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Controlling-dependent process steps using variable sample size control charts

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SUMMARY

The article considers the variable process control scheme for two dependent process steps. We construct variable sample size (VSS) $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts to effectively monitor the quality variable produced by the first process step and the quality variable produced by the second process step 2, respectively. The performance of the proposed VSS control charts is measured by the adjusted average time to signal (AATS) using a Markov chain approach. An example of producing cotton yarn system shows the application and performance of the proposed VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts in detecting shifts in process mean. Furthermore, the performance of the VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts and the fixed sample size (FSS) $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts are compared by numerical analysis results. These demonstrate that the former is much faster in detecting small shifts in mean. Copyright © 2006 John Wiley & Sons, Ltd.

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KEY WORDS: dependent process steps; control scheme; variable sample size

1. INTRODUCTION

Control charts are important tools in statistical quality control. They are used to effectively monitor and determine whether a process is in control or out of control. To monitor the process mean, Shewhart [1] developed the $\bar{X}$ control chart which is easy to implement and has been widely used for industrial processes. However, since Shewhart $\bar{X}$ control charts always monitor a process by taking samples of equal size at a fixed sampling interval, they are usually slow in signalling small to moderate shifts in the process mean. Consequently, several alternatives have been developed to improve the performance of $\bar{X}$ control charts in recent years. One of the useful approaches to improve the detecting ability is to use a variable sample size (VSS) and/or a variable sampling interval control chart instead of the traditional fixed sample size and/or fixed sampling interval. Whenever there is some indication that a process parameter may have

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changed, the next sample should be larger and/or the next sampling interval should be shorter. On the other hand, if there is no indication of a parameter change, then the next sample should be smaller and/or the next sampling interval should be longer.

The properties of the $\bar{X}$ chart with VSSs were studied by Daudin [2], Prabhu *et al.* [3], Costa [4] and Zimmer *et al.* [5]. Tagaras [6] reviewed the literature on adaptive control charts. However, these articles assume that there is only a single process step, whereas many products are currently produced in several dependent process steps. Consequently, it is not appropriate to monitor these process steps by utilizing a control chart for each individual process step. Zhang [7] proposes the simple cause-selecting control chart to control the specific quality in the current process by adjusting the effect of in-coming quality variable (X) on out-going quality variable (Y), since the in-coming quality variable on the first process step and the out-going quality variable on the second process step are dependent. The cause-selecting values (e) are Y minus the effect of X, and the cause-selecting control chart is constructed accordingly. Wade and Woodall [8] reviewed and analysed the cause-selecting control chart and examined the relationship between the cause-selecting control chart and the Hotelling T^2 control chart. In their opinion the cause-selecting control chart outperforms Hotelling T^2 control chart, since it is easy to distinguish whether the second step of the process is out of control. Therefore, it seems reasonable to develop variable control schemes to control dependent process steps. However, the properties of the VSS charts used to control two dependent process steps have not yet been addressed. Therefore, to study the performance of the VSS control charts on two dependent process steps is reasonable. In this paper, the VSS control charts are proposed to control two dependent process steps with two assignable causes. In next section, the performance of the proposed control charts is measured by the adjusted average time to signal (AATS), which is derived using a Markov chain approach. Finally, we illustrate the application of the proposed control charts using a real example, and compare the performance between the VSS control charts and fixed sample size (FSS) control charts.

2. DESCRIPTION OF THE JOINT VSS $Z_{\bar{X}}$ AND $Z_{\bar{e}}$ CHARTS

Consider a process with two dependent steps controlled by VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts. Let X be the measurable in-coming quality variable on the first process step. Assume further that this process starts in a state of statistical control, that is, X follows a normal distribution with the mean at its target value, μ_X, and the standard deviation at its target value σ_X; let Y be the measurable out-going quality characteristic of interest for the second step, and follow a normal distribution condition on X. Since the two process steps are dependent, and the second process step is affected by the first process step, then following Wade and Woodall [8], the relationship between X and Y is generally expressed as

$$Y_i \mid X_i = f(X_i) + \varepsilon_i, \quad i = 1, 2, 3, \ldots, m \tag{1}$$

where, it is assumed that $\varepsilon_i \sim \mathrm{NID}(0, \sigma^2)$. Let Y instead of $Y \mid X$. If the function $f(X_i)$ is known, the values of the standardized error term $\varepsilon_i^* = (Y_i - f(X_i))/\sigma$ are called the cause-selecting values since they are the values of Y_i adjusted for the effects of X_i. In practice, the true function $f(X_i)$ is usually unknown and thus must be estimated using the m observations obtained from the initial m samples of size one, and thus, the estimate for $f(X_i)$ will be $\hat{Y}_i$. The residuals, $e_i = Y_i - \hat{Y}_i$, are generated by the model used. Hence, $e_i \sim \mathrm{NID}(0, \sigma_\varepsilon^2)$. Consequently, the standard residuals $e_i^* = (Y_i - \hat{Y}_i)/\sigma_\varepsilon$ are called the cause-selecting values.

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However, in our study the chosen sample size is not one. Instead rational samples of variable size (n_q) are taken from the process; the standardized sample means, $Z_{\bar{X}}$ and $Z_{\bar{e}}$, are

$$Z_{\bar{X}_i} = \frac{\bar{X}_i - \mu_X}{\sigma_X / \sqrt{n_q}} \sim N(0,1) \quad \text{and} \quad Z_{\bar{e}_i} = \frac{\bar{e}_i}{\sigma_\varepsilon / \sqrt{n_q}} \sim N(0,1) \tag{2}$$

where

$$\bar{X}_i = \frac{\sum_{j=1}^{n_q} X_{ij}}{n_q}, \quad \bar{e}_i = \frac{\sum_{j=1}^{n_q} e_{ij}}{n_q} \quad \text{and} \quad e_{ij} = Y_{ij} - \hat{Y}_{ij}, \;\; i = 1,2,3,\ldots,m, \;\; j = 1,2,3,\ldots,n_q$$

The sample means, $Z_{\bar{e}_i}$, where $Z_{\bar{e}_i} = \overline{e_i}/(\sigma_\varepsilon/\sqrt{n_q})$ are called cause-selecting values.

Also assume that the first step is only subjected to assignable cause 1 such that the mean of X shifts from μ_X to $\mu_X + \delta_1\sigma_X$ ($\delta_1 \neq 0$), and the second step is only subjected to assignable cause 2 such that the mean of the specific quality shifts from 0 to δ_2 ($\delta_2 \neq 0$). Let T_i be the time until the occurrence of assignable cause i, where $i = 1,2$, and follow an exponential distribution of the form

$$f(t_i) = \lambda_i \exp(-\lambda_i t_i), \quad t_i > 0, \;\; i = 1,2 \tag{3}$$

where $1/\lambda_i$ is the mean time that the process step i remains in a state of statistical control.

An in-control state analysis for the VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts is performed since the shifts in the process mean on step 1 and/or step 2 do not occur when the process is just starting, but occur at some time in the future. The standardized samples $z_{\bar{X}_i}$ and $z_{\bar{e}_i}$ are plotted on the VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts with warning limits of the form $\pm w_{\bar{X}}$ and $\pm w_{\bar{e}}$, and action limits of the form $\pm k_{\bar{X}}$ and $\pm k_{\bar{e}}$, respectively, where $0 \leqslant w_{\bar{X}} < k_{\bar{X}}$ and $0 \leqslant w_{\bar{e}} < k_{\bar{e}}$ (see Figure 1).

The search for the assignable cause 1 and/or 2 is undertaken when the sample $z_{\bar{X}_i}$ falls outside the interval $(-k_{\bar{X}}, k_{\bar{X}})$ and/or when the sample $z_{\bar{e}_i}$ falls outside the interval $(-k_{\bar{e}}, k_{\bar{e}})$, that is, when the $Z_{\bar{X}}$ and/or $Z_{\bar{e}}$ charts produce a signal. For a discontinuous process, the process is stopped to search for and eliminate the assignable cause after a signal is obtained from a control chart, and then the process is brought back to an in-control state.

The position of the current sample in each chart constructs the size of the next sample.

We divide the proposed VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts into the following two regions:

$$I_{\bar{X}_1} = [-w_{\bar{X}}, w_{\bar{X}}], \quad I_{\bar{e}_1} = [-w_{\bar{e}}, w_{\bar{e}}]$$

$$I_{\bar{X}_2} = (-k_{\bar{X}}, -w_{\bar{X}}) \cup (w_{\bar{X}}, k_{\bar{X}}), \quad I_{\bar{e}_2} = (-k_{\bar{e}}, -w_{\bar{e}}) \cup (w_{\bar{e}}, k_{\bar{e}})$$

Three VSSs are adopted, $1 < n_1 < n_2 < n_3 < \infty$. If the sample, $z_{\bar{X}_i}$, falls within the interval $I_{\bar{X}_1}$ and the sample, $z_{\bar{e}_i}$, falls within the interval $I_{\bar{e}_1}$, then the next sample size should be small (n_1). If the samples, $z_{\bar{X}_i}$ and $z_{\bar{e}_i}$, fall within the interval ($I_{\bar{X}_1}$ and $I_{\bar{e}_2}$) or ($I_{\bar{X}_2}$ and $I_{\bar{e}_1}$), then the next sample

$$\begin{array}{ll}
\mathrm{UCL}_{z_{\bar{X}}} = k_{\bar{X}} & \mathrm{UCL}_{z_{\bar{e}}} = k_{\bar{e}} \\
UWL_{z_{\bar{X}}} = w_{\bar{X}} & \mathrm{UWL}_{z_{\bar{e}}} = w_{\bar{e}} \\
\mathrm{CL}_{z_{\bar{X}}} = 0 & \mathrm{CL}_{z_{\bar{e}}} = 0 \\
\mathrm{LWL}_{z_{\bar{X}}} = -w_{\bar{X}} & \mathrm{LWL}_{z_{\bar{e}}} = -w_{\bar{e}} \\
\mathrm{LCL}_{z_{\bar{X}}} = -k_{\bar{X}} & \mathrm{LCL}_{z_{\bar{e}}} = -k_{\bar{e}}
\end{array}$$

Figure 1. The control limits of VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts.

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size should be in the middle (n_2). If the samples, $z_{\bar{X}_i}$ and $z_{\bar{e}_i}$, fall within the interval $I_{\bar{X}_2}$ and $I_{\bar{e}_2}$, then the next sample size should be large (n_3).

The relationship between the next sample size and fixed sampling interval ($t, n_q,\ q = 1, 2, 3$) and the position of the current samples ($z_{\bar{X},i-1}, z_{\bar{e},i-1}$) is expressed as follows:

$$(t, n(i)) = \begin{cases} (t, n_1) & \text{if } z_{\bar{X},i-1} \in I_{\bar{X}_1} \cap z_{\bar{e},i-1} \in I_{\bar{e}_1} \\ (t, n_2) & \text{if } z_{\bar{X},i-1} \in I_{\bar{X}_1} \cap z_{\bar{e},i-1} \in I_{\bar{e}_2} \\ (t, n_2) & \text{if } z_{\bar{X},i-1} \in I_{\bar{X}_2} \cap z_{\bar{e},i-1} \in I_{\bar{e}_1} \\ (t, n_3) & \text{if } z_{\bar{X},i-1} \in I_{\bar{X}_2} \cap z_{\bar{e},i-1} \in I_{\bar{e}_2} \end{cases}, \quad i = 2, 3, \ldots \tag{4}$$

Following Costa [9], the first sample size taken from the process when it is just starting is chosen randomly. When the process is in control, all samples, including the first one, should have a probability of p_0 of being small, a probability of p_1 of being median, and a probability of $1 - p_0 - p_1$ of being large, where p_0 and p_1 are given by the following:

$$\begin{aligned} p_0 &= P_r(|Z_{\bar{X}}| < w_{\bar{X}} \,|\, |Z_{\bar{X}}| < k_{\bar{X}}) \cdot P_r(|Z_{\bar{e}}| < w_{\bar{e}} \,|\, |Z_{\bar{e}}| < k_{\bar{e}}) \\ p_1 &= P_r(|Z_{\bar{X}}| < w_{\bar{X}} \,|\, |Z_{\bar{X}}| < k_{\bar{X}}) \cdot (1 - P_r(|Z_{\bar{e}}| < w_{\bar{e}} \,|\, |Z_{\bar{e}}| < k_{\bar{e}})) \\ &\quad + (1 - P_r(|Z_{\bar{X}}| < w_{\bar{X}} \,|\, |Z_{\bar{X}}| < k_{\bar{X}})) \cdot P_r(|Z_{\bar{e}}| < w_{\bar{e}} \,|\, |Z_{\bar{e}}| < k_{\bar{e}}) \end{aligned} \tag{5}$$

To facilitate the computation of the performance measures, $w_{\bar{X}}$, $k_{\bar{X}}$, $w_{\bar{e}}$ and $k_{\bar{e}}$ will be specified with the constraint that the probability of a sample falling in the warning limits is same for both the $Z_{\bar{X}}$ and $Z_{\bar{e}}$ charts when the process is in control. Thus,

$$P_r(|Z_{\bar{X}}| < w_{\bar{X}} \,|\, |Z_{\bar{X}}| < k_{\bar{X}}) = P_r(|Z_{\bar{e}}| < w_{\bar{e}} \,|\, |Z_{\bar{e}}| < k_{\bar{e}}) \tag{6}$$

implying, $w_{\bar{X}} = w_{\bar{e}} = w$ and $k_{\bar{X}} = k_{\bar{e}} = k$.

If both $w_{\bar{X}} = w_{\bar{e}} = 0$ and $n_1 = n_2 = n_3 = n_0$, then the VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ charts reduce to the $Z_{\bar{X}}$ and $Z_{\bar{e}}$ charts with FSS n_0.

3. COMPARISON OF CONTROL CHARTS

Sampling schemes should be compared under equal conditions; that is, VSS and FSS schemes should demand the same average sample size under the in-control period. That is,

$$E[n_q | \delta_1 = 0, \delta_2 = 0, |Z_{\bar{X}}| < k, |Z_{\bar{e}}| < k] = n_0 \tag{7}$$

Based on Equation (7), the following equation can be formulated as

$$\begin{aligned} & n_1 \cdot P(Z_{\bar{X},i-1} \in I_{\bar{X}_1} | \delta_1 = 0, \delta_2 = 0) \cdot P(Z_{\bar{e},i-1} \in I_{\bar{e}_1} | \delta_1 = 0, \delta_2 = 0) \\ & + n_2 \cdot P(Z_{\bar{X},i-1} \in I_{\bar{X}_1} | \delta_1 = 0, \delta_2 = 0) \cdot P(Z_{\bar{e},i-1} \in I_{\bar{e}_2} | \delta_1 = 0, \delta_2 = 0) \\ & + n_2 \cdot P(Z_{\bar{X},i-1} \in I_{\bar{X}_2} | \delta_1 = 0, \delta_2 = 0) \cdot P(z_{\bar{e},i-1} \in I_{\bar{e}_1} | \delta_1 = 0, \delta_2 = 0) \\ & + n_3 \cdot P(Z_{\bar{X},i-1} \in I_{\bar{X}_2} | \delta_1 = 0, \delta_2 = 0) \cdot P(Z_{\bar{e},i-1} \in I_{\bar{e}_2} | \delta_1 = 0, \delta_2 = 0) \\ & + 0 \cdot P(\text{false alarms}) \\ & = n_0 (2\Phi(k) - 1)^2 \end{aligned} \tag{8}$$

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Simplifying,

$$\begin{aligned}&4\Phi(w)^2[n_1 - 2n_2 + n_3] + 4\Phi(w)[-n_1 + 2n_2\Phi(k) + n_2 - 2n_3\Phi(k)]\\&= n_0(2\Phi(k) - 1)^2 - n_1 + 4n_2\Phi(k) - 4n_3(\Phi(k))^2\end{aligned} \tag{9}$$

where $\Phi(\cdot)$ denotes the standard normal cumulative function.

The warning limit is derived as follows:

$$w = \Phi^{-1}\left(\frac{-4B_1 \pm \sqrt{16B_1^2 - 16A_1C_1}}{8A_1}\right) \tag{10}$$

where

$$A_1 = n_1 - 2n_2 + n_3$$

$$B_1 = -n_1 + 2n_2\Phi(k) + n_2 - 2n_3\Phi(k)$$

$$C_1 = -[n_0(2\Phi(k) - 1)^2 - n_1 + 4n_2\Phi(k) - 4n_3(\Phi(k))^2]$$

However, to obtain w and let $w < k$, the constraint $1 < n_1 < n_2 < n_0 < n_3 < \infty$ is required.

Thus, the warning limit can be obtained by using Equation (10) and choosing a combination of the three VSSs, (n_1, n_2, n_3), and the FSS, n_0.

In this paper, the VSS scheme is compared with the FSS scheme and one sampling scheme was considered to be better than another when it allows the joint $Z_{\bar{X}}$ and $Z_{\bar{e}}$ charts to detect changes in the process means on steps 1 and 2 faster.

4. PERFORMANCE MEASUREMENT

The speed with which a control chart detects process shifts measures the chart's statistical efficiency. For a VSS, the detection speed is measured by the average time from either mean shifting until either $Z_{\bar{X}}$ or $Z_{\bar{e}}$ chart or both signal, which is known as the adjusted AATS. That is, the AATS is the mean time when the process remains out of control.

Since $T_i \sim \exp(-\lambda_i t_i)t_i > 0$, $i = 1, 2$, the occurrence time, T, until the first assignable cause occurs is

$$T \sim \exp(\lambda_1 + \lambda_2) \quad \text{where } T = \min(T_1, T_2)$$

Hence,

$$\text{AATS} = \text{ATC} - \frac{1}{\lambda_1 + \lambda_2} \tag{11}$$

The average time of the cycle (ATC) is the average time from the start of process until the first signal obtained from one of the proposed charts. The Markov chain approach is allowed to compute the ATC due to the memory-less property of the exponential distribution. Thus, at each sampling, one of 17 states is assigned based on whether the process step is in or out of control and the position of samples (see Table I for the 17 states of the process).

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Table I. The 17 process states.

State	Does assignable cause 1 occur?	The location of sample statistic Z_X	Is an alarm in the first process?	Does assignable cause 2 occur?	The location of sample statistic Z_e	Is an alarm in the second process?	Transient state or absorbing state?
1	No	$I_{\bar{X}_1}$	No alarm	No	$I_{\bar{e}_1}$	No alarm	Transient state
2	No	$I_{\bar{X}_1}$		Yes	$I_{\bar{e}_1}$		
3	Yes	$I_{\bar{X}_1}$		No	$I_{\bar{e}_1}$		
4	Yes	$I_{\bar{X}_1}$		Yes	$I_{\bar{e}_1}$		
5	No	$I_{\bar{X}_2}$		No	$I_{\bar{e}_1}$		
6	No	$I_{\bar{X}_2}$		Yes	$I_{\bar{e}_1}$		
7	Yes	$I_{\bar{X}_2}$		No	$I_{\bar{e}_1}$		
8	Yes	$I_{\bar{X}_2}$		Yes	$I_{\bar{e}_1}$		
9	No	$I_{\bar{X}_1}$		No	$I_{\bar{e}_2}$		
10	No	$I_{\bar{X}_1}$		Yes	$I_{\bar{e}_2}$		
11	Yes	$I_{\bar{X}_1}$		No	$I_{\bar{e}_2}$		
12	Yes	$I_{\bar{X}_1}$		Yes	$I_{\bar{e}_2}$		
13	No	$I_{\bar{X}_2}$		No	$I_{\bar{e}_2}$		
14	No	$I_{\bar{X}_2}$		Yes	$I_{\bar{e}_2}$		
15	Yes	$I_{\bar{X}_2}$		No	$I_{\bar{e}_2}$		
16	Yes	$I_{\bar{X}_2}$		Yes	$I_{\bar{e}_2}$		
17	Except	The above		Situations			Absorbing state

The status of the process when the $(i+1)$th sample is taken, and the position of the ith sample on the $Z_{\bar{X}}$ and $Z_{\bar{e}}$ charts defines the transient states of the Markov chain. The VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ charts produce a signal when at least one of the samples falls outside the control limits. If the current state is any one of the States 1–16, then there is no signal. The absorbing state (State 17) is reached when a signal occurs.

Let $\mathbf{P}$ be the transition probability matrix, where $\mathbf{P}$ is a square matrix of order 17. Let $P_{i,j}(t, n_q)$ be the transition probability from prior state i to the current state j with sample size n_q, where n_q is determined by the prior state i, $i = 1, 2, \ldots, 17$ and $q = 1, 2, 3$. The transition probability, for example, from state 1 to state 4 with sample size n_1 and fixed sampling interval t is calculated as

$$\begin{aligned} p_{1,4}(t, n_1) &= P[|Z_{\bar{X}}| < w|\delta_1] \cdot P[|Z_{\bar{e}}| < w|\delta_2] \cdot (1 - \mathrm{e}^{-\lambda_1 t}) \cdot (1 - \mathrm{e}^{-\lambda_2 t}) \\ &= (\Phi(w - \delta_1\sqrt{n_1}) - \Phi(-w - \delta_1\sqrt{n_1})) \\ &\quad \cdot (\Phi(w - \delta_2\sqrt{n_1}) - \Phi(-w - \delta_2\sqrt{n_1})) \cdot (1 - \mathrm{e}^{-\lambda_1 t}) \cdot (1 - \mathrm{e}^{-\lambda_2 t}) \end{aligned}$$

The calculation of all transition probabilities is shown in Appendix.

From the elementary properties of Markov chains (see, e.g. [10]), the ATC is derived as follows:

$$\mathrm{ATC} = \mathbf{b}'(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{t} \tag{12}$$

where $\mathbf{b}' = (p_0, 0, 0, 0, p_1/2, 0, 0, 0, p_1/2, 0, 0, 0, p_2, 0, \ldots, 0)$ is the vector of starting probabilities for states $1, 2, 3, \ldots, 16$, where the first sample has probability p_0 (see Equation (5) for calculation) of being small (or state 1 with probability p_0), the probability $p_1/2$ of being median (or state 5 and state 9 with probability $p_1/2$) and the probability p_2 ($p_2 = 1 - p_0 - p_1$) of being large (or state 13 with probability p_2), $\mathbf{I}$ is the identity matrix of order 16, $\mathbf{Q}$ is the transition

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probability matrix where elements represent the transition probability, $P_{i,j}(t, n_q)$, from transient state $i, i = 1, \ldots, 16$, to transient state $j, j = 1, \ldots, 16$, and $\mathbf{t}' = (t, t, t, t, t, t, t, t, t, t, t, t, t, t, t, t)$ is the vector of the fixed sampling intervals for state 1–state 16.

5. AN EXAMPLE

An example of process control for producing cotton yarn is presented, and the data of the process are measurements of the cotton yarn. The variables X = fibre length and Y = skein strength were measured at the end of the second process step. The skein strength is influenced by the fibre length. Presently, the FSS Shewhart-type control charts are used to monitor the process steps per hour. To construct reasonable subgroups and reduce the large costs of sampling and inspection associated with measurements (X_{ij}, Y_{ij}), 30 samples of size five are collected from historical data under the stable process steps. The relationship of quality variables X and Y is expressed by a linear regression model. The fitted model is

$$\hat{Y}_{ij} = 66.8 + 0.639X_{ij}, \quad i = 1, 2, \ldots, 30, \quad j = 1, 2, \ldots, 5 \tag{13}$$

Thus, the residuals (e) are obtained by $Y_{ij} - \hat{Y}_{ij}$. The estimated means and standard deviations of variables X_{ij} and e_{ij} are $(\hat{\mu}_X = 210.1, \hat{\sigma}_X = 1.23)$ and $(\hat{\mu}_\varepsilon = 0, \hat{\sigma}_\varepsilon = 1.11)$, respectively. That is, when both process steps are in control, $\overline{X_i} \sim N(210.1, (1.23/\sqrt{5})^2)$ and $\overline{e_i} \sim N(0, (1.11/\sqrt{5})^2)$. Two machines could fail in the process. From historical data, the estimated failure frequency for machine 1 is 0.03 times per hour and 0.04 times per hour for machine 2. The failures of machines 1 and 2 are independent and only influence the means of X and Y, but the standard deviations are unaffected. The failure of machine 1 would shift the mean of X to $\hat{\mu}_X + \delta_1\hat{\sigma}_X$, where $\delta_1 = 0.25$; the failure of machine 2 would shift the mean of e to $\delta_2\hat{\sigma}_e$, where $\delta_2 = 0.5$. Hence, for out-of-control process step 1, $\bar{X} \sim N(210.1 + 1.23 \cdot 0.25, (1.23/\sqrt{5})^2)$; for out-of-control process step 2, $\bar{e} \sim N(1.11 \cdot 0.5, (1.11/\sqrt{5})^2)$.

The FSS standard $\bar{X}$ and $\bar{e}$ charts have control limits placed at ± 3, respectively. Thus, approximately 5.4 false alarms are expected per 1000 samples have in-control average run length (ARL) of 185 h and AATS = 30.42 h. The slowness with which the FSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts detect shifts in the process ($\delta_1 = 0.25$ and $\delta_2 = 0.5$) has led the quality manager to propose building the $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts with VSSs. The construction and application of the proposed VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts are illustrated. The following are the guidelines for using the proposed charts:

Step 1: Let the factor of control limits, $k = 3$, to maintain the average false alarm rate at around 5.4 per 1000 samples. The reciprocal of 5.4 false alarms is also the ARL, but for the in-control case of $\delta_1 = \delta_2 = 0$.

Step 2: Since $1 < n_1 < n_2 < n_0 < n_3 < \infty$ is required, and for saving the cost of sampling we take $n_3 = 20$ and the combination ($n_1 = 2$, $n_2 = 3$, $n_3 = 20$, $t = 1.0$ h) is adopted.

Step 3: Letting $n_1 = 2$, $n_2 = 3$, $n_3 = 20$, $n_0 = 5$ and $k = 3$ in Equations (8) and (10) leads to $w = 0.88$.

Consequently, the structure of the proposed VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts are given in Figure 2.

With the design parameters determined, the VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts can be used for controlling the two dependent process steps for producing cotton yarn. According to the VSS scheme, if both samples, $(z_{\bar{X}}, z_{\bar{e}})$, fall within the warning limits, then a sample of size $n_1 = 2$ is

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$\text{UCL}_{z_{\bar{X}}} = 3$	$\text{UCL}_{z_{\bar{e}}} = 3$
$UWL_{z_{\bar{X}}} = 0.88$	$\text{UWL}_{z_{\bar{e}}} = 0.88$
$\text{CL}_{z_{\bar{X}}} = 0$	$\text{CL}_{z_{\bar{e}}} = 0$
$\text{LWL}_{z_{\bar{X}}} = -0.88$	$\text{LWL}_{z_{\bar{e}}} = -0.88$
$\text{LCL}_{z_{\bar{X}}} = -3$	$\text{LCL}_{z_{\bar{e}}} = -3$

Figure 2. The VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control limits.

taken. If one of the samples falls within the warning limits but the other falls between warning and control limits, then a sample of size $n_2 = 3$ is taken. If both samples fall between warning and control limits, then a sample of size $n_3 = 20$ is taken.

An example using the VSS is introduced now. When the process starts, a random procedure decides the first sample of size three, and the three paired observations of (X_{1j}, Y_{1j}) are $(209, 201)$, $(212, 203)$ and $(208, 199)$. The first samples are $(\bar{x}_1 = 209.69, \bar{e}_1 = 0.22)$ and the values of $Z_{\bar{X}}$ and $Z_{\bar{e}}$ are $(-0.61, 0.34)$. Both samples fall within the warning limits. The second sample will be observed adopting a sample of size two after 1 h. The two observed data of (X_{2j}, Y_{2j}) are $(210, 200)$ and $(208, 199)$. Since $z_{\bar{X}} = -1.26$ and $z_{\bar{e}} = -1.08$, the samples fall between the warning and control limits. The next sample now adopts a sample of size 20. The process is stopped when one of the samples falls outside its control limits.

The AATS is used to measure the performance of the proposed VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts. The proposed Markov chain approach is used to obtain the ATC and calculate the AATS. There are 17 possible process states, as presented in Table I. The ATC is calculated to be 32.29 h using Equation (12). Hence, the AATS is 17.99 h according to Equation (11).

The VSS scheme improves the sensitivity of the FSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ charts. From the example, in order to detect a shift in the process mean, the AATS for the VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ charts has been reduced from 30.42 h to only 17.99 h.

6. PERFORMANCE COMPARISON BETWEEN ASS AND FSS SCHEMES

Tables II and III provide the AATS of VSS and FSS schemes, which are obtained under various combination of parameters, $\lambda_1 = 0.03, 0.05, \lambda_2 = 0.04, 0.05$, $\delta_1 = 0.25 \sim 1.5$, $\delta_2 = 0.25 \sim 1.5$, $t = 1.0$, $n_1 = 2, 3$, $n_2 = 3, 4$, $n_3 = 20, 30$ and $n_0 = 5$.

Comparing the AATS between the FSS and VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts it can be seen that the performance of the VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts is better for detecting small shifts ($0.25 < \delta_1 \leqslant 1$ or $0.25 < \delta_2 \leqslant 1$) in process means but the performance of the FSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts is better for detecting moderate shifts ($\delta_1 > 1$ and $\delta_2 > 1$) in process means.

7. CONCLUSIONS

The proposed VSS scheme controlling two dependent process steps substantially improves the performance of the FSS scheme by increasing the speed with which small shifts in the mean of

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Table II. The AATS for the ASS and FSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts when $\lambda_1 = 0.03$ and $\lambda_2 = 0.04$.

$\lambda_1 = 0.03, \lambda_2 = 0.04$				δ_1	0.25						0.5						0.75					
n_1	n_2	n_3	w	δ_2	0.25	0.5	0.75	1	1.25	1.5	0.25	0.5	0.75	1	1.25	1.5	0.25	0.5	0.75	1	1.25	1.5
2	3	20	0.8805		52.91	17.99	10.62	9.02	8.41	8.02	21.82	10.43	5.91	4.73	4.27	3.96	15.27	7.14	3.34	2.31	1.90	1.63
2	3	25	0.9605		49.64	16.78	10.57	9.15	8.50	8.03	20.60	9.62	5.70	4.66	4.16	3.80	15.13	6.79	3.39	2.46	2.02	1.70
2	3	30	1.0220		46.81	16.05	10.65	9.29	8.56	8.03	19.84	9.18	5.68	4.67	4.11	3.71	15.13	6.64	3.55	2.67	2.17	1.81
2	4	20	0.9436		53.90	18.35	10.64	8.94	8.28	7.88	22.18	10.68	5.98	4.72	4.22	3.90	15.32	7.26	3.33	2.24	1.79	1.52
2	4	25	1.0165		50.74	17.09	10.55	9.02	8.32	7.87	20.91	9.82	5.73	4.60	4.07	3.72	15.15	6.86	3.33	2.34	1.86	1.55
2	4	30	1.0727		47.98	16.32	10.59	9.12	8.34	7.85	20.12	9.34	5.66	4.57	3.98	3.60	15.11	6.67	3.45	2.49	1.97	1.63
3	4	20	1.0177		55.81	19.32	10.96	9.06	8.32	7.87	23.12	11.39	6.37	4.98	4.41	4.05	15.66	7.73	3.59	2.38	1.89	1.58
3	4	25	1.0934		53.09	18.09	10.87	9.15	8.36	7.86	21.90	10.58	6.14	4.87	4.26	3.87	15.52	7.36	3.58	2.48	1.94	1.60
3	4	30	1.1514		50.66	17.33	10.92	9.24	8.37	7.84	21.13	10.11	6.07	4.83	4.17	3.75	15.50	7.19	3.70	2.61	2.03	1.66
FSS control charts					67.54	30.42	15.23	10.29	8.54	7.86	33.90	20.65	11.23	7.33	5.86	5.26	19.94	13.16	7.16	4.23	3.04	2.54
2	3	20	0.8805		13.83	6.23	2.59	1.59	1.21	0.95	13.28	5.86	2.28	1.31	0.93	0.67	12.92	5.61	2.08	1.11	0.74	0.48
2	3	25	0.9605		13.86	5.98	2.71	1.82	1.40	1.10	13.27	5.58	2.38	1.51	1.10	0.80	12.85	5.29	2.14	1.29	0.88	0.59
2	3	30	1.0220		13.91	5.86	2.90	2.06	1.58	1.24	13.24	5.42	2.53	1.71	1.25	0.91	12.76	5.10	2.26	1.45	1.00	0.67
2	4	20	0.9436		13.79	6.29	2.53	1.48	1.06	0.79	13.19	5.88	2.20	1.17	0.75	0.49	12.82	5.63	1.99	0.97	0.55	0.30
2	4	25	1.0165		13.78	5.99	2.60	1.65	1.20	0.90	13.15	5.57	2.25	1.32	0.87	0.58	12.73	5.29	2.01	1.09	0.65	0.37
2	4	30	1.0727		13.79	5.83	2.75	1.83	1.33	1.00	13.09	5.37	2.35	1.46	0.97	0.65	12.64	5.06	2.10	1.22	0.73	0.42
3	4	20	1.0177		13.95	6.65	2.71	1.56	1.08	0.78	13.28	6.19	2.34	1.20	0.74	0.44	12.86	5.91	2.10	0.98	0.52	0.23
3	4	25	1.0934		13.97	6.38	2.77	1.71	1.20	0.87	13.25	5.90	2.37	1.34	0.84	0.52	12.79	5.59	2.11	1.09	0.60	0.28
3	4	30	1.1514		13.99	6.23	2.90	1.87	1.31	0.96	13.21	5.71	2.47	1.46	0.92	0.57	12.71	5.37	2.19	1.20	0.66	0.32
FSS control charts					15.34	9.90	4.98	2.48	1.43	0.99	13.72	8.64	4.04	1.70	0.72	0.30	13.08	8.13	3.65	1.37	0.40	0

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Table III. The AATS for the ASS and FSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts when $\lambda_1 = 0.03$ and $\lambda_2 = 0.05$.

$\lambda_1 = 0.03, \lambda_2 = 0.05$				δ_1	0.25						0.5						0.75					
n_1	n_2	n_3	w	δ_2	0.25	0.5	0.75	1	1.25	1.5	0.25	0.5	0.75	1	1.25	1.5	0.25	0.5	0.75	1	1.25	1.5
2	3	20	0.8805		53.27	16.76	9.03	7.35	6.72	6.31	23.08	10.61	5.66	4.36	3.86	3.52	16.75	7.78	3.59	2.45	2.00	1.70
2	3	25	0.9605		49.95	15.54	9.01	7.52	6.84	6.35	21.85	9.81	5.51	4.37	3.82	3.43	16.59	7.38	3.63	2.61	2.13	1.78
2	3	30	1.0220		47.08	14.82	9.13	7.70	6.93	6.38	21.09	9.38	5.53	4.43	3.82	3.38	16.57	7.21	3.80	2.82	2.28	1.88
2	4	20	0.9436		54.26	17.12	9.04	7.26	6.57	6.15	23.44	10.84	5.70	4.33	3.78	3.44	16.80	7.90	3.57	2.37	1.88	1.57
2	4	25	1.0165		51.06	15.85	8.98	7.37	6.64	6.17	22.17	10.00	5.51	4.28	3.70	3.31	16.62	7.47	3.57	2.48	1.96	1.61
2	4	30	1.0727		48.26	15.08	9.05	7.50	6.69	6.17	21.37	9.53	5.49	4.30	3.65	3.24	16.56	7.25	3.69	2.63	2.06	1.69
3	4	20	1.0177		56.20	18.11	9.35	7.36	6.59	6.12	24.37	11.55	6.05	4.54	3.91	3.53	17.14	8.39	3.83	2.50	1.95	1.61
3	4	25	1.0934		53.44	16.86	9.29	7.48	6.65	6.13	23.15	10.74	5.88	4.50	3.84	3.41	16.99	7.99	3.82	2.60	2.01	1.64
3	4	30	1.1514		50.98	16.10	9.36	7.60	6.69	6.13	22.37	10.29	5.86	4.51	3.79	3.33	16.96	7.79	3.94	2.74	2.10	1.69
FSS control charts					68.09	29.46	13.67	8.54	6.74	6.02	35.19	20.86	10.71	6.54	4.96	4.33	21.46	14.00	7.40	4.18	2.87	2.33
2	3	20	0.8805		15.34	6.95	2.93	1.84	1.41	1.12	14.80	6.61	2.67	1.59	1.17	0.89	14.45	6.38	2.48	1.42	1.00	0.73
2	3	25	0.9605		15.36	6.65	3.05	2.07	1.61	1.27	14.77	6.29	2.75	1.80	1.35	1.02	14.36	6.02	2.54	1.60	1.15	0.83
2	3	30	1.0220		15.38	6.50	3.24	2.30	1.78	1.40	14.72	6.10	2.91	2.00	1.49	1.12	14.25	5.80	2.66	1.77	1.27	0.91
2	4	20	0.9436		15.31	7.03	2.89	1.73	1.26	0.96	14.72	6.66	2.59	1.45	0.99	0.70	14.36	6.42	2.40	1.28	0.82	0.54
2	4	25	1.0165		15.28	6.69	2.95	1.90	1.39	1.07	14.66	6.30	2.63	1.60	1.11	0.79	14.25	6.04	2.42	1.41	0.92	0.61
2	4	30	1.0727		15.27	6.49	3.09	2.07	1.52	1.16	14.58	6.07	2.74	1.75	1.21	0.86	14.14	5.79	2.51	1.54	1.01	0.66
3	4	20	1.0177		15.48	7.42	3.08	1.80	1.27	0.95	14.82	7.00	2.74	1.49	0.98	0.66	14.41	6.74	2.53	1.30	0.79	0.47
3	4	25	1.0934		15.48	7.11	3.13	1.96	1.39	1.03	14.78	6.67	2.78	1.63	1.08	0.72	14.33	6.38	2.54	1.41	0.87	0.52
3	4	30	1.1514		15.49	6.93	3.25	2.12	1.50	1.11	14.72	6.45	2.87	1.76	1.16	0.78	14.24	6.14	2.62	1.53	0.93	0.56
FSS control charts					16.92	10.91	5.48	2.73	1.57	1.08	15.31	9.71	4.63	2.05	0.96	0.49	14.67	9.21	4.26	1.75	0.68	0.23

process are detected. We have found that the VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts always work better (in the cases examined) than the FSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts for small δ_1 and δ_2 values.

This paper considered two dependent process steps with two independent assignable causes. However, a study of the VSS $Z_{\bar{X}}$ and $Z_{\bar{e}}$ control charts under two dependent process steps with two dependent assignable causes, whereby one shifts the process mean and the other changes the process variance, is an interesting topic for future research. Other important extensions of the proposed model can also be developed. It is straightforward to extend the proposed model to study AP control charts or other control charts, such as attribute charts, exponentially weighted moving average (EWMA)-charts or multivariate charts.

APPENDIX

The calculation of all transition probabilities is as follows:

$$p_{im}(t,n_j), \quad i=1,\ldots,17, \quad m=1,\ldots,17$$

$$p_{1,1}(t,n_1) = P[|Z_{\bar{X}}|<w_{\bar{X}}]*P[|Z_{\bar{e}}|<w_{\bar{e}}]*\mathrm{e}^{-\lambda_1 t}*\mathrm{e}^{-\lambda_2 t}$$

$$p_{1,2}(t,n_1) = P[|Z_{\bar{X}}|<w_{\bar{X}}]*P[|Z_{\bar{e}}|<w_{\bar{e}}|\delta_2]*\mathrm{e}^{-\lambda_1 t}*(1-\mathrm{e}^{-\lambda_2 t})$$

$$p_{1,3}(t,n_1) = P[|Z_{\bar{X}}|<w_{\bar{X}}|\delta_1]*P[|Z_{\bar{e}}|<w_{\bar{e}}]*(1-\mathrm{e}^{-\lambda_1 t})*\mathrm{e}^{-\lambda_2 t}$$

$$p_{1,4}(t,n_1) = P[|Z_{\bar{X}}|<w_{\bar{X}}|\delta_1]*P[|Z_{\bar{e}}|<w_{\bar{e}}|\delta_2]*(1-\mathrm{e}^{-\lambda_1 t})*(1-\mathrm{e}^{-\lambda_2 t})$$

$$p_{1,5}(t,n_1) = P[-k_{\bar{X}}<Z_{\bar{X}}\leqslant -w_{\bar{X}} \cup w_{\bar{X}}\leqslant Z_{\bar{X}}<k_{\bar{X}}]*P[|Z_{\bar{e}}|<w_{\bar{e}}]*\mathrm{e}^{-\lambda_1 t}*\mathrm{e}^{-\lambda_2 t}$$

$$p_{1,6}(t,n_1) = P[-k_{\bar{X}}<Z_{\bar{X}}\leqslant -w_{\bar{X}} \cup w_{\bar{X}}\leqslant Z_{\bar{X}}<k_{\bar{X}}]*P[|Z_{\bar{e}}|<w_{\bar{e}}|\delta_2]*\mathrm{e}^{-\lambda_1 t}*(1-\mathrm{e}^{-\lambda_2 t})$$

$$p_{1,7}(t,n_1) = P[-k_{\bar{X}}<Z_{\bar{X}}\leqslant -w_{\bar{X}} \cup w_{\bar{X}}\leqslant Z_{\bar{X}}<k_{\bar{X}}|\delta_1]*P[|Z_{\bar{e}}|<w_{\bar{e}}]*(1-\mathrm{e}^{-\lambda_1 t})*\mathrm{e}^{-\lambda_2 t}$$

$$p_{1,8}(t,n_1) = P[-k_{\bar{X}}<Z_{\bar{X}}\leqslant -w_{\bar{X}} \cup w_{\bar{X}}\leqslant Z_{\bar{X}}<k_{\bar{X}}|\delta_1]*P[|Z_{\bar{e}}|<w_{\bar{e}}|\delta_2]*(1-\mathrm{e}^{-\lambda_1 t})*(1-\mathrm{e}^{-\lambda_2 t})$$

$$P_{1,9}(t,n_1) = P[|Z_{\bar{X}}|<w_{\bar{X}}]*P[-k_{\bar{e}}<Z_{\bar{e}}\leqslant -w_{\bar{e}} \cup w_{\bar{e}}\leqslant Z_{\bar{e}}<k_{\bar{e}}]*\mathrm{e}^{-\lambda_1 t}*\mathrm{e}^{-\lambda_2 t}$$

$$P_{1,10}(t,n_1) = P[|Z_{\bar{X}}|<w_{\bar{X}}]*P[-k_{\bar{e}}<Z_{\bar{e}}\leqslant -w_{\bar{e}} \cup w_{\bar{e}}\leqslant Z_{\bar{e}}<k_{\bar{e}}|\delta_2]*\mathrm{e}^{-\lambda_1 t}*(1-\mathrm{e}^{-\lambda_2 t})$$

$$P_{1,11}(t,n_1) = P[|Z_{\bar{X}}|<w_{\bar{X}}|\delta_1]*P[-k_{\bar{e}}<Z_{\bar{e}}\leqslant -w_{\bar{e}} \cup w_{\bar{e}}\leqslant Z_{\bar{e}}<k_{\bar{e}}]*(1-\mathrm{e}^{-\lambda_1 t})*\mathrm{e}^{-\lambda_2 t}$$

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$$P_{1,12}(t,n_1)=P[|Z_{\bar{X}}|<w_{\bar{X}}|\delta_1]*P[-k_{\bar{e}}<Z_{\bar{e}}\leqslant -w_{\bar{e}}\cup w_{\bar{e}}\leqslant Z_{\bar{e}}<k_{\bar{e}}|\delta_2]*(1-e^{-\lambda_1 t})*(1-e^{-\lambda_2 t})$$

$$P_{1,13}(t,n_1)=P[-k_{\bar{X}}<Z_{\bar{X}}\leqslant -w_{\bar{X}}\cup w_{\bar{X}}\leqslant Z_{\bar{X}}<k_{\bar{X}}]*P[-k_{\bar{e}}<Z_{\bar{e}}\leqslant -w_{\bar{e}}\cup w_{\bar{e}}\leqslant Z_{\bar{e}}<k_{\bar{e}}]*e^{-\lambda_1 t}*e^{-\lambda_2 t}$$

$$P_{1,14}(t,n_1)=P[-k_{\bar{X}}<Z_{\bar{X}}\leqslant -w_{\bar{X}}\cup w_{\bar{X}}\leqslant Z_{\bar{X}}<k_{\bar{X}}]*P[-k_{\bar{e}}<Z_{\bar{e}}\leqslant -w_{\bar{e}}\cup w_{\bar{e}}\leqslant Z_{\bar{e}}<k_{\bar{e}}|\delta_2]*e^{-\lambda_1 t}*(1-e^{-\lambda_2 t})$$

$$P_{1,15}(t,n_1)=P[-k_{\bar{X}}<Z_{\bar{X}}\leqslant -w_{\bar{X}}\cup w_{\bar{X}}\leqslant Z_{\bar{X}}<k_{\bar{X}}|\delta_1]*P[-k_{\bar{e}}<Z_{\bar{e}}\leqslant -w_{\bar{e}}\cup w_{\bar{e}}\leqslant Z_{\bar{e}}<k_{\bar{e}}]*(1-e^{-\lambda_1 t})*e^{-\lambda_2 t}$$

$$P_{1,16}(t,n_1)=P[-k_{\bar{X}}<Z_{\bar{X}}\leqslant -w_{\bar{X}}\cup w_{\bar{X}}\leqslant Z_{\bar{X}}<k_{\bar{X}}|\delta_1]*P[-k_{\bar{e}}<Z_{\bar{e}}\leqslant -w_{\bar{e}}\cup w_{\bar{e}}\leqslant Z_{\bar{e}}<k_{\bar{e}}|\delta_2]*(1-e^{-\lambda_1 t})*(1-e^{-\lambda_2 t})$$

$$P_{1,17}(t,n_1)=1-\sum_{m=1}^{16}P_{1,m}(t,n_1)$$

$$p_{2,1}(t,n_1)=0$$

$$p_{2,2}(t,n_1)=P[|Z_{\bar{X}}|<w_{\bar{X}}]*P[|Z_{\bar{e}}|<w_{\bar{e}}|\delta_2]*e^{-\lambda_1 t}$$

$$p_{2,3}(t,n_1)=0$$

$$p_{2,4}(t,n_1)=P[|Z_{\bar{X}}|<w_{\bar{X}}|\delta_1]*P[|Z_{\bar{e}}|<w_{\bar{e}}|\delta_2]*(1-e^{-\lambda_1 t})$$

$$p_{2,5}(t,n_1)=0$$

$$p_{2,6}(t,n_1)=P[-k_{\bar{X}}<Z_{\bar{X}}\leqslant -w_{\bar{X}}\cup w_{\bar{X}}\leqslant Z_{\bar{X}}<k_{\bar{X}}]*P[|Z_{\bar{e}}|<w_{\bar{e}}|\delta_2]*e^{-\lambda_1 t}$$

$$p_{2,7}(t,n_1)=0$$

$$p_{2,8}(t,n_1)=P[-k_{\bar{X}}<Z_{\bar{X}}\leqslant -w_{\bar{X}}\cup w_{\bar{X}}\leqslant Z_{\bar{X}}<k_{\bar{X}}|\delta_1]*P[|Z_{\bar{e}}|<w_{\bar{e}}|\delta_2]*(1-e^{-\lambda_1 t})$$

$$p_{2,9}(t,n_1)=0$$

$$p_{2,10}(t,n_1)=P[|Z_{\bar{X}}|<w_{\bar{X}}]*P[-k_{\bar{e}}<Z_{\bar{e}}\leqslant -w_{\bar{e}}\cup w_{\bar{e}}\leqslant Z_{\bar{e}}<k_{\bar{e}}|\delta_2]*e^{-\lambda_1 t}$$

$$p_{2,11}(t,n_1)=0$$

$$p_{2,12}(t,n_1)=P[|Z_{\bar{X}}|<w_{\bar{X}}|\delta_1]*P[-k_{\bar{e}}<Z_{\bar{e}}\leqslant -w_{\bar{e}}\cup w_{\bar{e}}\leqslant Z_{\bar{e}}<k_{\bar{e}}|\delta_2]*(1-e^{-\lambda_1 t})$$

$$p_{2,13}(t,n_1)=0$$

$$p_{2,14}(t,n_1)=P[-k_{\bar{X}}<Z_{\bar{X}}\leqslant -w_{\bar{X}}\cup w_{\bar{X}}\leqslant Z_{\bar{X}}<k_{\bar{X}}]*P[-k_{\bar{e}}<Z_{\bar{e}}\leqslant -w_{\bar{e}}\cup w_{\bar{e}}\leqslant Z_{\bar{e}}<k_{\bar{e}}|\delta_2]*e^{-\lambda_1 t}$$

$$p_{2,15}(t,n_1)=0$$

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$$p_{2,16}(t,n_1) = P[-k_{\bar{X}} < Z_{\bar{X}} \leqslant -w_{\bar{X}} \cup w_{\bar{X}} \leqslant Z_{\bar{X}} < k_{\bar{X}}|\delta_1]$$
$$* P[-k_{\bar{e}} < Z_{\bar{e}} \leqslant -w_{\bar{e}} \cup w_{\bar{e}} \leqslant Z_{\bar{e}} < k_{\bar{e}}|\delta_2] * (1 - \mathrm{e}^{-\lambda_1 t})$$

$$p_{2,17}(t,n_1) = 1 - \sum_{m=1}^{16} P_{2,m}(t,n_1)$$

$$p_{3,1}(t,n_1) = 0$$

$$p_{3,2}(t,n_1) = 0$$

$$p_{3,3}(t,n_1) = P[|Z_{\bar{X}}| < w_{\bar{X}}|\delta_1] * P[|Z_{\bar{e}}| < w_{\bar{e}}] * \mathrm{e}^{-\lambda_{12} t}$$

$$p_{3,4}(t,n_1) = P[|Z_{\bar{X}}| < w_{\bar{X}}|\delta_1] * P[|Z_{\bar{e}}| < w_{\bar{e}}|\delta_2] * (1 - \mathrm{e}^{-\lambda_2 t})$$

$$p_{3,5}(t,n_1) = 0$$

$$p_{3,6}(t,n_1) = 0$$

$$p_{3,7}(t,n_1) = P[-k_{\bar{X}} < Z_{\bar{X}} \leqslant -w_{\bar{X}} \cup w_{\bar{X}} \leqslant Z_{\bar{X}} < k_{\bar{X}}|\delta_1] * P[|Z_{\bar{e}}| < w_{\bar{e}}] * \mathrm{e}^{-\lambda_2 t}$$

$$p_{3,8}(t,n_1) = P[-k_{\bar{X}} < Z_{\bar{X}} \leqslant -w_{\bar{X}} \cup w_{\bar{X}} \leqslant Z_{\bar{X}} < k_{\bar{X}}|\delta_1] * P[|Z_{\bar{e}}| < w_{\bar{e}}|\delta_2] * (1 - \mathrm{e}^{-\lambda_2 t})$$

$$p_{3,9}(t,n_1) = 0$$

$$p_{3,10}(t,n_1) = 0$$

$$p_{3,11}(t,n_1) = P[|Z_{\bar{X}}| < w_{\bar{X}}|\delta_1] * P[-k_{\bar{e}} < Z_{\bar{e}} \leqslant -w_{\bar{e}} \cup w_{\bar{e}} \leqslant Z_{\bar{e}} < k_{\bar{e}}] * \mathrm{e}^{-\lambda_2 t}$$

$$p_{3,12}(t,n_1) = P[|Z_{\bar{X}}| < w_{\bar{X}}|\delta_1] * P[-k_{\bar{e}} < Z_{\bar{e}} \leqslant -w_{\bar{e}} \cup w_{\bar{e}} \leqslant Z_{\bar{e}} < k_{\bar{e}}|\delta_2] * (1 - \mathrm{e}^{-\lambda_2 t})$$

$$p_{3,13}(t,n_1) = 0$$

$$p_{3,14}(t,n_1) = 0$$

$$p_{3,15}(t,n_1) = P[-k_{\bar{X}} < Z_{\bar{X}} \leqslant -w_{\bar{X}} \cup w_{\bar{X}} \leqslant Z_{\bar{X}} < k_{\bar{X}}|\delta_1] * P[-k_{\bar{e}} < Z_{\bar{e}} \leqslant -w_{\bar{e}} \cup w_{\bar{e}} \leqslant Z_{\bar{e}} < k_{\bar{e}}] * \mathrm{e}^{-\lambda_2 t}$$

$$p_{3,16}(t,n_1) = P[-k_{\bar{X}} < Z_{\bar{X}} \leqslant -w_{\bar{X}} \cup w_{\bar{X}} \leqslant Z_{\bar{X}} < k_{\bar{X}}|\delta_1]$$
$$* P[-k_{\bar{e}} < Z_{\bar{e}} \leqslant -w_{\bar{e}} \cup w_{\bar{e}} \leqslant Z_{\bar{e}} < k_{\bar{e}}|\delta_2] * (1 - \mathrm{e}^{-\lambda_2 t})$$

$$p_{3,17}(t,n_1) = 1 - \sum_{m=1}^{16} P_{3,m}(t,n_1)$$

$$P_{4,1}(t,n_1) = P_{4,2}(t,n_1) = P_{4,3}(t,n_1) = 0$$

$$P_{4,4}(t,n_1) = P(|Z_{\bar{X}}| < w_{\bar{X}}|\delta_1) * P(Z_{\bar{e}} < w_{\bar{e}}|\delta_2)$$

$$P_{4,5}(t,n_1) = P_{4,6}(t,n_1) = P_{4,7}(t,n_1) = 0$$

$$P_{4,8}(t,n_1) = P(-k_{\bar{X}} < Z_{\bar{X}} \leqslant w_{\bar{X}} \cup w_{\bar{X}} \leqslant Z_{\bar{X}} < k_{\bar{X}}|\delta_1) * P(Z_{\bar{e}} < w_{\bar{e}}|\delta_2)$$

$$P_{4,9}(t,n_1) = P_{4,10}(t,n_1) = P_{4,11}(t,n_1) = 0$$

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$$P_{4,12}(t,n_1) = P(|Z_{\bar{X}}| < w_{\bar{X}}|\delta_1) * P(-k_{\bar{e}} < Z_{\bar{e}} \leqslant w_{\bar{e}} \cup w_{\bar{e}} \leqslant Z_{\bar{e}} < k_{\bar{e}}|\delta_2)$$

$$P_{4,13}(t,n_1) = P_{4,14}(t,n_1) = P_{4,15}(t,n_1) = 0$$

$$P_{4,16}(t,n_1) = P(-k_{\bar{X}} < Z_{\bar{X}} \leqslant w_{\bar{X}} \cup w_{\bar{X}} \leqslant Z_{\bar{X}} < k_{\bar{X}}|\delta_1) * P(-k_{\bar{e}} < Z_{\bar{e}} \leqslant w_{\bar{e}} \cup w_{\bar{e}} \leqslant Z_{\bar{e}} < k_{\bar{e}}|\delta_2)$$

$$P_{4,17}(t,n_1) = 1 - \sum_{m=1}^{16} P_{4,m}(t,n_1)$$

The transition probabilities for $P_{5,j}(t,n_2)$, $P_{6,j}(t,n_2)$, $P_{7,j}(t,n_2)$, $P_{8,j}(t,n_2)$, $P_{9,j}(t_2,n_2)$, $P_{10,j}(t,n_2)$, $P_{11,j}(t,n_2)$ and $P_{12,j}(t,n_2)$ are calculated by replacing n_2 on n_1 for $P_{1,j}(t,n_1)$, $P_{2,j}(t,n_1)$, $P_{3,j}(t,n_1)$ and $P_{4,j}(t,n_1)$, $j = 1,2,\ldots,17$.

That is,

$$P_{5,j}(t,n_2) = P_{1,j}(t,n_2), \quad j = 1,2,\ldots,17$$

$$P_{6,j}(t,n_2) = P_{2,j}(t,n_2), \quad j = 1,2,\ldots,17$$

$$P_{7,j}(t,n_2) = P_{3,j}(t,n_2), \quad j = 1,2,\ldots,17$$

$$P_{8,j}(t,n_2) = P_{4,j}(t,n_2), \quad j = 1,2,\ldots,17$$

$$P_{9,j}(t,n_2) = P_{1,j}(t,n_2), \quad j = 1,2,\ldots,17$$

$$P_{10,j}(t,n_2) = P_{2,j}(t,n_2), \quad j = 1,2,\ldots,17$$

$$P_{11,j}(t,n_2) = P_{3,j}(t,n_2), \quad j = 1,2,\ldots,17$$

$$P_{12,j}(t,n_2) = P_{4,j}(t,n_2), \quad j = 1,2,\ldots,17$$

The transition probabilities for $P_{13,j}(t,n_3)$, $P_{14,j}(t,n_3)$, $P_{15,j}(t,n_3)$ and $P_{16,j}(t,n_3)$ are calculated by replacing n_3 on n_1 for $P_{1,j}(t,n_1)$, $P_{2,j}(t,n_1)$, $P_{3,j}(t,n_1)$ and $P_{4,j}(t,n_1)$, $j = 1,2,\ldots,17$.

That is,

$$P_{13,j}(t,n_3) = P_{1,j}(t,n_3), \quad j = 1,2,\ldots,17$$

$$P_{14,j}(t,n_3) = P_{2,j}(t,n_3), \quad j = 1,2,\ldots,17$$

$$P_{15,j}(t,n_3) = P_{3,j}(t,n_3), \quad j = 1,2,\ldots,17$$

$$P_{16,j}(t,n_3) = P_{4,j}(t,n_3), \quad j = 1,2,\ldots,17$$

$$P_{i,j} = 0, \quad i = 17, \quad j = 1,2,\ldots,16 \text{ and } i \neq j$$

$$P_{i,i} = 1, \quad i = 17$$

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