On the Efficiency of Certain Nonparametric Tests

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摘　要

在統計上很熟悉的一個問題是如何檢定兩個母之分配是否相同。在此檢定中，一個時常被考慮的對立假設是：這兩個母體之分配中，其離勢不同而其他皆同。關於觀察值是一元的情況，已有甚多有母數或無母數檢定可用。然而觀察值是二元的情況，至今仍少有研究。在這篇論文中，導出了當母體具有二元常態時之可能性比率檢定，並研究其機率分配。也研究了當母體具有二元常態或二元均等分配時，兩種無母數檢定R及R*（Liu（1982）之論文中曾提出）與其競爭之有母數檢定之間的漸近相對效率。

ABSTRACT

A familiar problem is to test whether two samples have come from identical populations. A frequently considered alternative is that the populations differ only in dispersion. If the observations are univariate, several parametric or nonparametric tests have been proposed in the literature. However, the bivariate case seems to have been studied far less fully. In this paper, the likelihood ratio test is derived and its distribution is studied if the underlying distributions are bivariate normal. The asymptotic relative efficiencies of the nonparametric tests R and R* suggested in Liu (1982) with respect to the parametric competitors are also investigated for bivariate normal and bivariate uniform distributions.

1. Introduction

Consider a bivariate two-sample problem: Suppose that $(X_{11}, X_{21}), \ldots, (X_{1m}, X_{2m})$ and $(Y_{11}, Y_{21}), \ldots, (Y_{1n}, Y_{2n})$ are two independent bivariate random samples from populations with continuous distribution functions $F_{X_1, X_2}(x_1, x_2)$ and $G_{Y_1, Y_2}(y_1, y_2)$ respectively such that

$$G_{\underset{\sim}{Y}-\underset{\sim}{\nu}}(y_1, y_2) = F_{\underset{\sim}{X}-\underset{\sim}{\nu}}(\theta_1 y_1, \theta_2 y_2) \text{ for all } (y_1, y_2)$$
$$\text{and for some } \theta_1 > 0, \theta_2 > 0,$$

where $\underset{\sim}{X} = (X_1, X_2)$, $\underset{\sim}{Y} = (Y_1, Y_2)$ and $\underset{\sim}{\nu} = (\nu_1, \nu_2)$ is the common median. We would like to detect differences in variability or dispersion for the two populations.

Two nonparametric tests R and R* are suggested in Liu (1982).

If the common median $\underset{\sim}{\nu} = (\nu_1, \nu_2)$ is known, we define $R_{m,n}$ to be the Mann-Whitney (1947) test statistic for the two independent random samples

$$U_1, U_2, \ldots, U_m \text{ and } V_1, V_2, \ldots, V_n$$

i.e., $R_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} D_{ij}$,

where $D_{ij} = 1$ if $U_i > V_j$ for all $i = 1, 2, \ldots m$,
$= 0$ otherwise $\quad j = 1, 2, \ldots n$,

$U_i = [(X_{1i} - \nu_1)^2 + (X_{2i} - \nu_2)^2]^{½}$ for $i = 1, 2, \ldots, m$, and

$V_j = [(Y_{1j} - \nu_1)^2 + (Y_{2j} - \nu_2)^2]^{½}$ for $j = 1, 2, \ldots, n$.

If the common median $\underset{\sim}{\nu} = (\nu_1, \nu_2)$ is unknown, we define $R^*_{m,n}$ to be the Mann-Whitney test statistic for the two samples,

$$U^*_{1N}, U^*_{2N}, \ldots, U^*_{mN} \text{ and } V^*_{1N}, V^*_{2N}, \ldots, V^*_{nN}$$

i.e., $R^*_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} D^*_{ij}$,

where $D^*_{ji} = 1$ if $U^*_{iN} > V^*_{jN}$ for all $i = 1, 2, \ldots, m$,
$= 0 \quad j = 1, 2, \ldots, n$,

$U^*_{iN} = [(X_{1i} - M_{1N})^2 + (X_{2i} - M_{2N})^2]^{½}$,

$V^*_{jN} = [(Y_{1j} - M_{1N})^2 + (Y_{2j} - M_{2N})^2]^{½}$,

$N = m + n$, and

$\underset{\sim}{M}_N = (M_{1N}, M_{2N})$ is the combined sample median.

In this paper, we would like to seek appropriate parametric tests for bivariate normal and bivariate uniform distributions and investigate the asymptotic relative efficiencies (ARE) of the nonparametric tests R and R* with respect to the parametric competitors.

2. The Test Statistic $F^*_{m,n}$ Under Normal Theory

In the univariate case, the general distribution model of the scale problem for two independent random samples

$$X_1, \ldots, X_m \text{ and } Y_1, \ldots, Y_n$$

is $G_{Y-\mu}(t) = F_{X-\mu}(\theta t)$ where μ is the common location. The null hypothesis of identical distribution then is H: $\theta = 1$ against either one- or two-sided alternatives. Under normal theory, the parametric test for the scale problem is the

statistic $F_{m,n} = \dfrac{\sum_{i=1}^{m}(X_i - \bar{X})^2/(m-1)}{\sum_{j=1}^{n}(Y_j - \bar{Y})^2/(n-1)}$

We are now interested in seeking an appropriate parametric test for a bivariate normal two-sample scale-model so that later we can compare the efficiency of the nonparametric tests with it.

Let us consider the bivariate normal two-sample scale-model as follows:

Suppose $(X_{11}, X_{21}), \ldots, (X_{1m}, X_{2m})$ and $(Y_{11}, Y_{21}), \ldots, (Y_{1n}, Y_{2n})$ are two independent random samples from $N_2((\mu_1, \mu_2), \begin{pmatrix} a & \rho_1 a \\ \rho_1 a & a \end{pmatrix})$ and $N_2((\mu_1, \mu_2), \begin{pmatrix} b & \rho_2 b \\ \rho_2 b & b \end{pmatrix})$, respectively, where ρ_1 and ρ_2 are the known correlation coefficients, $\underset{\sim}{\mu} = (\mu_1, \mu_2)$ and $\underset{\sim}{\eta} = (\eta_1, \eta_2)$ are unknown means, a and b are unknown scale parameters. Then $\Omega = [(a,b,\mu_1,\mu_2,\eta_1,\eta_2) \mid 0 < a,b < \infty, -\infty < \mu_1,\mu_2,\eta_1,\eta_2 < \infty]$. The hypothesis H′: $a = b$, $\underset{\sim}{\mu}$ and $\underset{\sim}{\eta}$ unspecified, is to be tested against A′: $a \neq b$, $\underset{\sim}{\mu}$ and $\underset{\sim}{\eta}$ unspecified. Then $\omega = [(a,b,\mu_1,\mu_2,\eta_1,\eta_2) \mid 0 < a = b < \infty, -\infty < \mu_1,\mu_2, \eta_1,\eta_2 < \infty]$. We are going to derive the likelihood ratio test and study its distribution. The likelihood functions are

$$L(\Omega) = \frac{1}{(2\pi)^{m+n} a^m b^n (1-\rho_1^2)^{m/2} (1-\rho_2^2)^{n/2}}$$

$$\cdot \exp\left\{ - \frac{\sum_{i=1}^{m}[(x_{1i}-\mu_1)^2 - 2\rho_1(x_{1i}-\mu_1)(x_{2i}-\mu_2) + (x_{2i}-\mu_1)^2]}{2a(1-\rho_1^2)} \right.$$

$$
- \frac{\sum_{j=1}^{n} [(y_{1j}-\eta_1)^2 - 2\rho_2(y_{1j}-\eta_1)(y_{2j}-\eta_2) + (y_{2j}-\eta_2)^2]}{2b(1-\rho_2^{\,2})} \Bigg\}
$$

and $$L(\omega) = \frac{1}{(2\pi)^{m+n} a^{m+n} (1-\rho_1^{\,2})^{m/2} (1-\rho_2^{\,2})^{n/2}}$$

$$
\cdot \exp\Bigg\{ - \frac{1}{2a} \Bigg[\frac{\sum_{i=1}^{m} ((x_{1i}-\mu_1)^2 - 2\rho_1(x_{1i}-\mu_1)(x_{2i}-\mu_2) + (x_{2i}-\mu_2)^2)}{(1-\rho_1^{\,2})}
$$

$$
+ \frac{\sum_{j=1}^{n} ((y_{1j}-\eta_1)^2 - 2\rho_2(y_{1j}-\eta_1)(y_{2j}-\eta_2) + (y_{2j}-\eta_2)^2)}{(1-\rho_2^{\,2})} \Bigg] \Bigg\} .
$$

If $\frac{\partial \log L(\Omega)}{\partial a}$, $\frac{\partial \log L(\Omega)}{\partial b}$, $\frac{\partial \log L(\Omega)}{\partial \mu_1}$, $\frac{\partial \log L(\Omega)}{\partial \mu_2}$, $\frac{\partial \log L(\Omega)}{\partial \eta_1}$ and $\frac{\partial \log L(\Omega)}{\partial \eta_2}$ are equated to zero, then $\hat{\mu}_{1\Omega} = \sum_{i=1}^{m} X_{1i}/m = \overline{X}_1$, $\hat{\mu}_{2\Omega} = \sum_{i=1}^{m} X_{2i}/m = \overline{X}_2$, $\hat{\eta}_{1\Omega} = \sum_{j=1}^{n} Y_{1j}/n = \overline{Y}_1$

$$\hat{\eta}_{2\Omega} = \sum_{j=1}^{n} Y_{2j}/n = \overline{Y}_2,$$

$$
\hat{a}_\Omega = \frac{\sum_{i=1}^{m} [(X_{1i}-\overline{X}_1)^2 - 2\rho_1(X_{1i}-\overline{X}_1)(X_{2i}-\overline{X}_2) + (X_{2i}-\overline{X}_2)^2]}{2m(1-\rho_1^{\,2})}
$$

and $$\hat{b}_\Omega = \frac{\sum_{j=1}^{n} [(Y_{1j}-\overline{Y}_1)^2 - 2\rho_2(Y_{1j}-\overline{Y}_1)(Y_{2j}-\overline{Y}_2) + (Y_{2j}-\overline{Y}_2)^2]}{2n(1-\rho_2^{\,2})}$$ maximize $L(\Omega)$.

The maximum is $L(\hat{\Omega}) = \dfrac{1}{(2\pi)^{m+n}(\hat{a}_\Omega)^m(\hat{b}_\Omega)^n(1-\rho_1{}^2)^{m/2}(1-\rho_2{}^2)^{n/2}} \exp(-m-n)$.

If $\dfrac{\partial \log L(\omega)}{\partial a}, \dfrac{\partial \log L(\omega)}{\partial \mu_1}, \dfrac{\partial \log(\omega)}{\partial \mu_2}, \dfrac{\partial \log(\omega)}{\partial \eta_1}, \dfrac{\partial \log(\omega)}{\partial \eta_2}$ are equated to zero,

then $\hat{\mu}_{1\omega} = \overline{X}_1, \hat{\mu}_{2\omega} = \overline{X}_2, \hat{\eta}_{1\omega} = \overline{Y}_1, \hat{\eta}_{2\omega} = \overline{Y}_2$ and

$$\hat{a}_\omega = \frac{1}{2(m+n)}\left\{\frac{\sum_{i=1}^{m}[(X_{1i}-\overline{X}_1)^2 - 2\rho_1(X_{1i}-\overline{X}_1)(X_{2i}-\overline{X}_2) + (X_{2i}-\overline{X}_2)^2]}{1-\rho_1{}^2}\right.$$

$$\left. + \frac{\sum_{j=1}^{n}[(Y_{1j}-\overline{Y}_1)^2 - 2\rho_2(Y_{1j}-\overline{Y}_1)(Y_{2j}-\overline{Y}_2) + (Y_{2j}-\overline{Y}_2)^2]}{1-\rho_2{}^2}\right\}$$

maximize $L(\omega)$. The maximum is

$$L(\hat{\omega}) = \frac{1}{(2\pi)^{m+n}(\hat{a}_\omega)^{m+n}(1-\rho_1{}^2)^{m/2}(1-\rho_2{}^2)^{n/2}} \exp(-m-n)$$

Hence $\lambda = \dfrac{L(\hat{\omega})}{L(\hat{\Omega})} = \dfrac{(\hat{a}_\Omega)^m(\hat{b}_\Omega)^n}{(\hat{a}_\omega)^{m+n}}$

$$= \frac{\left\{\dfrac{\sum_{i=1}^{m}[(X_{1i}-\overline{X}_1)^2 - 2\rho_1(X_{1i}-\overline{X}_1)(X_{2i}-\overline{X}_2) + (X_{2i}-\overline{X}_2)^2]}{2m(1-\rho_1{}^2)}\right\}^m}{\left\{\dfrac{1}{2(m+n)}\left[\dfrac{\sum_{i=1}^{m}((X_{1i}-\overline{X}_1)^2 - 2\rho_1(X_{1i}-\overline{X}_1)(X_{2i}-\overline{X}_2) + (X_{2i}-\overline{X}_2)^2)}{1-\rho_1{}^2}\right.\right.}$$

$$\frac{\cdot\left\{\dfrac{\sum_{j=1}^{n}[(Y_{1j}-\overline{Y}_1)^2-2\rho_2(Y_{1j}-\overline{Y}_1)(Y_{2j}-\overline{Y}_2)+(Y_{2j}-\overline{Y}_2)^2]}{2n(1-\rho_2{}^2)}\right\}^{n}}{+\dfrac{\sum_{j=1}^{n}((Y_{1j}-\overline{Y}_1)^2-2\rho_2(Y_{1j}-\overline{Y}_1)(Y_{2j}-\overline{Y}_2)+(Y_{2j}-\overline{Y}_2)^2)}{1-\rho_2{}^2}]\Big\}^{m+n}}$$

$$=\frac{(m+n)^m}{m^m n^n}\;\frac{\left\{\dfrac{\sum_{i=1}^{m}[(X_{1i}-\overline{X}_1)^2-2\rho_1(X_{1i}-\overline{X}_1)(X_{2i}-\overline{X}_2)+(X_{2i}-\overline{X}_2)^2]}{1-\rho_1{}^2}\right\}^{m}}{\left\{\dfrac{\sum_{i=1}^{m}[(X_{1i}-\overline{X}_1)^2-2\rho_1(X_{1i}-\overline{X}_1)(X_{2i}-\overline{X}_2)+(X_{2i}-\overline{X}_2)^2]}{1-\rho_1{}^2}\right.}$$

$$\frac{\cdot\left\{\dfrac{\sum_{j=1}^{n}[(Y_{1j}-\overline{Y}_1)^2-2\rho_2(Y_{1j}-\overline{Y}_1)(Y_{2j}-\overline{Y}_2)+(Y_{2j}-\overline{Y}_2)^2]}{1-\rho_2{}^2}\right\}^{n}}{\left.+\dfrac{\sum_{j=1}^{n}[(Y_{1j}-\overline{Y}_1)^2-2\rho_2(Y_{1j}-\overline{Y}_1)(Y_{2j}-\overline{Y}_2)+(Y_{2j}-\overline{Y}_2)^2]}{1-\rho_2{}^2}\right\}^{m+n}}$$

$$=\frac{(m+n)^m}{m^m n^n}\cdot\frac{[(\frac{m-1}{n-1})F^*_{m,n}]^m}{[(\frac{m-1}{n-1})F^*_{m,n}+1]^{m+n}},\ \text{where}$$

$$F^*_{m,n}=\frac{\sum_{i=1}^{m}[(X_{1i}-\overline{X}_1)^2-2\rho_1(X_{1i}-\overline{X}_1)(X_{2i}-\overline{X}_2)+(X_{2i}-\overline{X}_2)^2]/[(1-\rho_1{}^2)(m-1)]}{\sum_{j=1}^{n}[(Y_{1j}-\overline{Y}_1)^2-2\rho_2(Y_{1j}-\overline{Y}_1)(Y_{2j}-\overline{Y}_2)+(Y_{2j}-\overline{Y}_2)^2]/[(1-\rho_2{}^2)(n-1)]}\ .$$

$$= \frac{(m+n)^{m+n}(m-1)^m(n-1)^n}{m^n n^n} \cdot \frac{(F^*_{m,n})^m}{[(m-1)F^*_{m,n} + (n-1)]^{m+n}}$$

Thus, the critical region $\lambda \leqslant \lambda_o$ is equivalent to $F^*_{m,n} \leqslant c_1$ or $F^*_{m,n} \geqslant c_2$. Finally, the likelihood ratio test for testing H′: a = b, $\underset{\sim}{\mu}$ and $\underset{\sim}{\eta}$ unspecified, against A′: a ≠ b, $\underset{\sim}{\mu}$ and $\underset{\sim}{\eta}$ unspecified, can be based on $F^*_{m,n}$.

In this normal-theory model, we can show that for every $\theta = (a/b)^{1/2} \epsilon\ \Omega = (o,\infty)$, the distribution of $F^*_{m,n}/\theta^2$ is F with 2(m−1) and 2(n−1) degrees of freedom. We first need the following lemmas.

Lemma 2.1: If a p-dimensional random vector $\underset{\sim}{X} = (X_1, \ldots, X_p)$ is distributed according to N_p $(\underset{\sim}{0}, \Sigma)$ (nonsingular), then $\underset{\sim}{X}\Sigma^{-1}\underset{\sim}{X}'$ is distributed according to the χ^2 distribution with p degrees of freedom.

Proof: See { Anderson (1958), p. 54 }.

Lemma 2.2: Suppose $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_N$ are independent, where $\underset{\sim}{X}_\alpha$ is distributed according to N_p $(\underset{\sim}{\mu}_\alpha, \Sigma)$. Let $C = (c_{\alpha\beta})$ be an orthogonal matrix. Then $\underset{\sim}{Y}_\alpha = \sum_{\beta=1}^{N} c_{\alpha\beta}\underset{\sim}{X}_\beta$ is distributed according to N_p $(\underset{\sim}{\upsilon}_\alpha, \Sigma)$, where $\underset{\sim}{\upsilon}_\alpha = \sum_{\beta=1}^{N} c_{\alpha\beta}\underset{\sim}{\mu}_\beta$ and $\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_N$ are independent.

Proof: See { Anderson (1958), p. 52 }.

Theorem 2.1: In the previous normal-theory model, for every $\theta = (a/b)^{1/2} \epsilon$ $\Omega = (o,\infty)$, we obtain

(1) $F^*_{m,n}/\theta^2$ has an F distribution with 2(m−1) and 2(n−1) degrees of freedom,

(2) $E(F^*_{m,n}) = \frac{(n-1)}{(n-2)}\theta^2$, and

(3) $Var(F^*_{m,n}) = \frac{(n-1)^2(m+n-3)}{(m-1)(n-2)^2(n-3)}\theta^4$.

Proof: (1) Fix $\theta = (a/b)^{1/2} \epsilon\ \Omega = (o,\infty)$.

$$F^*_{m,n} = \frac{\sum_{i=1}^{m} [(X_{1i}-\overline{X}_1)^2 - 2\rho_1(X_{1i}-\overline{X}_1)(X_{2i}-\overline{X}_2) + (X_{2i}-\overline{X}_2)^2]}{\sum_{j=1}^{n} [(Y_{1j}-\overline{Y}_1)^2 - 2\rho_2(Y_{1j}-\overline{Y}_1)(Y_{2j}-\overline{Y}_2) + (Y_{2j}-\overline{Y}_2)^2]} \quad \frac{/[(1-\rho_1{}^2)(m-1)]}{/[(1-\rho_2{}^2)(n-1)]}$$

$$= \left(\frac{a}{b}\right) \frac{\sum_{i=1}^{m} [(X_{1i}-\overline{X}_1)^2 - 2\rho_1(X_{1i}-\overline{X}_1)(X_{2i}-\overline{X}_2) + (X_{2i}-\overline{X}_2)^2]}{\sum_{j=1}^{n} [(Y_{ij}-\overline{Y}_1)^2 - 2\rho_2(Y_{1j}-\overline{Y}_1)(Y_{2j}-\overline{Y}_2) + (Y_{2j}-\overline{Y}_2)^2]} \quad \frac{/[a(1-\rho_1{}^2)2(m-1)]}{/[b(1-\rho_2{}^2)2(n-1)]}$$

$$= \theta^2 \frac{\sum_{i=1}^{m} (\underset{\sim}{X}_i-\overline{\underset{\sim}{X}})\Sigma_{\underset{\sim}{X}}{}^{-1}(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}})'/2(m-1)}{\sum_{j=1}^{n} (\underset{\sim}{Y}_j-\overline{\underset{\sim}{Y}})\Sigma_{\underset{\sim}{Y}}{}^{-1}(\underset{\sim}{Y}_j-\overline{\underset{\sim}{Y}})'/2(n-1)},$$

where
$$\underset{\sim}{X}_i = (X_{1i}, X_{2i}), \qquad \underset{\sim}{Y}_j = (Y_{1j}, Y_{2j}),$$
$$\overline{\underset{\sim}{X}} = (\overline{X}_1, \overline{X}_2), \qquad \overline{\underset{\sim}{Y}} = (\overline{Y}_1, \overline{Y}_2),$$
$$\Sigma_{\underset{\sim}{X}} = \begin{pmatrix} a & \rho_1 a \\ \rho_1 a & a \end{pmatrix}, \qquad \Sigma_{\underset{\sim}{Y}} = \begin{pmatrix} b & \rho_2 b \\ \rho_2 b & b \end{pmatrix}.$$

We may proceed to prove that $\sum_{i=1}^{m} (\underset{\sim}{X}_i-\overline{\underset{\sim}{X}})\Sigma_{\underset{\sim}{X}}{}^{-1}(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}})'$ has a χ^2 distribution with 2(m−1) degrees of freedom. Since $(X_{11}, X_{21}), \ldots, (X_{1m}, X_{2m})$ is a random sample from a bivariate normal population with mean $\underset{\sim}{\mu} = (\mu_1, \mu_2)$ and covariance matrix $\Sigma_{\underset{\sim}{X}} = \begin{pmatrix} a & \rho a \\ \rho a & a \end{pmatrix}$, there exists an m X m orthogonal matrix $C = (c_{ki})$ with the last row $(1/\sqrt{m}, \ldots, 1/\sqrt{m})$. Let $\mathbf{Z}_k = \sum_{i=1}^{m} c_{ki}\mathbf{X}_i$. Then, by Lemma 2.2, $\mathbf{Z}_k$ is distributed according to $N_p(\boldsymbol{\upsilon}_k, \boldsymbol{\Sigma}_{\mathbf{X}})$, where $\boldsymbol{\upsilon}_k = \sum_{i=1}^{m} c_{ki}\boldsymbol{\mu}$, and $\mathbf{Z}_1, \ldots, \mathbf{Z}_m$ are independent.

In particular $\mathbf{Z}_m = \sum_{i=1}^{m} c_{mi}\mathbf{X}_i = \sum_{i=1}^{m} (1/\sqrt{m})\mathbf{X}_i = \sqrt{m}\,\overline{\mathbf{X}}$.

Consider $\sum_{k=1}^{m} \mathbf{Z}_k \Sigma_{\mathbf{X}}^{-1} \mathbf{Z}_k'$

$$= \sum_{k=1}^{m} \Big(\sum_{i=1}^{m} c_{ki}\mathbf{X}_i\Big)\Sigma_{\mathbf{X}}^{-1}\Big(\sum_{j=1}^{m} c_{kj}\mathbf{X}_j\Big)'$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m}\Big(\sum_{k=1}^{m} c_{ki}c_{kj}\Big)\mathbf{X}_i\Sigma_{\mathbf{X}}^{-1}\mathbf{X}_j'$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m} \delta_{ij}\mathbf{X}_i\Sigma_{\mathbf{X}}^{-1}\mathbf{X}_j', \quad \text{where } \delta_{ij} = 1, \text{ if } i=j \text{ and } \delta_{ij} = 0, \text{ if } i \neq j$$

$$= \sum_{i=1}^{m} \mathbf{X}_i\Sigma_{\mathbf{X}}^{-1}\mathbf{X}_i'$$

Thus,

$$\sum_{i=1}^{m} (\mathbf{X}_i - \overline{\mathbf{X}})\Sigma_{\mathbf{X}}^{-1}(\mathbf{X}_i - \overline{\mathbf{X}})' = \sum_{i=1}^{m} \mathbf{X}_i\Sigma_{\mathbf{X}}^{-1}\mathbf{X}_i' - m\overline{\mathbf{X}}\Sigma_{\mathbf{X}}^{-1}\overline{\mathbf{X}}'$$

$$= \sum_{k=1}^{m} \mathbf{Z}_k\Sigma_{\mathbf{X}}^{-1}\mathbf{Z}_k' - \mathbf{Z}_m\Sigma_{\mathbf{X}}^{-1}\mathbf{Z}_m'$$

$$= \sum_{k=1}^{m-1} \mathbf{Z}_k\Sigma_{\mathbf{X}}^{-1}\mathbf{Z}_k'.$$

We note $E(\underset{\sim}{Z}_k) = \underset{\sim}{\nu}_k = \sum_{i=1}^{m} c_{ki}\underset{\sim}{\mu}$

$$= \sum_{i=1}^{m} c_{ki}(1/\sqrt{m})\sqrt{m}\underset{\sim}{\mu}$$

$$= \sum_{i=1}^{m} c_{ki}c_{mi}\sqrt{m}\underset{\sim}{\mu}$$

$$= 0, \text{ for } k \neq m.$$

By Lemma 2.1, $\sum_{k=1}^{m-1} \underset{\sim}{Z}_k \underset{\sim}{\Sigma}_X^{-1} \underset{\sim}{Z}_k' = \sum_{i=1}^{m} (\underset{\sim}{X}_i - \overline{\underset{\sim}{X}})\underset{\sim}{\Sigma}_X^{-1}(\underset{\sim}{X}_i - \overline{\underset{\sim}{X}})'$ has a χ^2 distribution with $2(m-1)$ degrees of freedom. Similarly, we can show that $\sum_{j=1}^{n} (\underset{\sim}{Y}_j - \overline{\underset{\sim}{Y}})\underset{\sim}{\Sigma}_Y^{-1} (\underset{\sim}{Y}_j - \overline{\underset{\sim}{Y}})'$ has a χ^2 distribution with $2(n-1)$ degrees of freedom. By the fact that $(X_{11}, X_{21}), \ldots, (X_{1m}, X_{2m})$ and $(Y_{11}, Y_{21}), \ldots, (Y_{1n}, Y_{2n})$ are two independent random samples, $\sum_{i=1}^{m} (\underset{\sim}{X}_i - \overline{\underset{\sim}{X}})\underset{\sim}{\Sigma}_X^{-1}(\underset{\sim}{X}_i - \overline{\underset{\sim}{X}})'$ and $\sum_{j=1}^{n} (\underset{\sim}{Y}_j - \overline{\underset{\sim}{Y}})\underset{\sim}{\Sigma}_Y^{-1}(\underset{\sim}{Y}_j - \overline{\underset{\sim}{Y}})'$ are independent. Therefore, $F^*_{m,n}/\theta^2$ has an F distribution with $2(m-1)$ and $2(n-1)$ degrees of freedom.

(2) and (3) are immediate consequences of (1) and the result in [Cramér (1946), p.242].

This completes the proof.

3. Evaluating the ARE(R,F*) for Bivariate Normal Distributions

The purpose of this section is to obtain the asymptotic relative efficiency of the nonparametric test R (or R^*) suggested in Liu (1982) with respect to the parametric test F^* for the normal-theory model.

Let us first define the asymptotic relative efficiency of test T with respect to test T^*.

Suppose we have two test statistics T_n and T_n^* for a hypothesis testing problem, H: $\theta \in \omega$ against A: $\theta \in \Omega$-ω. Let $[\theta_0, \theta_1, \theta_2, \ldots]$ be a sequence of constants such that θ_0 specifies a value in ω and the remaining $\theta_1, \theta_2, \ldots$ are in Ω-ω and that $\lim_{n \to \infty} \theta_n = \theta_0$. Let $\{\Phi_n\}$ and $\{\Phi_n^*\}$ be two sequences of tests all of the same size α, which are based on the test statistics T_n and T_n^*, respectively. Also let $\{n_i\}$ and $\{n_i^*\}$ be two monotonic increasing sequences of positive integers such that

$$\lim_{i \to \infty} P_{\Phi_{n_i}}(\theta_i) = \lim_{i \to \infty} P_{\Phi^*_{n_i^*}}(\theta_i),$$

with the two limits existing not equal to 0 or 1 (the limiting power of Φ_{n_i} at θ_i must be the same as the limiting power of $\Phi^*_{n_i^*}$ at θ_i). Then the asymptotic relative efficiency of test T with respect to test T^* is defined to be

$$\text{ARE}(T,T^*) = \lim_{i \to \infty} \frac{n_i}{n_i^*}, \text{ if this limit exists.}$$

We are going to apply the following theorem in evaluating the ARE(R,F^*).

Theorem 3.1: If T and T^* are two tests satisfying the four regular conditions, i.e., (analogous ones for T_n^*)

1. $dE(T_n)/d\theta$ exists and is nonzero for $\theta = \theta_o$, and is continuous at θ_o,
2. There exists a positive constant c such that

$$\lim_{n \to \infty} \frac{dE(T_n)/d\theta \mid \theta = \theta_o}{\sqrt{n\text{Var}(T_n)} \mid \theta = \theta_o} = c,$$

3. There exists a sequence of alternatives $[\theta_n]$ such that for some constant $d > 0$, we have

$$\theta_n = \theta_o + \frac{d}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{[dE(T_n)/d\theta] \mid \theta = \theta_n}{[dE(T_n)/d\theta] \mid \theta = \theta_o} = 1$$

$$\lim_{n\to\infty} \frac{\sqrt{Var(T_n)} \mid \theta = \theta_n}{\sqrt{Var(T_n)} \mid \theta = \theta_o} = 1,$$

4. $\lim_{n\to\infty} P\left(\frac{[T_n - E(T_n)] \mid \theta = \theta_n}{\sqrt{Var(T_n)} \mid \theta = \theta_n} \leqslant z \mid \theta = \theta_n\right) = \Phi(z)$, where $\Phi(z)$ is the stan-normal d.f.,

then the ARE of T with respect to T^* is

$$ARE(T,T^*) = \lim_{n\to\infty} \frac{e(T_n)}{e(T_n^*)}, \text{ where } e(T_n) = \frac{([dE(T_n)/d\theta] \mid \theta = \theta_o)^2}{[Var(T_n)] \mid \theta = \theta_o}$$

Proof: See {Fraser (1957), p, 273}.

Suppose $(X_{11},X_{21}), \ldots, (X_{1m},X_{2m})$ and $(Y_{11},Y_{21}), \ldots, (Y_{1n},Y_{2n})$ are two independent random samples from bivariate normal populations with p.d.f.'s

$$f(x_1,x_2) = \frac{1}{2\pi a\sqrt{1-\rho^2}} \exp\left\{-\frac{[(x_1-\mu_1)^2 - 2\rho(x_1-\mu_1)(x_2-\mu_2) + (x_2-\mu_2)^2]}{2a(1-\rho^2)}\right\},$$

$$-\infty < x_1, x_2 < \infty,$$

and $$g(y_1,y_2) = \frac{1}{2\pi b\sqrt{1-\rho^2}} \exp\left\{-\frac{[(y_1-\mu_1)^2 - 2\rho(y_1-\mu_1)(y_2-\mu_2) + (y_2-\mu_2)^2]}{2b(1-\rho^2)}\right\},$$

$$-\infty < y_1, y_2 < \infty,$$

respectively, where ρ is the known common correlation coefficient, $\underset{\sim}{\mu} = (\mu_1,\mu_2)$ is the known (or unknown) common mean, a and b are unknown scale parameters. Set $\theta = (a/b)^{1/2}$. Thus, we have two test statistics $R_{m,n}$ (or $R^*_{m,n}$) and $F^*_{m,n}$

for the hypothesis testing problem H': $\theta = 1$ against either one- or two-sided alternatives.

Now, it is easy to check that the tests F^* and R satisfy the four regular conditions stated in Theorem 3.1, and for every $\theta = (a/b)^{1/2} \in \Omega = (o.\infty)$, we have

$$E(F^*_{m,n}) = \frac{(n-1)}{(n-2)}\theta^2, \quad Var(F^*_{m,n}) = \frac{(n-1)^2(m+n-3)}{(m-1)(n-2)^2(n-3)}\theta^4 ,$$

$$E(R_{m,n}) = mN \, [\textstyle\int_o^\infty [\lambda_N S(u) + (1-\lambda_N)S(\theta u)]\, dS(u)\,] - m(m+1)/2 ,$$

$$Var(R_{m,n}) = 2m^2N(1-\lambda_N) \, \{ \iint_{o<u<v<\infty} S(\theta u)\,[1-S(\theta v)]\, dS(u)dS(v)$$

$$+ \frac{(1-\lambda_N)}{\lambda_N} \iint_{o<u<v<\infty} S(u)[1-S(v)]\, dS(\theta u)dS(\theta v)\,\} ,$$

where S(u) is the d.f. of U, and $U = [(X_1-\mu_1)^2 + (X_2-\mu_2)^2]^{1/2}$, by Theorem 2.1 and the Chernoff-Savage theorem (see {Chernoff and Savage (1958)}).

In particular, $[dE(R_{m,n})/d\theta] \mid_{\theta=1}$

$= [mn\int_o^\infty us(\theta u)s(u)du] \mid_{\theta=1}$, where s(u) is the p.d.f. of U,

$= mn\int_o^\infty u[s(u)]^2 du$, and

$[Var(R_{m,n})] \mid_{\theta=1}$

$= mn(m+n+1)/12.$

Let us derive the p.d.f. of $U = [(X_1-\mu_1)^2 + (X_2-\mu_2)^2]^{1/2}$ if the joint p.d.f. of X_1 and X_2 is

$$f(x_1,x_2) = \frac{1}{2\pi a\sqrt{1-\rho^2}} \exp\,[-\frac{[(x_1-\mu_1)^2 - 2\rho(x_1-\mu_1)(x_2-\mu_2) + (x_2-\mu_2)^2]}{2a(1-\rho^2)}] ,$$

$$-\infty < x_1, x_2 < \infty .$$

If we set $x_1-\mu_1 = u\cos t$ and $x_2-\mu_2 = u\sin t$, then the joint p.d.f of U and T is

$$g(u,t) = \frac{1}{2\pi a\sqrt{1-\rho^2}} \exp\left[-\frac{u^2(1-\rho\sin 2t)}{2a(1-\rho^2)}\right], \quad \begin{array}{l} 0 \leqslant u < \infty, \\ 0 \leqslant t < 2\pi . \end{array}$$

Thus the p.d.f. of U is

$$s(u) = \frac{1}{2\pi a\sqrt{1-\rho^2}} \int_0^{2\pi} \exp\left[-\frac{u^2(1-\rho\sin 2t)}{2a(1-\rho^2)}\right] dt, \quad 0 \leqslant u < \infty .$$

Hence $\int_0^\infty u[s(u)]^2 du$

$$= \left(\frac{1}{2\pi a\sqrt{1-\rho^2}}\right)^2 \int_0^\infty u^2 \left\{\int_0^{2\pi} \exp\left[-\frac{u^2(1-\rho\sin 2t)}{2a(1-\rho^2)}\right] dt\right\}^2 u\,du$$

$$= \frac{1}{8\pi^2 a^2(1-\rho^2)} \int_0^\infty r \left\{\int_0^{2\pi} \exp\left[-\frac{r(1-\rho\sin 2t)}{2a(1-\rho^2)}\right] dt\right\}^2 dr,$$

where we set $r = u^2$

$$= \frac{1}{8\pi^2 a^2(1-\rho^2)} \int_0^\infty aq \left\{\int_0^{2\pi} \exp\left[-\frac{q(1-\rho\sin 2t)}{2(1-\rho^2)}\right] dt\right\}^2 a\,dq,$$

where we set $q = \frac{r}{a}$

$$= \frac{1}{8\pi^2(1-\rho^2)} \int_0^\infty q \left\{\int_0^{2\pi} \exp\left[-\frac{q(1-\rho\sin 2t)}{2(1-\rho^2)}\right] dt\right\}^2 dq$$

$$= \frac{1}{8\pi^2(1-\rho^2)} \int_0^\infty q \left\{\int_0^{2\pi} \exp\left[-\frac{q(1-\rho\sin 2t_1)}{2(1-\rho^2)}\right] dt_1\right\}$$

$$\cdot \left\{\int_0^{2\pi} \exp\left[-\frac{q(1-\rho\sin 2t_2)}{2(1-\rho^2)}\right] dt_2\right\} dq$$

$$= \frac{1}{8\pi^2(1-\rho^2)} \int_0^{2\pi}\int_0^{2\pi} q \exp\left[-\frac{q(2-\rho\sin 2t_1-\rho\sin 2t_2)}{2(1-\rho^2)}\right] dq\,dt_2\,dt_1$$

$$= \frac{1}{8\pi^2(1-\rho^2)} \int_0^{2\pi}\int_0^{2\pi}\int_0^{\infty} q \exp\left[-\frac{q(2-\rho\sin 2t_i-\rho\sin 2t_2)}{2(1-\rho^2)}\right] dq dt_2 dt_1$$

$$= \frac{1}{8\pi^2(1-\rho^2)} \int_0^{2\pi}\int_0^{2\pi} \frac{4(1-\rho^2)^2}{(2-\rho\sin 2t_1-\rho\sin 2t_2)^2} dt_2 dt_1$$

$$= \frac{(1-\rho^2)}{2\pi^2} \int_0^{2\pi}\int_0^{2\pi} \frac{1}{(2-\rho\sin 2t_1-\rho\sin 2t_2)^2} dt_2 dt_1$$

In order to simplify this integral, we need the following lemma.

Lemma 3.1: For a and b real, $a > |b|$, we have

$$\int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}} .$$

Proof: Setting $z = \exp(i\theta)$, we see that $dz = i\exp(i\theta)d\theta = izd\theta$.

Thus, $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)^2} = \int_C \frac{dz/iz}{[a+b(z-1/z)/(2i)]^2}$,

since $\sin\theta = [\exp(i\theta)-\exp(-i\theta)]/(2i) = (z-1/z)/(2i)$, and where C is the unit circle,

$$= \frac{4i}{b^2}\int_C \frac{zdz}{[z^2+(2ai/b)z-1]^2} .$$

The function $f(z) = \dfrac{z}{[z^2+(2ai/b)z-1]^2}$ has poles of order 2 at $z_1 = \dfrac{(-a+\sqrt{a^2-b^2})i}{b}$, and $z_2 = \dfrac{(-a-\sqrt{a^2-b^2})i}{b}$ with z_1 being inside the unit circle and z_2 outside. The residue of f(z) at z_1 (see {Silverman (1975), p. 254}) is

$$\lim_{z\to z_1} \frac{d^{2-1}}{dz^{2-1}} [(z-z_1)^2 f(z)] = \lim_{z\to z_1} \frac{d}{dz} [(z-z_1)^2 \frac{z}{(z-z_1)^2(z-z_2)^2}]$$

$$= \lim_{z\to z_1} \frac{d}{dz} [\frac{z}{(z-z_2)^2}]$$

$$= -\frac{(z_1+z_2)}{(z_1-z_2)^3} = \frac{ab^2}{4i^2(a^2-b^2)^{3/2}}.$$

By the Residue Theorem (see {Silverman (1975), p. 253 }),

$$\int_C f(z)dz = \int_C \frac{zdz}{[z^2+(2ai/b)z-1]^2}$$

$$= 2\pi i \mathrm{Res}(z_1)$$

$$= 2\pi i \cdot \frac{ab^2}{4i^2(a^2-b^2)^{3/2}}$$

$$= \frac{\pi ab^2}{2i(a^2-b^2)^{3/2}}$$

Hence $$\int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)^2} = \frac{4i}{b^2} \int_C \frac{zdz}{[z^2+(2ai/b)z-1]^2}$$

$$= \frac{4i}{b^2} \cdot \frac{\pi ab^2}{2i(a^2-b^2)^{3/2}}$$

$$= \frac{2\pi a}{(a^2-b^2)^{3/2}}.$$

This completes the proof.

Using Lemma 3.1, we obtain

$$\int_0^\infty u[s(u)]^2 du = \frac{(1-\rho^2)}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{(2-\rho\sin 2t_1-\rho\sin 2t_2)^2}dt_2dt_1$$

$$= \frac{(1-\rho^2)}{2\pi^2}\int_0^{2\pi}\frac{2\pi(2-\rho\sin 2t_1)}{[(2-\rho\sin 2t_1)^2-\rho^2]^{3/2}}dt_1 .$$

Therefore, $[dE(R_{m,n}) / d\theta] \mid \theta = 1$

$$= mn\int_0^\infty u\,[s(u)]^2 du$$

$$= mn \cdot \frac{(1-\rho^2)}{2\pi^2}\int_c^{2\pi}\frac{2\pi(2-\rho\sin 2t_1)}{[(2-\rho \mathrm{sint})^2-\rho^2]^{3/2}}dt_1$$

$$= \frac{mn(1-\rho^2)}{\pi}\int_0^{2\pi}\frac{(2-\rho\sin t)}{[(2-\rho\sin 2t\ )^2-\rho^2]^{3/2}}dt.$$

We now summarize the results in the following theorem.

Theorem 3.2: Let $(X_{11},X_{21}), \ldots, (X_{1m},X_{2m})$ and $(Y_{11},Y_{21}), \ldots, (Y_{1n},Y_{2n})$ be two independent random samples from bivariate normal populations with p.d.f.'s

$$f(x_1,x_2) = \frac{1}{2\pi a\sqrt{1-\rho^2}}\exp\left\{-\frac{[(x_1-\mu_1)^2-2\rho(x_1-\mu_1)(x_2-\mu_2)+(x_2-\mu_2)^2]}{2a(1-\rho^2)}\right\},$$

$$-\infty < x_1, x_2 < \infty,$$

and $$g(y_1,y_2) = \frac{1}{2\pi b\sqrt{1-\rho^2}}\exp\left\{-\frac{[(y_1-\mu_1)^2-2\rho(y_1-\mu_1)(y_2-\mu_2)+(y_2-\mu_2)^2]}{2b(1-\rho^2)}\right\},$$

$$-\infty < y_1, y_2 < \infty,$$

respectively, where ρ is the known common correlation coefficient, $\underset{\sim}{\mu} = (\mu_1, \mu_2)$ is the common mean (either known or unknown), a and b are unknown scale parameters. Set $\theta = (a/b)^{1/2}$. For the hypothesis testing problem H': $\theta = 1$ against either one- or two-sided alternatives, we have

$$ARE(R,F^*) = ARE(R^*,F^*) = \frac{3(1-\rho^2)^2}{\pi^2}\left[\int_0^{2\pi} \frac{(2-\rho \sin t)}{[(2-\rho \sin t)^2-\rho^2]^{3/2}}\, dt\right]^2 .$$

Particularly, if $\rho = 0$ then $ARE(R,F^*) = ARE(R^*,F^*) = 3/4$.

Proof: Since
$$e(F^*_{m,n}) = \frac{([dE(F^*_{m,n})/d\theta] \mid \theta = 1)^2}{[Var(F^*_{m,n})] \mid \theta = 1}$$

$$= \frac{[2(n-1)/(n-2)]^2}{(n-1)^2(m+n-3)/[(m-1)(n-2)^2(n-3)]}$$

$$= \frac{4(m-1)(n-3)}{m+n-3}$$

and
$$e(R_{m,n}) = \frac{([dE(R_{m,n})/d\theta] \mid \theta = 1)^2}{[Var(R_{m,n})] \mid \theta = 1}$$

$$= \frac{\left[\frac{mn(1-\rho^2)}{\pi}\int_0^{2\pi} \frac{(2-\rho \sin t)}{[(2-\rho \sin t)^2-\rho^2]^{3/2}}\, dt\right]^2}{mn(m+n+1)/12}$$

$$= \frac{12(1-\rho^2)^2 mn}{\pi^2(m+n+1)}\left[\int_0^{2\pi} \frac{(2-\rho \sin t)}{[(2-\rho \sin t)^2-\rho^2]^{3/2}}\, dt\right]^2$$

then
$$ARE(R,F^*) = \lim_{m,n\to\infty} \frac{e(R_{m,n})}{e(F^*_{m,n})}$$

$$= \frac{3(1-\rho^2)^2}{\pi^2} [\int_0^{2\pi} \frac{(2-\rho \sin t)}{[(2-\rho \sin t)^2-\rho^2]^{3/2}} dt]^2$$

by Theorem 3.1.

Applying the technique used in Liu (1981), we can prove that $R_{m,n}$ and $R^*_{m,n}$ have the same limiting distribution under both H′ and A′ if the underlying populations are bivariate normal. Hence the tests R and R^* have the same ARE with respect to the test F^*, which concludes the proof.

4. Evaluating the ARE(R,F^{**}) for Bivariate Uniform Distributions

We are now interested in evaluating the asymptotic relative efficiency of the nonparametric test R (or R^*) with respect to an appropriate parametric test if the underlying populations have bivariate uniform distributions.

First we note that if a random vector (X_1,X_2) is uniformly distributed over a disc with p.d.f.

$$f(x_1,x_2) = 1/(\pi c^2), \quad 0 \leqslant (x_1-\mu_1)^2 + (x_2-\mu_2)^2 \leqslant c^2,$$

then $E(X_1) = \mu_1$, $E(X_2) = \mu_2$, $Var(X_1) = Var(X_2) = c^2/4$, $\rho = 0$, and the random variables X_1 and X_2 are dependent.

Consider the bivariate uniform two-sample scale-model as follows:

Suppose $(X_{11},X_{21}), \ldots, (X_{1m},X_{2m})$ and $(Y_{11},Y_{21}), \ldots, (Y_{1n},Y_{2n})$ are two independent random samples from bivariate uniform populations with p.d.f.'s

$$f(x_1,x_2) = 1/(\pi c_1^2), \qquad 0 \leqslant (x_1-\mu_1)^2 + (x_2-\mu_2)^2 \leqslant c_1^2,$$

and $$g(y_1,y_2) = 1/(\pi c_2^2), \qquad 0 \leqslant (y_1-\mu_1)^2 + (y_2-\mu_2)^2 \leqslant c_2^2,$$

where $\underset{\sim}{\mu} = (\mu_1,\mu_2)$ is the known common mean, c_1 and c_2 are unknown scale parameters. Set $\theta = c_1/c_2$. Then the problem is to test the hypothesis H′: $\theta = 1$ against either one- or two-sided alternatives. Since $E((X_1-\mu_1)^2 + (X_2-\mu_2)^2) = c_1^2/2$, and $E((Y_1-\mu_1)^2 + (Y_2-\mu_2)^2) = c_2^2/2$, we would like to compare the nonparametric test statistic $R_{m,n}$ with the test statistic

$$F^{**}_{m,n} = \frac{\sum_{i=1}^{m} [(X_{1i}-\mu_1)^2 + (X_{2i}-\mu_2)^2]/m}{\sum_{j=1}^{n} [(Y_{1j}-\mu_1)^2 + (Y_{2j}-\mu_2)^2]/n}$$ if the common mean (μ_1, μ_2) is known.

For $i=1, \ldots, m$ and $j=1, \ldots, n$, let

$$X'_i = [(X_{1i}-\mu_1)^2 + (X_{2i}-\mu_2)^2]/c_1^2,$$

$$Y'_j = [(Y_{1j}-\mu_1)^2 + (Y_{2j}-\mu_2)^2]/c_2^2.$$

Then $X'_1, \ldots, X'_m$ and $Y'_1, \ldots, Y'_n$ are two independent random samples from the same population and

$$F^{**}_{m,n} = \frac{\sum_{i=1}^{m} [(X_{1i}-\mu_1)^2 + (X_{2i}-\mu_2)^2] / m}{\sum_{j=1}^{n} [(Y_{1j}-\mu_1)^2 + (Y_{2j}-\mu_2)^2] / n}$$

$$= \frac{c_1^2}{c_2^2} \cdot \frac{\sum_{i=1}^{m} [(X_{1i}-\mu_1)^2 + (X_{2i}-\mu_2)^2] / (c_1^2 m)}{\sum_{j=1}^{n} [(Y_{1j}-\mu_1)^2 + (Y_{2j}-\mu_2)^2] / (c_2^2 n)}$$

$$= \theta^2 \frac{\overline{X'}}{\overline{Y'}} .$$

By the fact that $2\overline{Y'} \xrightarrow[n \to \infty]{P} 1$, the Central Limit Theorem, and the result in { Cramer (1946), p. 254 }, it is easy to see that $F^{**}_{m,n}$, appropriately normed, is asymptotically normal for every $\theta = c_1/c_2 \in \Omega = (0,\infty)$.

The following lemma will give us the approximate values of $E(F^{**}_{m,n})$ and $Var(F^{**}_{m,n})$ to order m^{-1} and n^{-1}.

Lemma 4.1: Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be two independent random samples from the same population which has finite moments with mean μ and variance σ^2. Then

$$(1)\quad E\left(\frac{\overline{X}}{\overline{Y}}\right) \doteq 1 + \frac{\sigma^2}{n\mu^2} \text{ and}$$

$$(2)\quad \operatorname{Var}\left(\frac{\overline{X}}{\overline{Y}}\right) \doteq \frac{\sigma^2}{\mu^2}\left(\frac{1}{m} + \frac{1}{n}\right),$$

which are approximate values to order m^{-1} and n^{-1}.

Proof: Consider $\frac{\overline{X}}{\overline{Y}} = \frac{\overline{X}}{\mu + (\overline{Y}-\mu)} = \frac{\overline{X}}{\mu}\left[1 - \frac{(\overline{Y}-\mu)}{\mu} + \frac{(\overline{Y}-\mu)^2}{\mu^2} - \frac{(\overline{Y}-\mu)^3}{\mu^3} + \dots\right]$

Thus, $\left(\frac{\overline{X}}{\overline{Y}}\right)^2 = \frac{\overline{X}^2}{\mu^2}\left[1 - \frac{2(\overline{Y}-\mu)}{\mu} + \frac{3(\overline{Y}-\mu)^2}{\mu^2} - \frac{4(\overline{Y}-\mu)^3}{\mu^3} + \dots\right]$

Accordingly, $E\left(\frac{\overline{X}}{\overline{Y}}\right) = \frac{E(\overline{X})}{\mu}\left[1 - \frac{E(\overline{Y}-\mu)}{\mu} + \frac{E[(\overline{Y}-\mu)^2]}{\mu^2} + \frac{E[(\overline{Y}-\mu)^3]}{\mu^3} + \dots\right]$

$= 1\left[1 - 0 + \frac{\sigma^2}{n\mu^2} - \frac{m_3}{n^2\mu^3} + \dots\right]$, where m_3 is the third central moment,

$= 1 + \frac{\sigma^2}{n\mu^2}$, and

$$\operatorname{Var}\left(\frac{\overline{X}}{\overline{Y}}\right) = E\left[\left(\frac{\overline{X}}{\overline{Y}}\right)^2\right] - \left[E\left(\frac{\overline{X}}{\overline{Y}}\right)\right]^2$$

$$= E\left(\frac{\overline{X}^2}{\mu^2}\right)\left[1 - \frac{2E(\overline{Y}-\mu)}{\mu} + \frac{3E[(\overline{Y}-\mu)^2]}{\mu^2} - \frac{4E[(\overline{Y}-\mu)^3]}{\mu^3}\right.$$

$$+\ldots] - \left(1+\frac{\sigma^2}{n\mu^2}-\frac{m_3}{n^2\mu^3}+\ldots\right)^2$$

$$=\frac{(\sigma^2/m+\mu^2)}{\mu^2}\left[1-0+\frac{3\sigma^2}{n\mu^2}-\frac{4m_3}{n^2\mu^3}+\ldots\right]$$

$$-\left(1+\frac{2\sigma^2}{n\mu^2}-\frac{2m_3}{n^2\mu^3}+\frac{\sigma^4}{n^2\mu^4}+\ldots.\right)$$

$$=\frac{\sigma^2}{\mu^2}\left(\frac{1}{m}+\frac{1}{n}\right),$$

when both m and n are so large that terms in m and n of order less than −1 are regarded as negligible. This completes the proof.

Next let us derive the p.d.f. of $U = [(X_1-\mu_1)^2 + (X_2-\mu_2)^2]^{1/2}$ if the joint p.d.f. of X_1 and X_2 is

$$f(x_1,x_2) = 1/(\pi c^2), \qquad 0 \leqslant (x_1-\mu_1)^2 + (x_2-\mu_2)^2 \leqslant c^2 .$$

If we set $x_1 - \mu_1$ = ucos t and $x_2 - \mu$ = usin t, then the joint p.d.f. of U and T is

$$g(u,t) = u/(\pi c^2), \qquad 0 \leqslant u \leqslant c \text{ and } 0 \leqslant t < 2\pi .$$

Thus the p.d.f. of U is

$$s(u) = \int_0^{2\pi} u/(\pi c^2)dt = 2u/c^2, \qquad 0 \leqslant u \leqslant c.$$

We are ready to evaluate the ARE(R,F**) for bivariate uniform distributions.

Theorem 4.1: Let $(X_{11},X_{21}), \ldots, (X_{1m},X_{2m})$ and $(Y_{11},Y_{21}), \ldots, (Y_{1n},Y_{2n})$ be two independent random samples from bivariate uniform populations with p.d.f.'s

$$f(x_1,x_2) = 1/(\pi {c_1}^2), \qquad 0 \leqslant (x_1-\mu_1)^2 + (x_1-\mu_2)^2 \leqslant {c_1}^2$$

and $g(y_1,y_2) = 1/(\pi c_2^2)$, $\quad 0 \leqslant (y_1-\mu_1)^2 + (y_2-\mu_2)^2 \leqslant c_2^2$

respectively, where $\underset{\sim}{\mu} = (\mu_1,\mu_2)$ is the known common mean, c_1 abd c_2 are unknown scale parameters. Set $\theta = c_1/c_2$. For the hypothesis testing problem H′: $\theta=1$ against either one- or two-sided alternatives, we have $ARE(R,F^{**}) = ARE(R^*,F^{**}) = 1$.

Proof: Since $[dE(R_{m,n})/d\theta] \mid \theta = 1$

$$= mn \int_0^\infty u[s(u)]^2 du$$

$$= mn \int_0^{c_1} u(2u/c_1^2)^2 du$$

$= mn$, and

$[Var(R_{m,n})] \mid \theta = 1$

$= mn(m+n+1)/12$, it implies that

$$e(R_{m,n}) = \frac{([dE(R_{m,n})/d\theta] \mid \theta = 1)^2}{Var(R_{m,n}) \mid \theta = 1}$$

$$= \frac{(mn)^2}{mn(m+n+1)/12}$$

$$= 12mn/(m+n+1).$$

By using the fact that $E([(X_1-\mu_1)^2 + (X_2-\mu_2)^2] / c_1^2)$

$= E([(Y_1-\mu_1)^2 + (Y_2-\mu_2)^2] / c_2^2) = 1/2$, $Var([(X_1-\mu_1)^2 + (X_2-\mu_2)^2] / c_1^2)$

$= Var([(Y_1-\mu_1)^2 + (Y_2-\mu_2)^2] / c_2^2) = 1/12$, and Lemma 4.1, we obtain

$$E(F^{**}_{m,n}) = E\left(\theta^2 \frac{\sum_{i=1}^{m} [(X_{1i}-\mu_1)^2 + (X_{2i}-\mu_2)^2] / (c_1^2 m)}{\sum_{j=1}^{n} [(Y_{1j}-\mu_1)^2 + (Y_{2j}-\mu_2)^2] / (c_2^2 n)}\right)$$

$$\doteq \theta^2\left(1 + \frac{1/12}{n(1/2)^2}\right)$$

$$= \theta^2\left(1 + \frac{1}{3n}\right), \text{ and}$$

$$[\mathrm{Var}(F^{**}_{m,n})]_{\theta=1} = \mathrm{Var}\left(\frac{\sum_{i=1}^{m} [(X_{1i}-\mu_1)^2 + (X_{2i}-\mu_2)^2] / (c_1^2 m)}{\sum_{j=1}^{n} [(Y_{1j}-\mu_1)^2 + (Y_{2j}-\mu_2)^2] / (c_2^2 n)}\right)$$

$$\doteq \frac{1/12}{(1/2)^2}\left(\frac{1}{m} + \frac{1}{n}\right)$$

$$= \frac{1}{3}\left(\frac{1}{m} + \frac{1}{n}\right).$$

Hence
$$e(F^{**}_{m,n}) = \frac{([dE(F^{**}_{m,n})/d\theta]\,|_{\theta=1})^2}{[\mathrm{Var}(F^{**}_{m,n})]\,|_{\theta=1}} \doteq \frac{4\left(1+\frac{1}{3n}\right)^2}{\frac{1}{3}\left(\frac{1}{m}+\frac{1}{n}\right)}$$

$$= 12mn\left(1 + \frac{1}{3n}\right)^2 / (m+n).$$

Therefore
$$\mathrm{ARE}(R, F^{**}) = \lim_{m,n \to \infty} \frac{e(R_{m,n})}{e(F^{**}_{m,n})}$$

$$= \lim_{m,n \to \infty} \frac{12mn/(m+n+1)}{12mn\left(1 + \frac{1}{3n}\right)^2 / (m+n)}$$

$$= 1.$$

Applying the technique used in Liu (1981), we can prove that $R_{m,n}$ and $R^*_{m,n}$ have the same limiting distribution under both H′ and A′ if the underlying populations are bivariate uniform. Hence the tests R and R^* have the same ARE with respect to the test F^{**}, which concludes the proof.

References

Anderson, T. W., An Introduction to Multivariate Statistical Analysis, John Wiley and Sons, Inc., New York, 1958.

Chernoff, H. and Savage, I. R., Asymptotic Normality and Efficiency of Certain Nonparametric Test Statistics, Ann. Math. Stat., 29(1958), 972-994.

Cramér, H., Mathematical Methods of Statistics, Princeton University Press, Princeton, 1946.

Fraser, D. A. S., Nonparametric Methods in Statistics, John Wiley and Sons, Inc., New York, 1957.

Liu, M. R., On the Asymptotic Normality of Certain Nonparametric Test Statistics, J. National Chengchi University, 43(1981), 65-78.

Liu, M. R., On the Large Sample Properties of Centain Nonparametric Tests for Dispersion, J. National Chengchi University, 45(1982), 25-42.

Silverman, H., Complex Variables, Houghton Mifflin Company, Boston, 1975.