h_{i+3}\beta_{10}^{i}\right) + \cdots\cdots \\
&\quad + (1-q)P\left(\sum_{i=1}^{\infty} h_{i+1}\beta_{01}^{i}\right) + (1-q)P(1-P)\left(\sum_{i=1}^{\infty} h_{i+2}\beta_{01}^{i}\right) \\
&\quad + (1-q)P(1-P)^2\left(\sum_{i=1}^{\infty} h_{i+3}\beta_{01}^{i}\right) + \cdots\cdots \\
&= \sum_{i=1}^{\infty} h_i(1-P)^{i-1} + \alpha T_f \sum_{i=1}^{\infty}(1-p)^i + T_{sr}P\sum_{i=1}^{\infty}(1-P)^{i-1} \\
&\quad + qP\left[\sum_{j=1}^{\infty}(1-P)^{j-1}\sum_{i=1}^{\infty} h_{i+j}\beta_{10}^{i}\right] \\
&\quad + (1-q)P\left[\sum_{j=1}^{\infty}(1-P)^{j-1}\sum_{i=1}^{\infty} h_{i+j}\beta_{01}^{i}\right]
\end{aligned}$$

By Eq. (2),

$$\begin{aligned}
E(T) &= h_1\sum_{i=1}^{\infty}\left[i^{1/\theta} - (i-1)^{1/\theta}\right](1-P)^{i-1} + \alpha T_f\frac{(1-P)}{p} + T_{sr} \\
&\quad + qPh_1\left\{\sum_{j=1}^{\infty}(1-P)^{j-1}\sum_{i=1}^{\infty}\left[(i+j)^{1/\theta} - (i+j-1)^{1/\theta}\right]\beta_{10}^{i}\right\} \\
&\quad + (1-q)Ph_1\left\{\sum_{j=1}^{\infty}(1-P)^{j-1}\sum_{i=1}^{\infty}\left[(i+j)^{1/\theta} - (i+j-1)^{1/\theta}\right]\beta_{01}^{i}\right\}.
\end{aligned}$$

Appendix 2: The Procedure to Obtain Approximated Value

Step 1. Use initial values P, θ, and β_{10}. Set the value of tolerance $\mathrm{TOL} = 10^{-5}$ and the sum of the infinite series is SUM.

Step 2. Set $j=1$, SUM1 $=0$, SUM2 $=0$.

Step 3. Set $i=1$, SUM $=(1-P)^{-1}\left[(i+j)^{1/\theta}-(i+j-1)^{1/\theta}\right]\beta_{10}^{i}$.

Step 4. Set $i=i+1$, SUM $=$ SUM $+(1-P)^{-1}\left[(i+j)^{1/\theta}-(i+j-1)^{1/\theta}\right]\beta_{10}^{i}$.

Step 5. If $|\text{SUM}-\text{SUM1}|<\text{TOL}$, then go to Step 6; otherwise, set SUM1 $=$ SUM and go to Step 4.

Step 6. If $|\text{SUM}-\text{SUM2}|<\text{TOL}$, then print out SUM and stop calculating. Otherwise, set SUM2 $=$ SUM, $j=j+1$, and go to Step 3.

Appendix 3: Derivation of $E(C)$

By Eqs. (20) and (21),

$$\begin{aligned}E(C) = {} & b+(h_1-\tau_1)PM[qD_1+(1-q)D_2]\\ & + C_{sr}P+\frac{bPq\beta_{10}}{1-\beta_{10}}+\frac{bP(1-q)\beta_{01}}{1-\beta_{01}}\\ & + D_1MqP\sum_{i=1}^{\infty}h_{i+1}\beta_{10}^{i}+D_2M(1-q)\\ & \times P\sum_{i=1}^{\infty}h_{i+1}\beta_{01}^{i}+D_0Mh_1(1-P)\\ & + D_0MP\tau_1+(1-P)\alpha C_f+(1-P)E(C_1),\end{aligned}$$

and

$$\begin{aligned}E(C_{j-1}) = {} & b+(h_j-\tau_j)PM[qD_1+(1-q)D_2]\\ & + C_{sr}P+\frac{bPq\beta_{10}}{1-\beta_{10}}+\frac{bP(1-q)\beta_{01}}{1-\beta_{01}}\\ & + D_1MqP\sum_{i=1}^{\infty}h_{i+j}\beta_{10}^{i}+D_2M(1-q)\\ & \times P\sum_{i=1}^{\infty}h_{i+j}\beta_{01}^{i}+D_0Mh_j(1-P)\\ & + D_0MP\tau_j+(1-P)\alpha C_f+(1-P)E(C_j), \quad j=2,3,\ldots.\end{aligned}$$

By solving the recursive system, we may obtain the expected cycle cost, that is

$$
\begin{aligned}
E(C) = {} & b + b(1-P) + b(1-P)^2 + \cdots\cdots \\
& + [qD_1 + (1-q)D_2]MP(h_1 - \tau_1) \\
& + [qD_1 + (1-q)D_2]MP(1-P)(h_2 - \tau_2) \\
& + [qD_1 + (1-q)D_2]MP(1-P)^2(h_3 - \tau_3) + \cdots\cdots \\
& + \left(C_{sr} + \frac{bq\beta_{10}}{1-\beta_{10}} + \frac{b(1-q)\beta_{01}}{1-\beta_{01}}\right)P \\
& + \left(C_{sr} + \frac{bq\beta_{10}}{1-\beta_{10}} + \frac{b(1-q)\beta_{01}}{1-\beta_{01}}\right)P(1-P) \\
& + \left(C_{sr} + \frac{bq\beta_{10}}{1-\beta_{10}} + \frac{b(1-q)\beta_{01}}{1-\beta_{01}}\right)P(1-P)^2 + \cdots\cdots \\
& + D_1MqP\sum_{i=1}^{\infty} h_{i+1}\beta_{10}^{i} + D_1MqP(1-P) \\
& \times \sum_{i=1}^{\infty} h_{i+2}\beta_{10}^{i} + D_1MqP(1-P)^2\sum_{i=1}^{\infty} h_{i+3}\beta_{10}^{i} + \cdots\cdots \\
& + D_2M(1-q)P\sum_{i=1}^{\infty} h_{i+1}\beta_{01}^{i} + D_2M(1-q)P(1-P) \\
& \times \sum_{i=1}^{\infty} h_{i+2}\beta_{10}^{i} + D_2M(1-q)P(1-P)^2\sum_{i=1}^{\infty} h_{i+3}\beta_{10}^{i} + \cdots\cdots \\
& + D_0Mh_1(1-P) + D_0Mh_2(1-P)^2 + D_0Mh_3(1-P)^3 + \cdots\cdots \\
& + D_0MP\tau_1 + D_0MP(1-P)\tau_2 + D_0MP(1-P)^2\tau_3 + \cdots\cdots \\
& + \alpha C_f(1-P) + \alpha C_f(1-P)^2 + \alpha C_f(1-P)^3 + \cdots\cdots \\
= {} & b\sum_{i=1}^{\infty}(1-P)^{i-1} + [qD_1 + (1-q)D_2]MP\sum_{i=1}^{\infty}(1-P)^{i-1}(h_i - \tau_i) \\
& + \left(C_{sr} + \frac{bq\beta_{10}}{1-\beta_{10}} + \frac{b(1-q)\beta_{01}}{1-\beta_{01}}\right)P\sum_{i=1}^{\infty}(1-P)^{i-1} \\
& + D_1MqP\left[\sum_{j=1}^{\infty}(1-P)^{j-1}\sum_{i=1}^{\infty} h_{i+j}\beta_{10}^{i}\right] + D_2M(1-q)P \\
& \times \left[\sum_{j=1}^{\infty}(1-P)^{j-1}\sum_{i=1}^{\infty} h_{i+j}\beta_{01}^{i}\right] + D_0M\sum_{i=1}^{\infty} h_i(1-P)^i \\
& + D_0MP\sum_{i=1}^{\infty}(1-P)^{i-1}\tau_i + \alpha C_f\sum_{i=1}^{\infty}(1-P)^i
\end{aligned}
$$

By Eq. (2),

$$
\begin{aligned}
E(C) = {} & \frac{b}{p} + [qD_1 + (1-q)D_2]MP \sum_{i=1}^{\infty} \left\{ h_1\left[i^{1/\theta} - (i-1)^{1/\theta}\right] - \tau_i \right\} \\
& \times (1-P)^{i-1} + C_{sr} + \frac{bq\beta_{10}}{1-\beta_{10}} + \frac{b(1-q)\beta_{01}}{1-\beta_{01}} \\
& + D_1 MqPh_1 \left[\sum_{j=1}^{\infty} (1-P)^{j-1} \sum_{i=1}^{\infty} \left[(i+j)^{1/\theta} - (i+j-1)^{1/\theta}\right]\beta_{10}^{i} \right] \\
& + D_2 M(1-q)Ph_1 \left[\sum_{j=1}^{\infty} (1-P)^{j-1} \right. \\
& \left. \times \sum_{i=1}^{\infty} \left[(i+j)^{1/\theta} - (i+j-1)^{1/\theta}\right]\beta_{01}^{i} \right] \\
& + D_0 M h_1 \sum_{i=1}^{\infty} \left[i^{1/\theta} - (i-1)^{1/\theta}\right](1-P)^{i} \\
& + D_0 MP \sum_{i=1}^{\infty} \tau_i (1-P)^{i-1} + \alpha C_f \frac{(1-P)}{P}
\end{aligned}
$$

Appendix 4. The Optimal Design Parameters and Minimal Cost

The effects of various λ and θ on the optimal design parameters and minimal cost.

λ	θ	h_1	k_1	k_2	$E(C)/E(T)$	α	$1-\beta_{10}$	$1-\beta_{01}$
0.00002	2	8.00	2.49	2.49	839.76	0.026	0.700	0.700
0.0002	2	8.00	2.49	2.49	992.29	0.026	0.700	0.700
0.002	2	4.44	2.49	2.33	2008.80	0.032	0.700	0.751
0.00002	3	8.00	2.49	2.49	983.32	0.026	0.700	0.700
0.0002	3	5.15	2.49	2.49	1630.61	0.026	0.700	0.700
0.002	3	2.92	2.06	1.86	2730.88	0.100	0.837	0.878
0.00002	4	5.80	2.49	2.49	1434.87	0.026	0.700	0.700
0.0002	4	3.65	2.49	2.46	2163.57	0.026	0.700	0.708
0.002	4	2.31	2.05	1.86	3186.89	0.100	0.839	0.877

Appendix 5. Cost Parameters and Optimal Design Parameters

The effect of the cost parameters on the optimal design.

T_f	λ	θ	h_1	k_1	k_2	EC/ET	α	$1-\beta_{10}$	$1-\beta_{01}$
0.1	0.002	3	2.922	2.063	1.857	2730.88	0.100	0.163	0.122
0.12	0.002	3	2.906	2.062	1.857	2717.18	0.100	0.163	0.122
0.15	0.002	3	2.88	2.060	1.858	2696.45	0.100	0.163	0.122
T_{sr}									
0.4	0.002	3	2.922	2.063	1.857	2730.88	0.100	0.163	0.122
1.0	0.002	3	2.924	2.060	1.859	2536.32	0.100	0.163	0.122
b									
20	0.002	3	2.922	2.063	1.857	2730.88	0.100	0.163	0.122
30	0.002	3	2.950	2.051	1.865	2756.99	0.100	0.161	0.123
100	0.002	3	3.114	2.013	1.892	2923.17	0.100	0.152	0.128
C_f									
250	0.002	3	2.922	2.063	1.857	2730.88	0.100	0.163	0.122
500	0.002	3	2.943	2.487	2.487	2755.60	0.026	0.3	0.3
1000	0.002	3	2.978	2.487	2.487	2786.35	0.026	0.3	0.3
C_{sr}									
500	0.002	3	2.923	2.062	1.857	2666.95	0.100	0.163	0.122
1000	0.002	3	2.922	2.063	1.857	2730.88	0.100	0.163	0.122
2000	0.002	3	2.921	2.064	1.856	2858.73	0.100	0.164	0.122
A_0									
1	0.002	3	2.922	2.063	1.857	2730.88	0.100	0.163	0.122
1.2	0.002	3	2.965	2.065	1.855	2903.35	0.100	0.164	0.121
1.5	0.002	3	3.025	2.069	1.853	3159.13	0.100	0.165	0.121
A_1									
1.2	0.002	3	2.922	2.063	1.857	2730.88	0.100	0.163	0.122
1.8	0.002	3	2.789	2.066	1.855	3530.83	0.100	0.164	0.121
2	0.002	3	2.759	2.067	1.854	3794.21	0.100	0.164	0.121
δ_{10}									
2.5	0.002	3	2.961	1.967	1.931	2694.96	0.100	0.281	0.136
3.0	0.002	3	2.922	2.063	1.857	2730.88	0.100	0.163	0.122
4.0	0.002	3	2.831	2.379	1.727	2817.43	0.100	0.048	0.100
δ_{01}									
2.5	0.002	3	3.061	2.206	1.783	2606.88	0.100	0.198	0.230
3.0	0.002	3	2.922	2.063	1.857	2730.88	0.100	0.163	0.122
4.0	0.002	3	2.664	1.801	2.163	2990.98	0.100	0.110	0.031
M									
20	0.002	3	2.998	2.487	2.487	1468.64	0.026	0.300	0.300
40	0.002	3	2.922	2.063	1.857	2730.88	0.100	0.163	0.122
60	0.002	3	2.876	2.070	1.852	3973.34	0.100	0.156	0.121

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