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The economic design of control charts when there are dependent process steps

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Introduction

Control charts are important tools of statistical quality control. These charts are used to decide whether a process has achieved a state of statistical control and to maintain current control of a process. Today, most products are produced by several different process steps. In multiple step processes a Shewhart control chart is often used at each individual step. If the steps of the process are independent, then using a Shewhart control chart at each individual step is a meaningful procedure. However, in many processes the steps are not independent and thus the charts are difficult to interpret. One approach to solve this problem is to use a multivariate control chart such as Hotelling T^2. The disadvantages of using a T^2 chart are that one must assume that the process quality characteristics are multivariate normal random variables and, once an out-of-control signal is given, it is often difficult to determine which component of the process is out of control. Another alternative to this problem was proposed by Zhang (1984). He calls his charts "cause-selecting charts". The cause-selecting control chart is constructed for a variable only after the observations have been adjusted for the effect of some other random variables. Zhang's cause-selecting control charts are the concepts of overall quality and specific quality. Zhang defines overall quality as that quality due to the current subprocess and any previous subprocesses. Specific quality is that quality which is due only to the current subprocess. The cause-selecting charts are designed to further distinguish between controllable assignable causes and uncontrollable assignable causes. Controllable assignable causes are those assignable causes that affect the current subprocess but no previous subprocesses. The uncontrollable assignable causes are those assignable causes affecting previous processes that cannot be controlled at the current process level. The advantage of this approach is that once an out-of-control signal is given, it is often easy to determine which component of the processes is out of control. Wade and Woodall (1993) review the basic principles of the cause-selecting chart, the simple case of a two step process and give an example to illustrate the use of the cause-selecting chart. They also examine the relationship between the cause-selecting chart and the multivariate T^2 chart. In their opinion the cause-selecting control chart has some advantages over the T^2 chart.

To use any control charts, three design parameters must be specified; the sample size, the sampling interval, and the number of standard deviation above

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or below the centre line of the control chart. The choice of these design parameters influence the costs of sampling and testing, costs of searching and repairing and costs due to the production of nonconforming items. Therefore, it is logical to consider the design of control charts from an economic viewpoint.

Duncan (1956) first proposed an economic model for the optimal economic design of the $\bar{X}$ control chart. He recommended the use of a concept which he called an economic design to obtain the optimal design. The pioneering work of Duncan was later extended by others, including the $\bar{X}$ and R charts employed jointly (Saniga, 1977, 1979; Yang, 1993). Rahim, Lashkari and Banerjee (1988) discussed the uses of joint $\bar{X}$ and S^2 charts according to economic considerations when sample sizes are moderately large. Collani and Sheil (1989) proposed the economic design of the S chart when the assignable cause could only influence the process variance. However, the economic model for dependent subprocesses has not been addressed. In this paper the multiple assignable-cause economic model for an individual X chart and cause-selecting control chart the simple case of a two step process will be proposed.

The derivation of cost model

Let X represent the quality measurement of interest for the first step of the process and let Y represent the quality measurement of interest for the second step. Suppose that a sample with size one is taken at the end of the second process every h hours and observations (X_i, Y_i) are measured on the same item of production. It is assumed that the X_i values are independent and $X_i \sim N(\mu_0, \sigma_0^2)$ when the process is in control. The relationship between X and Y should be known before the cause-selecting control chart is constructed. The cause-selecting chart is then based on values of the outgoing quality Y that have been adjusted for the value of in-coming quality X. The individual X chart is based on values of the in-coming quality X.

The model relating the two variables can take many forms. We assume the model is the simple linear regression model:

$$E(Y_i \mid X_i) = a_0 + a_1 X_i = \mu_i. \quad (1)$$

The model need not be linear for constructing the cause-selecting chart. That is, the cause-selecting technique can be applied to a nonlinear model. The values Y are also independent and $Y_i \sim N(\mu_i, \sigma_1^2)$ given X_i when the process is in control. We use the individual X chart on the X variable and the cause-selecting chart on the Y variable. The centre line, upper control limit and lower control limit of the individual X chart are set at μ_0, $\mu_0 + k_1 \sigma_0$ and $\mu_0 - k_1\sigma_0$ respectively, where k_1 is the number of standard deviation above or below the centre line of the individual X chart. The cause-selecting chart is a Shewhart type of control chart for the cause-selecting values, Z_i, where $Z_i = (Y_i - \mu_i)/\sigma_1$, are the values of Y_i adjusted for the effects of X_i. Thus, the Z_is are independent $N(0, 1)$ random variables. The centre line, upper control limit, and lower control limit for the cause-selecting control chart are 0, k_2, and $-k_2$ respectively, where k_2 is the number of standard deviations above or below the centre line of the cause-selecting chart.

A process is out of control when it is influenced by assignable causes. We assume that there are uncontrollable assignable cause and controllable assignable cause, say A_1 and A_2. A_1 can only affect the previous process and cannot be controlled at the current process. A_2 can only affects the current process but no previous subprocess. Once the previous process is influenced by A_1, the process mean of X shifts from μ_0 to μ_{01} (= $\mu_0 + \delta_0\sigma_0$) and the process mean of Y given X shifts from μ_i to μ_{i1}, and the variances of X and Y given X are unchanged. If the process is only influenced by A_2 then the process mean of Y given X shifts from μ_i to μ_{i2} (= $\mu_i + \delta_1\, \sigma_1$) and the distribution of X is unchanged. Assignable causes A_1 and A_2 would be allowed to occur in the first step and the second step of the process simultaneously; the process mean of X would shift from μ_0 to μ_{01}, and the process mean of Y given X would shift from μ_1 to μ_{i3} (= $\mu_{i1} + \delta_1\sigma_1$). Other assumptions and the nature of the operation condition are summarized as follows.

(1) The time (T_{Ai}) until the occurrence of assignable cause (A_i) is assumed exponential distribution with parameter λ_i, i = 1, 2. T_{A1} and T_{A2} are independent.

(2) When the process goes out of control it will not improve. That is, the process mean can only shift to worse values of the process parameters.

(3) The time of taking a sample, inspection, and charting are negligible.

(4) The search and repair time is a constant T_{sr} when there is at least one true alarm for the individual X chart and the cause-selecting chart. The search and repair time is a constant T_f when there is at least one false alarm for the two charts.

(5) The search and repair cost is a constant C_{sr} when there is at least one true alarm for the two charts. The search and repair cost is a constant C_f when there is at least one false alarm for the two charts.

(6) A quality cycle is defined as the time between the start of successive in-control periods. Then the process is expressed as a series of independent and identical cycles. That is, the process is a renewal process. The accumulated cost per cycle is the cycle cost. The cycle costs are independent and identically distributed. Such a process is known as a renewal reward process (see Ross, 1989).

(7) The cost of sampling and testing is b, $b > 0$.

(8) The process is discontinuous. That is, the process ceases during the search state.

The cost model is thus derived using the renewal theory approach (Banerjee and Rahim, 1987). Some notations used are defined as follows:

$E(T)$: The expected cycle time.

$E(C)$: The expected cycle cost.

α: The probability that at least one of the control charts has a false alarm.

$\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$, where

α_1 is the probability that the individual X chart has a false alarm.

$\alpha_1 = P(X_i < \mu 0 = k_1 \sigma_0 | M_0) + P(X_i < \mu_0 - k_1 \sigma_0 | \mu_0)$
$= 2(1 - \Phi(k_2))$,

where $\Phi(.)$ is a cumulative standard normal distribution,

α_2 is the probability that the cause-selecting chart has a false alarm.

$\alpha_2 = P(Y_i > \mu_i + k_2 \sigma_1 | \mu_1) + P(Y_i < \mu_i - k_2 \sigma_1 | \mu_i)$
$= P(Z_i > k_2) + P(Z_i < -k_2) = 2(1 - \Phi(k_2))$.

β_{01}: The probability that there is no alarm for the charts given the previous process is in control and the current process is out of control.

$\beta_{01} = (1 - \alpha_1)\beta_2$, where

β_2: is the probability that the cause-selecting chart has no alarm given that the current process is out of control and the previous process is in control.

$\beta_2 = P(-k_2 < Z_i < k_2 | \delta_1 \sigma_1) = \Phi(k_2 - \delta_1) - \Phi(-k_2 - \delta_1)$,

where

$Z_i' = (Y_i - \mu_i)/\sigma_1 \sim N(\delta_1, 1)$ because of $Y_i \sim N(\mu_{i2}, \sigma_2^1)$ given X_i and $X_i \wedge N(\mu_0, \sigma_0^2)$.

β_{10}: The probability that there is no alarm for the charts given that the previous process is out of control and the current process is in control.

$\beta_{10} = (1 - \alpha_2)\beta_1$,

where

β_1 is the probability that the individual X chart has no alarm given that the previous process is out of control.

$\beta_1 = P(\mu_0 - k_1 \sigma_0 < X_i < \mu_0 + k_1 \sigma_0 | \mu_{01}) = \Phi(-\delta_0 + k_1) - \Phi(-\delta_0 - k_1)$.

β_{11}: The probability that there is no alarm for the charts given that the previous process and the current process are all out of control.

$\beta_{11} = \beta_1 \beta_2$,

where

β_3 is the probability that the cause-selecting chart has no alarm given that the previous process and the current process are all out of control.

$\beta_3 = P(-k_2 < Z_i'' < k_2 | \delta_1 \sigma_1) = \Phi(k_2 - \delta_1) - \Phi(-k_2 - \delta_1)$, where $Z_i'' = (Y_i - \mu_{i1})/\sigma_1 \sim N(\delta_1, 1)$ because of $Y_i \sim N(\mu_{i3}, \sigma_1^2)$ given X_i, and $X_i \sim N(\sigma_{01}, \sigma_0^{2\}})$. In fact, $\beta_3 = \beta_2$.

C_0: Quality cost/hour while production is in control.

C_1: Quality cost/hour while the previous process is only disturbed by the assignable cause A_1.

C_2: Quality cost/hour while the current process is only disturbed by the assignable cause A_2.

C_{12}: Quality cost/hour while the previous process and current process are only disturbed by the assignable causes A_1 and A_2.

τ_i: The expected arrival time of the assignable cause A_i given that it occurred in the first sampling and testing interval, $i = 1, 2$. That is,

$$\tau_i = E(T_{Ai} \mid T_{Ai} < h) = [1 - e^{-\lambda_i h} - \lambda_i h e^{-\lambda_i h})] / [\lambda_i (1 - e^{-\lambda_i h})], \; i = 1, 2.$$

$\tau_{(i)}$: The expected arrival time of the ith arrived assignable cause given that A_1 and A_2 occurred in the first sampling and testing interval, $i = 1, 2$. That is,

$$\tau_1 = [e^{-(\lambda_1 + \lambda_2)h} (h + 1/\lambda_1 + 1/\lambda_2 - 1/(\lambda_1 + \lambda_2)) - e^{-\lambda_1 h}/\lambda_2 - e^{-\lambda_2 h}/\lambda_1 + 1/(\lambda_1 + \lambda_2)] / [(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})],$$

$$\tau_{(2)} = [e^{-(\lambda_1 + \lambda_2)h} (h + 1/(\lambda_1 + \lambda_2)) - e^{-\lambda_1 h}(h + 1/\lambda_1) - e^{-\lambda_2 h} (h + 1/\lambda_2) + 1/\lambda_1 + 1/\lambda_2 - 1/(\lambda_1 + \lambda_2)] \; [(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})],$$

(proof, see Appendix).

In order to obtain the expression for the expected cycle time (E(T)), we decomposed the cycle into the following three components:

(1) the in-control period;

(2) the time to obtain a true alarm given that the process is out of control;

(3) the time needed to find and repair the assignable causes.

To use the renewal equation approach we have to study the possible states at the end of the first sampling and testing. Depending on the state of the system, one can compute the expected residual cycle length and the expected residual cost. These values, together with the associated probabilities, lead us to formulate the renewal equation. The analysis developed below depends on the possible states at the end of the first sampling and testing. These states are defined as follows (Table I).

State 1: the previous process and the current process are all in control, and there is no alarm for the individual X chart and the cause-selecting chart.

State 2: the previous process and the current process are all in control, but there are at least one alarm for the charts.

State 3: the previous process is out of control and the current process is in control, but there is no alarm for the charts.

State 4: the previous process is out of control and the current process is in control, and there is at least one alarm for the charts.

State 5: the previous process is in control but the current process is out of control, but there is no alarm for the charts.

State 6: the previous process is in control but the current process is out of control, and there is at least one alarm for the charts.

State 7: the previous process and the current process are all out of control, but there is no alarm for the charts.

State 8: the previous process and the current process are all out of control, and there is at least one alarm for the charts.

Table II displays the possible states of the system, and the expected residual cycle length with the associated probability of being in each respective state.

Consequently,

$$E(T) = h + P_1 E(T) + P_2 (E(T) + T_f) + \sum_{i=3}^{8} P_i R_i .$$

Simplifying this we get

$$E(T) = [(h + P_2 T_f)/(1 - P_1 - P_2)] + [\sum_{i=3}^{8} P_i R_i /(1 - P_1 - P_2)]$$

State number	Previous process In control?	Current process In control?	At least one alarm for cause selecting chart and individual X chart?
1	Yes	Yes	No
2	Yes	Yes	Yes
3	No	Yes	No
4	No	Yes	Yes
5	Yes	No	No
6	Yes	No	Yes
7	No	No	No
8	No	No	Yes

Table I. Definition for each state

State	Probability	Expected residual cycle time
1	$P_1 = e^{-\lambda_1 h} e^{-\lambda_2 h} (1 - \alpha)$	$R_1 = E(T)$
2	$P_2 = e^{-\lambda_1 h} e^{-\lambda_2 h} \alpha$	$R_2 = T_f + E(T)$
3	$P_3 = (1 - e^{-\lambda_1 h}) e^{-\lambda_2 h} \beta_{10}$	$R_3 = h/(1 - \beta_{10}) + T_{sr}$
4	$P_4 = (1 - e^{-\lambda_1 h}) e^{-\lambda_2 h} (1 - \beta_{10})$	$R_4 = T_{sr}$
5	$P_5 = e^{-\lambda_1 h} (1 - e^{-\lambda_2 h}) \beta_{01}$	$R_5 = h/(1 - \beta_{01}) + T_{sr}$
6	$P_6 = e^{-\lambda_1 h} (1 - e^{-\lambda_2 h}) (1 - \beta_{01})$	$R_6 = T_{sr}$
7	$P_7 = (1 - e^{-\lambda_1 h}) (1 - e^{-\lambda_2 h}) \beta_{11}$	$R_7 = h/(1 - \beta_{11}) + T_{sr}$
8	$P_8 = (1\ e^{-\lambda_1 h}) (1 - e^{-\lambda_2 h}) (1 - \beta_{11})$	$R_8 = T_{sr}$

Table II. Probability and expected residual cycle time for each state

$$= [h + e^{-(\lambda_1 + \lambda_2)h} \alpha T_f + (1 - e^{(\lambda_1 + \lambda 2)h}) T_{sr} + (1 - e^{-\lambda_1 h})\ e^{-\lambda_2 h} h\beta_{10} /(1 - \beta_{10})$$
$$+ e^{-(\lambda_1 h}(1 - e^{-(\lambda_2 h})h\beta_{01} /(1 - \beta_{01}) + (1 - e^{-(\lambda_1 h})(1 - e^{-(\lambda_2 h})h\beta_{11} /(1 - \beta_{11})] /$$
$$[1 - e^{-(\lambda_1 + \lambda_2)h}] \tag{2}$$

In order to obtain an expression for the expected cycle cost (E(C)), we decomposed the cycle cost into the following two components:

(1) the cost occurred in the first sampling and testing;

(2) the expected residual cost, which is the cost occurred from the time that the process is influenced by any assignable cause to when all assignable causes are repaired.

We present the possible states of the system, and the costs occurred in the first sampling and testing with the expected residual costs in Table III.

Consequently:

$$E(C) = P_1[(b + C_0 h) + E(C)] + P_2[(b + C_0 h) + E(C) + C_f] + \sum_{i=3}^{8} P_i R_i'$$
$$= (P_1 + P_2)E(C) + (P_1 + P_2)(b + C_0 h) + P_2 C_f + \sum_{i=3}^{8} P_i R_i'.$$

Simplifying

$$E(C) = [(P_1 + P_2)(b + C_0 h) + P_2 C_f]/(1 - P_1 - P_2)$$
$$+ [\sum_{i=3}^{8} PiR_i' /(1 - P_1 - P_2)]$$
$$= \{b + C_0 h e^{-(\lambda_1 + \lambda_2)h} + e^{-(\lambda_1 + \lambda_2)h} \alpha C_f + (1 - e^{-(\lambda_1 + \lambda_2)h} C_{sr} + [C_0 \tau_1$$

Table III.
Cost for each state

State	Cost in the first sampling and testing	+	Expected residual cost
1	$R_1' = b + C_0 h$	+	$E(C)$
2	$R_2' = b + C_0 h$	+	$C_f = T_f + E(C)$
3	$R_3' = b + C_0\tau_1 + C_1(h - \tau_1)$	+	$hC_1/(1 - \beta_{10}) + C_{sr}$
4	$R_4' = b + C_0\tau_1 + C_1(h - \tau_1)$	+	C_{sr}
5	$R_5' = b + C_0\tau_2 + C_2(h - \tau_2)$	+	$hC_2/(1 - \beta_{01}) + C_{sr}$
6	$R_6' = b + C_0\tau_2 + C_2(h - \tau_2)$	+	C_{sr}
7	$R_7' = b + C_0\tau_1 + (\tau_2 - \tau_1)(C_1\lambda_1 + C_2\lambda_2)/(\lambda_1 + \lambda_2) + C_{12}(h - \tau_2)$	+	$hC_{12}/(1 - \beta_{11}) + C_{sr}$
8	$R_8' = b + C_0\tau_{(1)} + (\tau_{(2)} - \tau_{(1)})(C_1\lambda_1 + C_2\lambda_2)/(\lambda_1 + \lambda_2)$	+	$C_{12}(h - \tau_2) + C_{sr}$

$$
\begin{aligned}
&+C_1(h-\tau_1)](1-e^{-\lambda_1 h})e^{-\lambda_2 h}+(1-e^{-\lambda_1 h})e^{-\lambda_2 h}h\beta_{10}C_1/(1-\beta_{10})\\
&+[C_0\tau_2+C_2(h-\tau_2)](1-e^{-\lambda_2 h})e^{-\lambda_1 h}+(1-e^{-\lambda_2 h})\\
&\quad e^{-\lambda_1 h}h\beta_{01}C_2/(1-\beta_{01})+[C_0\tau_{(1)}+(\tau_{(2)}-\tau_{(1)})(C_1\lambda_1+C_2\lambda_2)/(\lambda_1+\lambda_2)\\
&+C_{12}(h-\tau_{(2)})](1-e^{-\lambda_1 h})(1-e^{-\lambda_2 h})+(1-e^{-\lambda_1 h})(1-e^{-\lambda_2 h})\cdot\\
&\quad h\beta_{11}C_{12}/(1-\beta_{11})\}/(1-e^{(-\lambda_1\lambda_2)h})
\end{aligned}
\tag{3}
$$

Applying the property of renewal reward process (Ross, 1989), the objective function, the expected cost per unit time ($E(V\infty)$) is derived by taking the ratio of the expected cycle cost ($E(C)$) and the expected cycle time ($E(T)$); $E(V\infty) = E(C)/E(T)$. The objective function is the function of design parameters h, k_1 and k_2. Hence, the optimal design parameters can be determined by minimizing the objective function.

An example

An example is given to illustrate the method proposed. Suppose that the combination of cost and process parameters is $\delta_0 = 3.5$, $\delta_1 = 3.0$, $\lambda_1 = 0.01$, $\lambda_2 = 0.05$, $b = 5$, $\mu_0 = 0.0$, $\sigma_0 = 1.0$, $\sigma_1 = 0.25$, $T_{sr} = 0.8$, $T_f = 0.2$, $C_0 = 5$, $C_1 = 10$, $C_2 = 15$, $C_{12} = 25$, $C_f = 30$, $C_{sr} = 50$. In the process of obtaining the approximate optimal values h^*, k_1^* and k_2^*, we treat h, k_1 and k_2 as discrete variables and assume that the values of h, k_1 and k_2 are within the ranges between 0.0 and 8.0 ($0 < h <= 8.0$ and the unit length of h is 0.1), 0.0 and 4.0 ($0 < k_1 <= 4.0$, $0 < k_2 <= 4.0$ and the unit lengths of k_1 and k_2 are 0.1) respectively. We also add a constraint ($\alpha <= 0.1$) to the model because in many economic designs the probability of Type I error of control chart is much higher than that in a statistical design, and this will result in more false alarms than expected (Woodall, 1987). The algorithm used to obtain the approximate values h^*, k_1^* and k_2^* of the design variables h, k_1 and k_2 is the simple grid search method. Consequently, $h^* = 7.7$, $k_1^* = 2.6$, $k_2^* = 1.7$, $E(V\infty) = 5.7$, $\alpha^* = 0.098$, $\beta_{10}^* = 0.168$, $\beta_{01}^* = 0.096$, and, $\beta_{11}^* = 0.018$. That is, the upper and lower control limits of the economic individual X chart should be set at 2.6 and –2.6 respectively. The upper and lower control limits of the economic cause-selecting chart should be set at 1.7 and –1.7 respectively. To

Combination number	Individual X chart signal?	Cause-selecting chart signal?	Action process stops?
1	No	No	No
2	No	Yes	Yes, search and repair A_2
3	Yes	No	Yes, search and repair A_1
4	Yes	Yes	Yes, search and repair A_1 and A_2

Table IV. Decision rules

monitor the process state, every 7.7 hours a sample with size one (X_i, Y_i) is taken and tested. There are four possible testing results for the two processes. These outcomes with associated various actions are given in Table IV. Combination 1 means that X_i falls inside the individual X chart and cause-selecting value, Z_i, also falls inside the cause-selecting chart, so the process continues and the next sample is taken after 7.7 hours. Interpretations for combination 2, 3 and 4 are similar to combination 1.

Conclusions and suggestions

The cause-selecting chart could be used in conjunction with individual X chart for the two processes. They may effectively distinguish the uncontrollable assignable cause and controllable assignable cause. The method of designing the economic cause-selecting chart and individual X chart simultaneously has been proposed. If the engineers would like to maintain the processes with minimum cost and determine effectively which component of the process is out of control then the economic cause-selecting chart is preferable.

In practice, the true relationship between X and Y is always known. Hence, the mean of Y given X, $E(Y|X)$, and the variance of Y given X, $V(Y|X)$, have to be estimated from an initial sample of n observations. For model-fitting methods and diagnosis see Montgomery and Park (1982), Weisberg (1985), and others. The cause-selecting values, Z_i, the residuals generated by model used are $Z_i = Y_i - \hat{\mathbf{Y}}_\mathbf{i}$, where $\hat{\mathbf{Y}}_\mathbf{i}$, is the fitted value of Yi given Xi. The centre line for the economic cause-selecting chart is $\bar{\mathbf{Z}} = 0$. The control limits for the economic cause-selecting chart are given by UCL = k2 $\hat{\sigma}_1$ is the square root of the mean square error. Alternatively, UCL = $k_2 \overline{MR}$ and LCL = $k_2 \overline{MR}$, where $\overline{MR}$ =

$$\sum_{i=1}^{n-1} \overline{MR}_i / (n-1),$$

and

$$\overline{MR}_i = | Z_{i+1} - Z |.$$

The method proposed can be extended to the case of multiple assignable causes occurring in the current process and previous processes.

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Appendix

The source for this proof is Yang and Jeng (1994)

$\tau_{(1)}$: $= E(T_{(1)} \mid T_{(2)} < h) = E(T_{(1)}')$

$= [e^{-(\lambda_1+\lambda_2)h}(h + 1/\lambda_1 + 1/\lambda_2^{-1}/((\lambda_1 + \lambda_2)) - e^{-\lambda_1 h}/\lambda_2 - e^{-\lambda 2h}/\lambda_1 + 1/(\lambda_1 + \lambda 2)]/[(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})]$,

$\tau_{(2)}$: $= E(T_{(2)} \mid T_{(2)} < h) = E(T_{(2)}')$

$= [e^{-(\lambda_1+\lambda_2)h}(h + 1/(\lambda_1 + \lambda_2)) - e^{-\lambda_1 h}(h + 1/(\lambda_1) - e^{-\lambda_2 h}(h + 1/\lambda_2) + 1/\lambda_1 + 1/\lambda_2 - 1/(\lambda_1 + \lambda_2)]/[(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2} h)]$,

where $T_{(1)} = \min(T_{A1}, T_{A2})$, $T_{(2)} = \max(T_{A1}, T_{A2})$,

$T_{(1)}' = \min(T_1', T_2')$, $T_{(2)}' = \max(T_1', T_2')$, and

T_i' is the truncated random variable of T_{Ai}, i = 1,2.

Proof:

$T_{Ai} \sim \exp(\lambda_i)$. Let T_{Ai} have p.d.f. $f(t_i)$, i = 1, 2.

The joint p.d.f. of T_1' and T_2' is

$h(t_1', t_2') = [f(t_1)/p(0 < T_{A1} < h)] \cdot [f(t_2)/p(0 < T_{A2} < h)] = h(t_1')h(t_2')$

$= [\lambda_1 e^{-\lambda_1 t_1'}/(1 - e^{-\lambda_1 h})] \cdot [\lambda_2 e^{-\lambda_2 t_2'}/(1 - e^{-\lambda_2 h})]$, $0 < t_1', t_2' < h$.

The joint p.d.f. of $T_{(1)}'$ and $T_{(2)}'$ is

$g(t_{(1)}', t_{(2)}') = h(t_{(1)}', t_{(2)}') + h(t_{(2)}', t_{(1)}')$

$= [\lambda_1 e^{-\lambda_1 t(1)'}\lambda 2 e^{-\lambda_2 t_{(2)}'}]/[(1 - e^{-\lambda_1} h)(1 - e^{-\lambda 2h})]$

$+ [\lambda_1 e^{-\lambda_1 t_{(2)}'}\lambda_2 e^{-\lambda_2 t_{(1)}'}]/[(1 - e^{-\lambda_1} h)(1 - e^{-\lambda_2 h})]$, $0 < t_{(1)}', t_{(2)}' < h$.

So we may derive the result

$\tau_{(1)}$: $= \int_0^h \int_{t(1)}^h g(t_{(1)}', t_{(2)}'), dt_{(2)}' \, dt_{(1)}'$

$= [e^{-(\lambda_1+\lambda_2)h}(h + 1/\lambda_1 + 1/\lambda_2 - 1/(\lambda_1 + \lambda_2)) - e^{-\lambda_1} h/\lambda_2 - e^{-\lambda_2 h}/\lambda_1 + 1/(\lambda_1 + \lambda_2)]/[(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})]$

Similarly,

$\tau_{(2)}$: $= \int_0^h \int_0 t'_{(2)} \, g(t_{(1)}', t_{(2)}'), dt_{(1)}' \, dt_{(2)}'$

$= [e^{-(\lambda_1+\lambda_2)h}(h + 1/\lambda_1 + \lambda_2)) - e^{-\lambda_1 h}(h + 1/\lambda_1) - e^{-\lambda_2 h}/(h + 1/\lambda_2) + 1/(\lambda_1 + 1 \ \lambda_2 - 1/(\lambda_1 + \lambda_2)]/[(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})]$.