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Adaptive control schemes for two dependent process steps

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Abstract

The article considers the statistical adaptive process control for two dependent process steps. We construct adaptive sample size and sampling interval (ASSI) $Z_{\overline{X}}$ and $Z_{\overline{e}}$ control charts to monitor the quality variables produced by the first process step and the second process step at the end of the second process step and effectively distinguish which process step is out-of-control. The performance of the proposed ASSI control charts is measured by adjusted average time to signal (AATS) using Markov chain approach. An example of automobile braking system shows the application and performance of the proposed ASSI $Z_{\overline{X}}$ and $Z_{\overline{e}}$ control charts in detecting shifts in process mean. Furthermore, the performance of the ASSI $Z_{\overline{X}}$ and $Z_{\overline{e}}$ control charts and the nonadaptive sample size and sampling interval $Z_{\overline{X}}$ and $Z_{\overline{e}}$ control charts are compared by some numerical analysis results. It demonstrates that the former is much faster than the latter in detecting small shifts in mean.

Keywords: Dependent process steps; Control charts; Adaptive sample size; Adaptive sampling interval

1. Introduction

Control charts are important tools in statistical quality control. They are used to effectively monitor and determine whether a process is in-control or out-of-control. Shewhart (1931) first develops the $\overline{X}$ control chart to monitor the process mean. The control chart is easy to implement, hence it has been widely used for industrial process. However, Shewhart $\overline{X}$ control charts always monitor a process by taking equal samples of size at a fixed sampling interval. It is not effective to control an out-of-control process, since it takes long time to detect it. A useful approach to improving the ability to detect process changes is to use a variable sample size and sampling interval control chart instead of the traditional fixed sample size and sampling interval. Whenever there is some indication that a process parameter may have changed, the next sample should be large or/and it should be sampled after a short interval of time. On the other hand, if there is no indication of a parameter change, then the next sample should be small or/and it should be sampled after a long interval of time. There have been several alternatives developed to improve this problem in recent years.

The properties of the $\overline{X}$ chart with adaptive sampling intervals (ASI) were studied by Reynolds, Amin, Arnold, and Nachlas (1988). Their paper has been extended by several others: Amin and Miller (1993); Baxley (1995); Reynolds and Arnold (1989, 2001); Reynolds, Amin, and Arnold (1990); Reynolds, Arnold, and Baik (1996); Runger and Montgomery (1993); Runger and Pignatiello (1991); Saccucci, Amin, and Lucas (1992). The properties of the $\overline{X}$ chart with adaptive sample sizes (ASS) were studied by Daudin (1992), Costa (1994) and Prabhu, Runger, and Keats (1993). The properties of the adaptive sample size and sampling interval (ASSI) $\overline{X}$ chart were studied by Prabhu, Montgomery, and Runger (1994) and Costa (1997). Costa (1999) considered the AP $\overline{X}$ chart with all design parameters variable, including the k factor used in determining the width of the control limits. Tagaras (1998) reviewed the papers about adaptive control charts. However, most work on developing ASI, ASS, ASSI, and AP control charts has been down for the problem of monitoring mean of a single process. Recently, most products are produced by several different dependent process steps. For monitoring a process with two dependent process steps, a powerful and popular approach

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proposed by Zhang (1984) is using cause-selecting control chart. The cause-selecting control chart is constructed after adjusting the effect of in-coming quality variable on out-going quality variable, since the in-coming quality variable on the first process step and the out-going quality variable on the second process step are dependent. The advantage of this approach is that once there is a signal from the cause-selecting control chart, it is easy to determine if the second step of the process is out-of-control. Wade and Woodall (1993) review and analyze the cause-selecting control chart and examine the relationship between the cause-selecting control chart and the Hotelling T^2 control chart. In their opinion the cause-selecting control chart outperforms Hotelling T^2 control chart. Therefore, it seems reasonable to develop adaptive control charts to monitor dependent process steps. For controlling two dependent process steps, the properties of the joint VSI Z_X and Z_e charts were studied by Yang and Su (2005). The properties of the joint VSSI charts used to monitor two dependent process steps have not been addressed. Therefore, to study the performance of the joint VSSI $Z_{\overline{X}}$ and $Z_{\bar{e}}$ charts on two dependent process steps is reasonable. In this paper, a process with two process steps and two assignable causes is considered. The occurrence of two assignable causes shifts the means on the two dependent process steps. The model has been adopted to study the joint economic design of X and e charts on two dependent process steps (see Yang, 1998).

2. Description of the joint VSSI $Z_{\overline{X}}$ and $Z_{\bar{e}}$ charts on controlling two dependent process steps

Consider a process with two dependent process steps monitored by joint ASSI $Z_{\overline{X}}$ and $Z_{\bar{e}}$ control charts. Let X be the single measurable quality characteristic of interest for the first process step. Assume further that this process starts in a state of statistical control, that is, X follows a normal distribution with the mean at its target value, μ_X, and the standard deviation at its target value σ_X; let Y be the single measurable quality characteristic of interest for the second step, and follow a normal distribution. Since the two process steps are dependent, and the second process step is affected by the first process step, i.e. the distribution of Y is affected by the distribution of X, following Wade and Woodall (1993), the relationship between X and Y is generally expressed as

$$Y|X = f(X) + \varepsilon. \tag{2.1}$$

The variable ε is the error term and $\varepsilon \sim NID\left(0, \sigma_\varepsilon^2\right)$ when the second process step is in-control. Using the X and Y control charts to monitor the two dependent process steps is incorrect and inefficient (see Zhang, 1984; Wade and Woodall, 1993). To monitor the second process step correctly, the quality characteristic of the second process step should be specified by adjusting the effect of X on Y; i.e. the specific quality is presented by the residual values or cause-selecting values, $e = Y|X - \hat{Y}|X$, where $\hat{Y}|X$, is the fitted model of the paired observations (x,y) using least square error method. Hence, $e \sim N\left(0, \sigma_\varepsilon^2\right)$. Consequently, the X and e charts are used to instead of X and Y charts. Also assume that the first step is only subject to assignable cause 1 such that the process mean shifts from μ_X to $\delta_1\sigma_X(\delta_1 \neq 0)$, and the second step is only subject to assignable cause 2 such that the process mean shifts from 0 to $\delta_2\sigma_\varepsilon(\delta_2 \neq 0)$. Let T_i be the time until the occurrence of assignable cause i, $i = 1,2$. Many papers (see, e.g. Ho & Case, 1994; Montgomery, 1997; Rahim, 1989; Rahim, Lashkari, & Banerjee, 1988; Saniga, 1977, 1979, 1989; Saniga & Montgomery, 1981) assume that T_i follows an exponential distribution. Following their paper, this paper assumes that T_i follows an exponential distribution of the form

$$f(t_i) = \lambda_i \exp(-\lambda_i t_i), \quad t_i > 0, i = 1,2,$$

where $1/\lambda_i$ is the mean time that the process step i remains in a state of statistical control.

An in-control state analysis for the joint ASSI $Z_{\overline{X}}$ and $Z_{\bar{e}}$ control charts are performed since the shift in the process mean on step 1 or/and step 2 does not occur when the process is just starting, but occurs at some time in the future. Rational samples of variable size are taken from the in-control process; the standardized sample means, $Z_{\overline{X}}$ and $Z_{\bar{e}}$, given by

$$Z_{\overline{X}} = \frac{\overline{X} - \mu_X}{\sigma_X/\sqrt{n_j}}$$

and

$$Z_{\bar{e}} = \frac{\bar{e}/\sigma_\varepsilon}{\sqrt{n_j}},$$

are plotted on the control charts with warning limits of the form $\pm w_{\overline{X}}$ and $\pm w_{\bar{e}}$ and action limits of the form $\pm k_{\overline{X}}$ and $\pm k_{\bar{e}}$, respectively, where $0 \leqslant w_{\overline{X}} < k_{\overline{X}}$ and $0 \leqslant w_{\bar{e}} < k_{\bar{e}}$ (see Fig. 1).

For process step 1 and process step 2, let the sample points plotted on the standard $\overline{X}$ and $\bar{e}$ charts be the sample mean points, $Z_{\overline{X}}$ and $Z_{\bar{e}}$. The search for the assignable cause 1 is undertaken when the sample mean point $Z_{\overline{X}}$ falls outside the interval $(-k_{\overline{X}},k_{\overline{X}})$ or/and when the sample mean point $Z_{\bar{e}}$ falls outside the interval $(-k_{\bar{e}},k_{\bar{e}})$, i.e. when $Z_{\overline{X}}$ or/and $Z_{\bar{e}}$ charts produce a signal. When the process mean has no shift on step 1 or 2, the signal is a false alarm. When the process mean shifts on step 1 or 2, the signal is a true alarm. For a discontinuous process, whatever the signal is false or true, after a signal

$$\begin{array}{ll}
\mathrm{UCL}_{z_{\overline{X}}} = k_{\overline{X}} & \mathrm{UCL}_{z_{\bar{e}}} = k_{\bar{e}} \\
UWL_{z_{\overline{X}}} = w_{\overline{X}} & \mathrm{UWL}_{z_{\bar{e}}} = w_{\bar{e}} \\
\mathrm{CL}_{z_{\overline{X}}} = 0 & \mathrm{CL}_{z_{\bar{e}}} = 0 \\
\mathrm{LWL}_{z_{\overline{X}}} = -w_{\overline{X}} & \mathrm{LWL}_{z_{\bar{e}}} = -w_{\bar{e}} \\
\mathrm{LCL}_{z_{\overline{X}}} = -k_{\overline{X}} & \mathrm{LCL}_{z_{\bar{e}}} = -k_{\bar{e}}
\end{array}$$

Fig. 1. The ASSI $Z_{\overline{X}}$ and $Z_{\bar{e}}$ control charts.

the process is stopped to search for and eliminate the assignable cause and the process is brought back to an in-control state.

The position of the sample mean points obtained from the current sample in each chart constructs the size of the next sample and the time of its sampling.

We divide the proposed ASSI $Z_{\overline{X}}$ and $Z_{\overline{e}}$ control charts into the following four regions:

$$\begin{aligned}
&I_{\overline{X}1} = \left[-w_{\overline{X}}, w_{\overline{X}}\right], && I_{\overline{e}1} = [-w_{\overline{e}}, w_{\overline{e}}],\\
&I_{\overline{X}2} = \left(-k_{\overline{X}}, -w_{\overline{X}}\right) \cup \left(w_{\overline{X}}, k_{\overline{X}}\right), && I_{\overline{e}2} = (-k_{\overline{e}}, -w_{\overline{e}}) \cup (w_{\overline{e}}, k_{\overline{e}}),\\
&I_{\overline{X}3} = \left(-k_{\overline{X}}, k_{\overline{X}}\right), && I_{\overline{e}3} = (-k_{\overline{e}}, k_{\overline{e}}),\\
&I_{\overline{X}4} = \left(-\infty, -k_{\overline{X}}\right) \cup \left(k_{\overline{X}}, \infty\right), && I_{\overline{e}4} = (-\infty, -k_{\overline{e}}) \cup (k_{\overline{e}}, \infty).
\end{aligned}$$

If the sample mean point, $Z_{\overline{X}}$, falls within the interval $I_{\overline{X}1}$ and the sample mean point, $Z_{\overline{e}}$, falls within the interval $I_{\overline{e}1}$, then the next sample should be small (n_1) and it should be sampled after a long time interval (t_3). If the sample mean points, $Z_{\overline{X}}$ and $Z_{\overline{e}}$, fall within the interval $I_{\overline{X}1}$ and $I_{\overline{e}2}$ or $I_{\overline{X}2}$ and $I_{\overline{e}1}$, then the next sample should be middle (n_2) and it should be sampled after a middle time interval (t_2). If the sample mean points, $Z_{\overline{X}}$ and $Z_{\overline{e}}$, fall within the interval $I_{\overline{X}2}$ and $I_{\overline{e}2}$, then the next sample should be large (n_3) and it should be sampled after a short time interval (t_1).

The relationship between the next sample size and sampling interval ($t(j)$, $n(j)$, $j = 1,2,3$) and the position of the current sample statistics ($Z_{\overline{X}}, Z_{\overline{e}}$) is expressed as follows:

$$(t(j), n(j)) = \begin{cases} (t_3, n_1) & \text{if } Z_{\overline{X},i-1} \in I_{\overline{X}1} \cap Z_{\overline{e},i-1} \in I_{\overline{e}1},\\ (t_2, n_2) & \text{if } Z_{\overline{X},i-1} \in I_{\overline{X}1} \cap Z_{\overline{e},i-1} \in I_{\overline{e}2},\\ (t_2, n_2) & \text{if } Z_{\overline{X},i-1} \in I_{\overline{X}2} \cap Z_{\overline{e},i-1} \in I_{\overline{e}1},\\ (t_1, n_3) & \text{if } Z_{\overline{X},\text{i-1}} \in I_{\overline{X}2} \cap \mathrm{Z}_{\overline{e},\text{i-1}} \in I_{\overline{e}2}. \end{cases} \tag{2.2}$$

Following Costa (1997), the first sample size taken from the process when it is just starting is chosen randomly. If the chosen sample was large (n_3), then it should be sampled after a short time interval (t_1); If the chosen sample was middle (n_2), then it should be sampled after a middle time interval (t_2); If the chosen sample was small (n_1), then it should be sampled after a long time interval (t_3). During the in-control process, all samples, including the first one, should have a probability of p_0 of being small, a probability of p_1 of being middle, and a probability of $1-p_0-p_1$ of being large, where p_0 and p_1 are given by

$$p_0 = P_r(|Z_{\overline{X}}| < w_{\overline{X}} \| Z_{\overline{X}}| < k_{\overline{X}}) * P_r(|Z_{\overline{e}}| < w_{\overline{e}} \| Z_{\overline{e}}| < k_{\overline{e}}),$$

$$\begin{aligned} p_1 = {} & P_r(|Z_{\overline{X}}| < w_{\overline{X}} \| Z_{\overline{X}}| < k_{\overline{X}}) * (1 - P_r(|Z_{\overline{e}}| < w_{\overline{e}} \| Z_{\overline{e}}| < k_{\overline{e}}))\\ & + (1 - P_r(|Z_{\overline{X}}| < w_{\overline{X}} \| Z_{\overline{X}}| < k_{\overline{X}}))\\ & * P_r(|Z_{\overline{e}}| < w_{\overline{e}} \| Z_{\overline{e}}| < k_{\overline{e}}), \end{aligned}$$

where $Z_{\overline{X}} \sim N(0,1)$ and $Z_{\overline{e}} \sim N(0,1)$.

To facilitate the computation of the performance measures, $w_{\overline{X}}$, $k_{\overline{X}}$, $w_{\overline{e}}$ and $k_{\overline{e}}$ will be specified with the constraint that the probability of a sample point falling in the warning limits when the process is in control is same for both the $Z_{\overline{X}}$ and $Z_{\overline{e}}$ charts. Thus, $P_r(|Z_{\overline{X}}| < w_{\overline{X}} \| Z_{\overline{X}}| < k_{\overline{X}}) = P_r(|Z_{\overline{e}}| < w_{\overline{e}} \| Z_{\overline{e}}| < k_{\overline{e}})$. Alternatively, $w_{\overline{X}} = w_{\overline{e}} = w$ and $k_{\overline{X}} = k_{\overline{e}} = k$.

When $w_{\overline{X}} = w_{\overline{e}} = 0$, $n_1 = n_2 = n_3 = n_0$, and $t_1 = t_2 = t_3 = t_0$, the joint $Z_{\overline{X}}$ and $Z_{\overline{e}}$ charts have a fixed sample size of n_0 and a fixed sampling interval of t_0 and is called a joint FSSI $Z_{\overline{X}}$ and $Z_{\overline{e}}$ charts. When $n_1 = n_2 = n_3 = n_0$ and $t_1 < t_0 < t_2 < t_3$, the sample size is fixed at n_0 and the sampling interval is variable; hence the joint ASSI $Z_{\overline{X}}$ and $Z_{\overline{e}}$ charts are called joint ASI $Z_{\overline{X}}$ and $Z_{\overline{e}}$ charts. When $t_1 = t_2 = t_3 = t_0$ and $n_1 < n_2 < n_0 < n_3$, the sample size is variable and the sampling interval is fixed at t_0 and the joint ASSI $Z_{\overline{X}}$ and $Z_{\overline{e}}$ charts are called joint ASS $Z_{\overline{X}}$ and $Z_{\overline{e}}$ charts.

3. Comparing control charts

Sampling schemes should be compared under equal conditions; i.e. they should demand the same average sample size and the same average sampling interval of time under the in-control period, i.e.

$$E\left[n(j) | \delta_1 = 0, \delta_2 = 0, |Z_{\overline{X}}| < k, |Z_{\overline{e}}| < k\right] = n_0 \tag{3.1}$$

and

$$E\left[t(j) \middle| \delta_1 = 0, \delta_2 = 0, |Z_{\overline{X}}| < k, |Z_{\overline{e}}| < k\right] = t_0. \tag{3.2}$$

Based on the Eq. (3.1), Eq. (3.3) can be formulated as follows:

$$\begin{aligned} & n_1 * P\left(Z_{\overline{X},i-1} \in I_{\overline{X}1} | \delta_1 = 0, \delta_2 = 0\right) * P(Z_{\overline{e},i-1} \in I_{\overline{e}1} | \delta_1 = 0, \delta_2 = 0)\\ & \quad + n_2 * P\left(Z_{\overline{X},i-1} \in I_{\overline{X}1} | \delta_1 = 0, \delta_2 = 0\right) * P(Z_{\overline{e},i-1} \in I_{\overline{e}2} | \delta_1 = 0, \delta_2 = 0)\\ & \quad + n_2 * P\left(Z_{\overline{X},i-1} \in I_{\overline{X}2} | \delta_1 = 0, \delta_2 = 0\right) * P(Z_{\overline{e},i-1} \in I_{\overline{e}1} | \delta_1 = 0, \delta_2 = 0)\\ & \quad + n_3 * P\left(Z_{\overline{X},i-1} \in I_{\overline{X}2} | \delta_1 = 0, \delta_2 = 0\right) * P(Z_{\overline{e},i-1} \in I_{\overline{e}2} | \delta_1 = 0, \delta_2 = 0)\\ & \quad + 0 * P(\text{false alarms})\\ & = n_0(2\Phi(k) - 1)^2. \end{aligned} \tag{3.3}$$

Simplifying,

$$\begin{aligned} & 4\Phi(w)^2[n_1 - 2n_2 + n_3] + 4\Phi(w)[-n_1 + 2n_2\Phi(k) + n_2 - 2n_3\Phi(k)]\\ & \quad = n_0(2\Phi(k) - 1)^2 - n_1 + 4n_2\Phi(k) - 4n_3(\Phi(k))^2, \end{aligned} \tag{3.4}$$

where $\Phi(\cdot)$ denotes the standard normal cumulative function.

The warning limits are derived as follows:

$$w = \Phi^{-1}\left(\frac{-4B_1 \pm \sqrt{16B_1^2 - 16A_1C_1}}{8A_1}\right), \tag{3.5}$$

where

$$\begin{aligned} A_1 &= n_1 - 2n_2 + n_3,\\ B_1 &= -n_1 + 2n_2\Phi(k) + n_2 - 2n_3\Phi(k),\\ C_1 &= -[n_0(2\Phi(k) - 1)^2 - n_1 + 4n_2\Phi(k) - 4n_3(\Phi(k))^2]. \end{aligned}$$

Similarly, another expression for the warning limits (see Eq. (3.6)) can be obtained by simplifying Eq. (3.2).

It follows

$$w = \Phi^{-1}\left(\frac{-4B_2 \pm \sqrt{16B_2^2 - 16A_2C_2}}{8A_2}\right), \tag{3.6}$$

where

$$A_2 = t_3 - 2t_2 + t_1,$$
$$B_2 = -t_3 + 2t_2\Phi(k) + t_2 - 2t_1\Phi(k),$$
$$C_2 = -[t_0(2\Phi(k) - 1)^2 - t_3 + 4t_2\Phi(k) - 4t_1(\Phi(k))^2].$$

Thus, the warning limit can be obtained by using Eq. (3.5) and choosing a combination of the three variable sample sizes (n_1,n_2,n_3,), or using Eq. (3.6) and choosing a combination of the three variable sampling intervals of time (t_1,t_2,t_3). Prabhu, Montgomery, and Runger (1994) recommend that choosing the sample sizes to obtain the warning threshold w avoids the approximation error associated with rounding off to the nearest integer. Thus, two of the three parameters t_1, t_2 and t_3 need to be determined. We prefer to determine t_1 and t_2 based on two reasons. One is that the minimum sampling interval often depends on the type of inspection and sampling method. The other is that the values of t_1 should be less than t_0. By fixing parameters n_1, n_2 and n_3, we obtain w by using Eq. (3.5). Using the obtained w and the pre-determined t_1 and t_2, the last parameter t_3 is then determined by Eq. (3.6), i.e.

$$t_3 = \frac{-4(\Phi(w))^2(-2t_2 + t_1) - 4\Phi(w)(2\Phi(k)t_2 + t_2 - 2\Phi(k)t_1) + t_0(2\Phi(k) - 1)^2 + 4\Phi(k)t_2 - 4(\Phi(k))^2 t_1}{4(\Phi(w))^2 - 4\Phi(w) + 1}. \tag{3.7}$$

In this paper, the ASSI scheme is compared with the FSSI scheme; one sampling scheme was considered to be better than another when it allowed the joint $Z_{\overline{X}}$ and $Z_{\bar{e}}$ charts to detect changes in the process means on step 1 and step 2 faster.

4. Measurement of performance

The speed with which a control chart detects process shifts measures its statistical efficiency. When the sampling interval is variable, the detection speed is measured by the average time from any one mean of the two process steps shift until either $Z_{\overline{X}}$ or $Z_{\bar{e}}$ chart or both signals, which is known as the adjusted average time to signal (AATS). The AATS is the mean time the process remains out-of-control. Since $T_i \sim \exp(-\lambda_i t_i)$ $t_i > 0$, $i = 1, 2$, thus $T \sim \exp(\lambda_1 + \lambda_2)$, where T is the occurrence time until the first assignable cause occurs; $T = \min(T_1, T_2)$.

Hence,

$$AATS = ATC - \frac{1}{\lambda_1 + \lambda_2}. \tag{4.1}$$

The average time of the cycle (ATC) is the average ime from the start of process until the first signal from any one of the proposed control charts. The Markov chain approach is allowed to compute the ATC since memory-less property of the exponential distribution, thus, at each sampling, one of 16 transient state is assigned based on if the process step is in- or out-of-control and the position of sample means (or if the sample size is large, middle or small) (see Table 1 for the 17 states of the process).

The status of the process when the $(i+1)$th sample is taken, the position of the ith sample mean on the $Z_{\overline{X}}$ chart, and the position of the ith sample mean on the $Z_{\bar{e}}$ chart define the transient states of the Makov chain. The joint ASSI $Z_{\overline{X}}$ and $Z_{\bar{e}}$ charts produce a signal when at least one of the sample mean points fall outside the control limits. If the current state is any one of the States 1–16, then there is no signal. The absorbing state (State 17) is reached when a signal occurs.

Denote $\mathbf{P}$ be the transition probability matrix, and $\mathbf{P}$ is a square matrix of order 17. Denote $P_{i,m}(t_j,n_j)$ to be the transition probability from prior state i to the current state m with sample size n_j and sampling interval t_j, where n_j and t_j are determined by the prior state i, $i = 1,2,\ldots,17$, $m = 1,2,\ldots,17$ and $j = 1,2,3$. The transition probability, for example, from states 1 to 4 with sample size n_1 and sampling interval t_3 is calculated as

$$\begin{aligned} p_{1,4}(t_3, n_1) &= P[|Z_{\bar{x}}| < w|\delta_1] * P[|Z_{\bar{e}}| < w|\delta_2] \\ &\quad * (1 - \mathrm{e}^{-\lambda_1 t_3}) * (1 - \mathrm{e}^{-\lambda_2 t_3}) \\ &= \left(\Phi\left(w - \delta_1\sqrt{n_1}\right) - \Phi\left(-w - \delta_1\sqrt{n_1}\right)\right) \\ &\quad * \left(\Phi\left(w - \delta_2\sqrt{n_1}\right) - \Phi\left(-w - \delta_2\sqrt{n_1}\right)\right) \\ &\quad * (1 - \mathrm{e}^{-\lambda_1 t_3}) * (1 - \mathrm{e}^{-\lambda_2 t_3}). \end{aligned}$$

The calculation of all transition probabilities is shown in Appendix A.

From the elementary properties of Markov chains (see, e.g. Cinlar, 1975), the ATC is derived as

$$ATC = \mathbf{b}'(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{t}, \tag{4.2}$$

where $\mathbf{b}' = \left(p_0, 0, 0, 0, p_1/2, 0, 0, 0, p_1/2, 0, 0, 0, p_2, 0, \ldots, 0\right)$ is the vector of starting probabilities for state 1, 2,3,…,16, where the first sample has probability p_0 of being small (or state 1 with probability p_0), the probability $p_1/2$ of being middle (or state 5 or state 9 with probability $p_1/2$) and the probability p_2 of being large (or state 13 with probability p_2); $\mathbf{I}$ is the identity matrix of order 16; $\mathbf{Q}$ is the transition probability matrix where elements represent the

Table 1
The 17 process states

State	Does AC1 occur?	The location of sample statistic Z_X	Is an alarm in the first process?	Does AC2 occur?	The location of sample statistic Z_e	Is an alarm in the second process?	Transient state or absorbing state?
1	No	$I_{\overline{X}1}$	No alarm	No	$I_{\overline{e}1}$	No alarm	Transient state
2	No	$I_{\overline{X}1}$		Yes	$I_{\overline{e}1}$		
3	Yes	$I_{\overline{X}1}$		No	$I_{\overline{e}1}$		
4	Yes	$I_{\overline{X}1}$		Yes	$I_{\overline{e}1}$		
5	No	$I_{\overline{X}2}$		No	$I_{\overline{e}1}$		
6	No	$I_{\overline{X}2}$		Yes	$I_{\overline{e}1}$		
7	Yes	$I_{\overline{X}2}$		No	$I_{\overline{e}1}$		
8	Yes	$I_{\overline{X}2}$		Yes	$I_{\overline{e}1}$		
9	No	$I_{\overline{X}1}$		No	$I_{\overline{e}2}$		
10	No	$I_{\overline{X}1}$		Yes	$I_{\overline{e}2}$		
11	Yes	$I_{\overline{X}1}$		No	$I_{\overline{e}2}$		
12	Yes	$I_{\overline{X}1}$		Yes	$I_{\overline{e}2}$		
13	No	$I_{\overline{X}2}$		No	$I_{\overline{e}2}$		
14	No	$I_{\overline{X}2}$		Yes	$I_{\overline{e}2}$		
15	Yes	$I_{\overline{X}2}$		No	$I_{\overline{e}2}$		
16	Yes	$I_{\overline{X}2}$		Yes	$I_{\overline{e}2}$		
17	Except	The above		Situations			Absorbing state

transition probability, $P_{i,m}(t_j,n_j)$, from transient state i, $i = 1, \ldots, 16$, to transient state m, $m = 1, \ldots, 16$; and $\mathbf{t}' = (t_3,t_3,t_3,t_3,t_2,t_2,t_2,t_2,t_2,t_2,t_2,t_2,t_1,t_1,t_1,t_1)$ is the vector of the sampling intervals for states 1–16.

5. An example

An example of process control of a component part for the braking system of automobiles is presented. To detect changes in the means of process steps faster, the ASSI control charts are used. The construction and application of the proposed ASSI $Z_{\overline{X}}$ and $Z_{\overline{e}}$ control charts is illustrated. The data of the process are measurements of a component part for the braking system of automobiles. The variables $X = ROLLWT$ was measured from the first process step and $Y = BAKEWT$ was measured from the second process step. Each sampling, a sample of size n_j is taken at the end of the second process step, and the paired observations (X_i, Y_i), $i = 1,2, \ldots, n_j$, are measured. The out-of-control time (T_1) of machine 1 follows an exponential distribution with parameter $\lambda_1 = 0.03$, and the out-of-control time (T_2) of machine 2 follows an exponential distribution with parameter $\lambda_2 = 0.04$. The out-of-control machines 1 and 2 only influence the process means of X and Y, respectively, and the process standard deviations are unaffected.

Based on the history data analyses, both variables X and Y follow a normal distribution. The variable Y is affected by the variable X, and their relationship is expressed by a linear regression model using the collected samples. The model is

$$\hat{Y}|X = 66.8 + 0.639X. \tag{5.1}$$

Thus, the residual (e) is obtained by $Y|X - \hat{Y}|X$. The estimated means and standard deviations of variables X and e are $\hat{\mu}_X = 210.1$, $\hat{\sigma}_X = 1.23$, $\hat{\mu}_e = 0$ and $\hat{\sigma}_e = 1.11$, respectively. That is, when both process steps are in-control, $\overline{X}\sim N\left(210.1, \left(1.23/\sqrt{n_0}\right)^2\right)$ and $\overline{e}\sim N\left(0, \left(1.11/\sqrt{n_0}\right)^2\right)$ when a sample of size n_0 is taken. For out-of-control process, the estimated process mean shift on the first process step is $\delta_1\hat{\sigma}_X$, where $\hat{\delta}_1 = 0.25$; the estimated process mean shift on the second process step is $\delta_2\hat{\sigma}_e$, where $\hat{\delta}_1 = 0.5$. Hence, for out-of-control process step 1, $\overline{X}\sim N\left(210.1 + 1.23 * 0.25, \left(1.23/\sqrt{n_0}\right)\right)^2$; for out-of-control process step 2, $\overline{e}\sim N\left(1.11 * 0.5, 1.11/\sqrt{n_0}\right)^2$.

Presently, 5 items are sampled every 1 h ($n_0 = 5$, $t_0 = 1$), with the standardized $\bar{x}$ and $\overline{e}$ values plotted on the charts with control limits placed at ± 3, respectively. Thus, approximately 5.4 false alarms are expected for each 1000 samples.

The low speed with which the joint $Z_{\overline{X}}$ and $Z_{\overline{e}}$ control charts detects shift in the process ($\hat{\delta}_1 = 0.25$ and $\hat{\delta}_1 = 0.5$) has led the quality manager to propose building the $Z_{\overline{X}}$ and $Z_{\overline{e}}$ control charts with variable sample sizes and sampling intervals. The guidelines of using the proposed charts are as follows.

Step 1: Let $k = 3$, to maintain the average false alarm rate around 5.4 for 1000 samples.

Step 2: To determine $n_1 = 2$, $n_2 = 3$, $n_3 = 20$, $t_1 = 0.01$ h and $t_2 = 0.7$ h.

Step 3: Using Eqs. (2.5) and (2.6), it is calculated that $w = 0.88$ and $t_3 = 1.74$ h.

With the design parameters determined, the joint ASSI $Z_{\overline{X}}$ and $Z_{\overline{e}}$ control charts can be used for monitoring the two dependent process steps of braking system of automobiles. When the process starts, a random procedure

Table 2
Comparisons the AATS of the ASSI and FSSI control charts when $\lambda_1 = 0.03$ and $\lambda_2 = 0.04$

Parameters							δ_1	0.25					0.5					0.75				
t_1	t_2	t_3	n_1	n_2	n_3	w	δ_2	0.25	0.5	0.75	1	1.25	0.25	0.5	0.75	1	1.25	0.25	0.5	0.75	1	1.25
0.09	0.9	1.47	2	3	20	0.88		53.05	20.10	13.65	12.19	11.60	23.91	12.83	8.82	7.75	7.31	18.25	9.95	6.42	5.47	5.08
0.01	0.7	1.74	2	3	20	0.88		53.23	20.92	14.76	13.37	12.81	24.73	13.77	9.90	8.88	8.46	19.34	10.99	7.57	6.65	6.28
0.09	0.9	1.27	2	3	30	1.02		46.88	17.63	12.58	11.19	10.45	21.39	10.88	7.55	6.52	5.97	17.01	8.48	5.49	4.58	4.09
0.01	0.7	1.46	2	3	30	1.02		46.99	18.28	13.37	12.00	11.29	22.04	11.59	8.33	7.32	6.78	17.79	9.24	6.31	5.41	4.94
0.09	0.9	1.28	3	4	20	1.02		55.09	20.09	12.61	10.93	10.27	23.89	12.45	7.92	6.69	6.20	17.28	9.19	5.29	4.20	3.77
0.01	0.7	1.47	3	4	20	1.02		54.83	20.46	13.32	11.75	11.15	24.27	12.97	8.60	7.45	7.00	17.99	9.83	6.03	5.01	4.61
0.09	0.9	1.17	3	4	30	1.15		50.15	18.02	11.99	10.37	9.57	21.80	10.92	7.07	5.89	5.29	16.55	8.15	4.77	3.72	3.19
0.01	0.7	1.31	3	4	30	1.15		49.90	18.32	12.49	10.92	10.19	22.12	11.31	7.55	6.41	5.86	17.04	8.61	5.29	4.27	3.79
FSSI control charts								67.54	30.42	15.23	10.29	8.54	33.90	20.65	11.23	7.33	5.86	19.94	13.16	7.16	4.23	3.04
(AATS of FSSI charts)-min(AATS of ASSI Charts)								20.66	12.80	3.24	−0.08	−1.03	12.51	9.77	4.15	1.45	0.57	3.39	5.01	2.40	0.51	−0.16
(AATS of FSSI charts)-max(AATS of ASSI Charts)								12.44	9.50	0.47	−3.08	−4.27	9.16	6.89	1.33	−1.55	−2.60	0.60	2.17	−0.40	−2.42	−3.24

Parameters							δ_1	1					1.25				
t_1	t_2	t_3	n_1	n_2	n_3	w	δ_2	0.25	0.5	0.75	1	1.25	0.25	0.5	0.75	1	1.25
0.09	0.70	1.71	2	3	20	0.88		16.95	9.13	5.73	4.80	4.43	16.42	8.78	5.43	4.53	4.16
0.10	0.90	1.46	2	3	20	0.88		18.10	10.21	6.90	6.01	5.65	17.60	9.88	6.62	5.75	5.39
0.09	0.70	1.45	2	3	30	1.02		15.77	7.68	4.82	3.96	3.49	15.10	7.25	4.46	3.62	3.16
0.10	0.90	1.27	2	3	30	1.02		16.56	8.46	5.65	4.80	4.35	15.92	8.04	5.30	4.46	4.02
0.09	0.70	1.46	3	4	20	1.02		15.78	8.25	4.50	3.45	3.03	15.19	7.86	4.17	3.14	2.73
0.10	0.90	1.28	3	4	20	1.02		16.58	8.95	5.29	4.29	3.91	16.04	8.60	4.99	4.01	3.63
0.09	0.70	1.31	3	4	30	1.15		15.10	7.24	4.00	3.00	2.50	14.38	6.77	3.61	2.63	2.13
0.10	0.90	1.17	3	4	30	1.15		15.64	7.73	4.55	3.58	3.11	14.98	7.30	4.19	3.23	2.77
FSSI control charts								15.34	9.90	4.98	2.48	1.43	13.72	8.64	4.04	1.70	0.72
(AATS of FSSI charts)-min(AATS of ASSI Charts)								0.25	2.67	0.97	−0.52	−1.06	−0.66	1.88	0.43	−0.93	−1.42
(AATS of FSSI charts)-max(AATS of ASSI Charts)								−2.76	−0.31	−1.92	−3.53	−4.22	−3.88	−1.23	−2.58	−4.04	−4.68

Table 3
Comparisons the AATS of the ASSI and FSSI control charts when $\lambda_1 = 0.05$ and $\lambda_2 = 0.05$

Parameters							δ_1	0.25					0.5					0.75				
t_1	t_2	t_3	n_1	n_2	n_3	w	δ_2	0.25	0.5	0.75	1	1.25	0.25	0.5	0.75	1	1.25	0.25	0.5	0.75	1	1.25
0.09	0.9	1.47	2	3	20	0.88		51.32	19.47	13.24	11.80	11.21	19.47	11.79	8.56	7.60	7.18	13.24	8.56	5.95	5.14	4.80
0.01	0.7	1.74	2	3	20	0.88		51.10	19.95	14.01	12.63	12.07	19.95	12.41	9.30	8.38	7.98	14.01	9.30	6.75	5.98	5.65
0.09	0.9	1.27	2	3	30	1.02		45.28	17.38	12.55	11.17	10.43	17.38	10.34	7.63	6.70	6.18	12.55	7.63	5.36	4.59	4.16
0.01	0.7	1.46	2	3	30	1.02		45.09	17.78	13.08	11.72	11.00	17.78	10.81	8.15	7.23	6.73	13.08	8.15	5.92	5.16	4.74
0.09	0.9	1.28	3	4	20	1.02		53.83	19.77	12.52	10.86	10.20	19.77	11.61	7.97	6.88	6.42	12.52	7.97	5.14	4.23	3.84
0.01	0.7	1.47	3	4	20	1.02		53.24	19.87	12.97	11.41	10.81	19.87	11.90	8.40	7.37	6.95	12.97	8.40	5.62	4.76	4.41
0.09	0.9	1.17	3	4	30	1.15		48.89	17.92	12.14	10.53	9.73	17.92	10.45	7.34	6.27	5.71	12.14	7.34	4.82	3.94	3.48
0.01	0.7	1.31	3	4	30	1.15		48.39	18.02	12.43	10.87	10.14	18.02	10.66	7.62	6.59	6.08	12.43	7.62	5.15	4.29	3.86
FSSI control charts								67.68	31.09	15.85	10.82	9.04	31.09	20.38	11.89	8.11	6.64	15.85	11.89	7.48	4.94	3.82
(AATS of FSSI charts)-min(AATS of ASSI charts)								22.59	13.70	3.71	0.29	−0.69	13.70	10.04	4.55	1.84	0.93	3.71	4.55	2.66	1.00	0.35
(AATS of FSSI charts)-max(AATS of ASSI charts)								13.84	11.14	1.84	−1.81	−3.03	11.14	7.98	2.59	−0.27	−1.34	1.84	2.59	0.73	−1.04	−1.82

Parameters							δ_1	1					1.25				
t_1	t_2	t_3	n_1	n_2	n_3	w	δ_2	0.25	0.5	0.75	1	1.25	0.25	0.5	0.75	1	1.25
0.09	0.70	1.71	2	3	20	0.88		11.80	7.60	5.14	4.38	4.06	11.21	7.18	4.80	4.06	3.75
0.10	0.90	1.46	2	3	20	0.88		12.63	8.38	5.98	5.24	4.93	12.07	7.98	5.65	4.93	4.63
0.09	0.70	1.45	2	3	30	1.02		11.17	6.70	4.59	3.88	3.48	10.43	6.18	4.16	3.48	3.09
0.10	0.90	1.27	2	3	30	1.02		11.72	7.23	5.16	4.45	4.06	11.00	6.73	4.74	4.06	3.69
0.09	0.70	1.46	3	4	20	1.02		10.86	6.88	4.23	3.37	3.01	10.20	6.42	3.84	3.01	2.65
0.10	0.90	1.28	3	4	20	1.02		11.41	7.37	4.76	3.95	3.61	10.81	6.95	4.41	3.61	3.29
0.09	0.70	1.31	3	4	30	1.15		10.53	6.27	3.94	3.12	2.69	9.73	5.71	3.48	2.69	2.27
0.10	0.90	1.17	3	4	30	1.15		10.87	6.59	4.29	3.49	3.09	10.14	6.08	3.86	3.09	2.70
FSSI control charts								10.82	8.11	4.94	2.98	2.06	9.04	6.64	3.82	2.06	1.24
(AATS of FSSI charts)-min(AATS of ASSI charts)								0.29	1.84	1.00	−0.14	−0.63	−0.69	0.93	0.35	−0.63	−1.04
(AATS of FSSI charts)-max(AATS of ASSI Charts)								−1.81	−0.27	−1.04	−2.27	−2.87	−3.03	−1.34	−1.82	−2.87	−3.40

decides the first sample size and sampling intervals are scheduled at time 0.7 h using a sample of size 3, and the 3 paired observations of (X_i, Y_i) are (209, 201), (212, 203) and (208, 199). The sample statistics $(\overline{X}, \overline{e})$ of the first sample are (209.67, 0.22) and the values of $Z_{\overline{X}}$ and $Z_{\overline{e}}$ are (−0.61, 0.34). Both sample statistics fall in the warning regions. The second sample will be observed (i.e. at $0.7+1.74=2.44$ h) using a sample of size 2. The two observed data (X_i, Y_i) are (210, 200) and (208, 199). Since $Z_{\overline{X}}=-1.26$ and $Z_{\overline{e}}=-1.08$, the sample means fall inside the central regions. The next sample is now scheduled at time $2.44+0.01=2.45$ h using a sample of size 20. The process is stopped when at least one of the sample means fall outside its control limits.

The ASSI scheme improves the sensitivity of the joint $Z_{\overline{X}}$ and $Z_{\overline{e}}$ charts. From the example, in order to detect a shift in the process mean, the AATS for the joint $Z_{\overline{X}}$ and $Z_{\overline{e}}$ charts has been reduced from 30.42 h to only 20.1 h.

6. Performance comparisons between ASSI and FSSI schemes

Tables 2 and 3 provide the AATS of ASSI and FSSI schemes, which are obtained with parameters $\lambda_1=0.03$, 0.05, $\lambda_2=0.04$,0.05, $\delta_1=0.25$–1.25, $\delta_2=0.25$–1.25, $t_1=0.01$, 0.09, $t_2=0.7$,0.9, $n_1=2$,3, $n_2=3$,4 and $n_3=20$,30. The control limits for the matched FSSI charts are set at $k=3$ to give reasonable protection against average 5.4 false alarms per 1000 samples. Tables 2 and 3 give the following results:

1. The ASSI scheme detects changes in the mean on process step 1 and/or step 2 faster than the FSSI scheme for small δ_1 and δ_2 values ($\delta_1<0.75$ and $\delta_2<0.75$). The AATS for the ASSI scheme reduces detection time from 0.5 h to 22.6 h under $\delta_1<0.75$ and $\delta_2<0.75$.
2. The ASSI charts sensitivity improves when t_3 decreases.
3. When t_3 decreases, the parameter w increases. w is practically dependent on t_3.

7. Conclusions

Extension of the approach proposed in Yang (1998) for the VSSI scheme on monitoring two dependent process steps substantially improves the performance of the FSSI scheme by increasing the speed with which small shifts in the mean of process step 1 and/or step 2 are detected. We have found that the joint VSSI $Z_{\overline{X}}$ and $Z_{\overline{e}}$ control charts always works better than the joint FSSI $Z_{\overline{X}}$ and $Z_{\overline{e}}$ control charts for small δ_1 and δ_2 values.

This paper used two assignable causes on two dependent process steps. However, a study of the joint ASSI $Z_{\overline{X}}$ and $Z_{\overline{e}}$ control charts under two dependent process steps with two dependent assignable causes, whereby one shifts the process mean and the other changes the process variance is an interesting topic for extended research. Other important extensions of the developed model can also be developed. It is straightforward to extend the proposed model to study AP control charts or other control charts, like EWMA-charts, or CUSUM-charts.

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Appendix A

The calculation of all transition probabilities $p_{im}(t_j, n_j)$ $i=1,\ldots,17$, $m=1,\ldots,17$, $j=1,2,3$.

$$
\begin{aligned}
p_{1,1}(t_3,n_1) &= P\left[|Z_{\overline{X}}|<w_{\overline{X}}\right] * P[|Z_{\overline{e}}|<w_{\overline{e}}] * \mathrm{e}^{-\lambda_1 t_3} * \mathrm{e}^{-\lambda_2 t_3},\\
p_{1,2}(t_3,n_1) &= P\left[|Z_{\overline{X}}|<w_{\overline{X}}\right] * P[|Z_{\overline{e}}|<w_{\overline{e}}|\delta_2] * \mathrm{e}^{-\lambda_1 t_3} * (1-\mathrm{e}^{-\lambda_2 t_3}),\\
p_{1,3}(t_3,n_1) &= P\left[|Z_{\overline{X}}|<w_{\overline{X}}|\delta_1\right] * P[|Z_{\overline{e}}|<w_{\overline{e}}] * (1-\mathrm{e}^{-\lambda_1 t_3}) * \mathrm{e}^{-\lambda_2 t_3},\\
p_{1,4}(t_3,n_1) &= P\left[|Z_{\overline{X}}|<w_{\overline{X}}|\delta_1\right] * P[|Z_{\overline{e}}|<w_{\overline{e}}|\delta_2] * (1-\mathrm{e}^{-\lambda_1 t_3}) * (1-\mathrm{e}^{-\lambda_2 t_3}),\\
p_{1,5}(t_3,n_1) &= P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}} \cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}\right] * P[|Z_{\overline{e}}|<w_{\overline{e}}] * \mathrm{e}^{-\lambda_1 t_3} * \mathrm{e}^{-\lambda_2 t_3},\\
p_{1,6}(t_3,n_1) &= P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}} \cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}\right] * P[|Z_{\overline{e}}|<w_{\overline{e}}|\delta_2] * \mathrm{e}^{-\lambda_1 t_3} * (1-\mathrm{e}^{-\lambda_2 t_3}),\\
p_{1,7}(t_3,n_1) &= P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}} \cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}|\delta_1\right] * P[|Z_{\overline{e}}|<w_{\overline{e}}] * (1-\mathrm{e}^{-\lambda_1 t_3}) * \mathrm{e}^{-\lambda_2 t_2},\\
p_{1,8}(t_3,n_1) &= P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}} \cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}|\delta_1\right] * P[|Z_{\overline{e}}|<w_{\overline{e}}|\delta_2] * (1-\mathrm{e}^{-\lambda_1 t_3}) * (1-\mathrm{e}^{-\lambda_2 t_3}),\\
P_{1,9}(t_3,n_1) &= P\left[|Z_{\overline{X}}|<w_{\overline{X}}\right] * P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}} \cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}] * \mathrm{e}^{-\lambda_1 t_3} * \mathrm{e}^{-\lambda_2 t_3},\\
P_{1,10}(t_3,n_1) &= P\left[|Z_{\overline{X}}|<w_{\overline{X}}\right] * P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}} \cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}|\delta_2] * \mathrm{e}^{-\lambda_1 t_3} * (1-\mathrm{e}^{-\lambda_2 t_3}),\\
P_{1,11}(t_3,n_1) &= P\left[|Z_{\overline{X}}|<w_{\overline{X}}|\delta_1\right] * P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}} \cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}] * (1-\mathrm{e}^{-\lambda_1 t_3}) * \mathrm{e}^{-\lambda_2 t_3},
\end{aligned}
$$

$$P_{1,12}(t_3,n_1)=P\left[\left|Z_{\overline{X}}\right|<w_{\overline{X}}|\delta_1\right]*P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}}\cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}|\delta_2]*(1-\mathrm{e}^{-\lambda_1 t_3})*(1-\mathrm{e}^{-\lambda_2 t_3}),$$
$$P_{1,13}(t_3,n_1)=P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}}\cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}\right]*P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}}\cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}]*\mathrm{e}^{-\lambda_1 t_3}*\mathrm{e}^{-\lambda_2 t_3},$$
$$P_{1,14}(t_3,n_1)=P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}}\cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}\right]*P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}}\cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}|\delta_2]*\mathrm{e}^{-\lambda_1 t_1}*(1-\mathrm{e}^{-\lambda_2 t_1}),$$
$$P_{1,15}(t_3,n_1)=P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}}\cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}|\delta_1\right]*P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}}\cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}]*(1-\mathrm{e}^{-\lambda_1 t_3})*\mathrm{e}^{-\lambda_2 t_3},$$
$$P_{1,16}(t_3,n_1)=P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}}\cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}|\delta_1\right]*P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}}\cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}|\delta_2]*(1-\mathrm{e}^{-\lambda_1 t_3})*(1-\mathrm{e}^{-\lambda_2 t_3}),$$
$$P_{1,17}(t_3,n_1)=1-\sum_{m=1}^{16}P_{1,m}(t_3,n_1),$$

$$p_{2,1}(t_3,n_1)=0,$$
$$p_{2,2}(t_3,n_1)=P\left[\left|Z_{\overline{X}}\right|<w_{\overline{X}}\right]*P[|Z_{\overline{e}}|<w_{\overline{e}}|\delta_2]*\mathrm{e}^{-\lambda_1 t_3},$$
$$p_{2,3}(t_3,n_1)=0,$$
$$p_{2,4}(t_3,n_1)=P\left[\left|Z_{\overline{X}}\right|<w_{\overline{X}}|\delta_1\right]*P[|Z_{\overline{e}}|<w_{\overline{e}}|\delta_2]*(1-\mathrm{e}^{-\lambda_1 t_3}),$$
$$p_{2,5}(t_3,n_1)=0,$$
$$p_{2,6}(t_3,n_1)=P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}}\cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}\right]*P[|Z_{\overline{e}}|<w_{\overline{e}}|\delta_2]*\mathrm{e}^{-\lambda_1 t_3},$$
$$p_{2,7}(t_3,n_1)=0,$$
$$p_{2,8}(t_3,n_1)=P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}}\cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}|\delta_1\right]*P[|Z_{\overline{e}}|<w_{\overline{e}}|\delta_2]*(1-\mathrm{e}^{-\lambda_1 t_3}),$$
$$p_{2,9}(t_3,n_1)=0,$$
$$p_{2,10}(t_3,n_1)=P\left[\left|Z_{\overline{X}}\right|<w_{\overline{X}}\right]*P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}}\cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}|\delta_2]*\mathrm{e}^{-\lambda_1 t_3},$$
$$p_{2,11}(t_3,n_1)=0,$$
$$p_{2,12}(t_3,n_1)=P\left[\left|Z_{\overline{X}}\right|<w_{\overline{X}}|\delta_1\right]*P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}}\cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}|\delta_2]*(1-\mathrm{e}^{-\lambda_1 t_3}),$$
$$p_{2,13}(t_3,n_1)=0,$$
$$p_{2,14}(t_3,n_1)=P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}}\cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}\right]*P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}}\cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}|\delta_2]*\mathrm{e}^{-\lambda_1 t_3},$$
$$p_{2,15}(t_3,n_1)=0,$$
$$p_{2,16}(t_3,n_1)=P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}}\cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}|\delta_1\right]*P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}}\cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}|\delta_2]*(1-\mathrm{e}^{-\lambda_1 t_3}),$$
$$p_{2,17}(t_3,n_1)=1-\sum_{m=1}^{16}P_{2,m}(t_3,n_1),$$

$$p_{3,1}(t_3,n_1)=0,$$
$$p_{3,2}(t_3,n_1)=0,$$
$$p_{3,3}(t_3,n_1)=P\left[\left|Z_{\overline{X}}\right|<w_{\overline{X}}|\delta_1\right]*P[|Z_{\overline{e}}|<w_{\overline{e}}]*\mathrm{e}^{-\lambda_2 t_3},$$
$$p_{3,4}(t_3,n_1)=P\left[\left|Z_{\overline{X}}\right|<w_{\overline{X}}|\delta_1\right]*P[|Z_{\overline{e}}|<w_{\overline{e}}|\delta_2]*(1-\mathrm{e}^{-\lambda_2 t_3}),$$
$$p_{3,5}(t_3,n_1)=0,$$
$$p_{3,6}(t_3,n_1)=0,$$
$$p_{3,7}(t_3,n_1)=P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}}\cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}|\delta_1\right]*P[|Z_{\overline{e}}|<w_{\overline{e}}]*\mathrm{e}^{-\lambda_2 t_3},$$
$$p_{3,8}(t_3,n_1)=P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}}\cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}|\delta_1\right]*P[|Z_{\overline{e}}|<w_{\overline{e}}|\delta_2]*(1-\mathrm{e}^{-\lambda_2 t_3}),$$
$$p_{3,9}(t_3,n_1)=0,$$
$$p_{3,10}(t_3,n_1)=0,$$
$$p_{3,11}(t_3,n_1)=P\left[\left|Z_{\overline{X}}\right|<w_{\overline{X}}|\delta_1\right]*P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}}\cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}]*\mathrm{e}^{-\lambda_2 t_3},$$
$$p_{3,12}(t_3,n_1)=P\left[\left|Z_{\overline{X}}\right|<w_{\overline{X}}|\delta_1\right]*P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}}\cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}|\delta_2]*(1-\mathrm{e}^{-\lambda_2 t_3}),$$
$$p_{3,13}(t_3,n_1)=0,$$
$$p_{3,14}(t_3,n_1)=0,$$
$$p_{3,15}(t_3,n_1)=P\left[-k_{\overline{X}}<Z_{\overline{X}}\leqslant -w_{\overline{X}}\cup w_{\overline{X}}\leqslant Z_{\overline{X}}<k_{\overline{X}}|\delta_1\right]*P[-k_{\overline{e}}<Z_{\overline{e}}\leqslant -w_{\overline{e}}\cup w_{\overline{e}}\leqslant Z_{\overline{e}}<k_{\overline{e}}]*\mathrm{e}^{-\lambda_2 t_3},$$

$$p_{3,16}(t_3,n_1) = P\left[-k_{\overline{X}} < Z_{\overline{X}} \leqslant -w_{\overline{X}} \cup w_{\overline{X}} \leqslant Z_{\overline{X}} < k_{\overline{X}}|\delta_1\right] * P[-k_{\overline{e}} < Z_{\overline{e}} \leqslant -w_{\overline{e}} \cup w_{\overline{e}} \leqslant Z_{\overline{e}} < k_{\overline{e}}|\delta_2] * (1-\mathrm{e}^{-\lambda_2 t_3}),$$

$$p_{3,17}(t_3,n_1) = 1 - \sum_{m=1}^{16} P_{3,m}(t_3,n_1),$$

$$\begin{aligned}
&P_{4,1}(t_3,n_1) = P_{4,2}(t_3,n_1) = P_{4,3}(t_3,n_1) = 0,\\
&P_{4,4}(t_3,n_1) = P\left(\left|Z_{\overline{X}}\right| < w_{\overline{X}}|\delta_1\right) * P(Z_{\overline{e}} < w_{\overline{e}}|\delta_2),\\
&P_{4,5}(t_3,n_1) = P_{4,6}(t_3,n_1) = P_{4,7}(t_3,n_1) = 0,\\
&P_{4,8}(t_3,n_1) = P\left(-k_{\overline{X}} < Z_{\overline{X}} \leqslant w_{\overline{X}} \cup w_{\overline{X}} \leqslant Z_{\overline{X}} < k_{\overline{X}}|\delta_1\right) * P(Z_{\overline{e}} < w_{\overline{e}}|\delta_2),\\
&P_{4,9}(t_3,n_1) = P_{4,10}(t_3,n_1) = P_{4,11}(t_3,n_1) = 0,\\
&P_{4,12}(t_3,n_1) = P\left(\left|Z_{\overline{X}}\right| < w_{\overline{X}}|\delta_1\right) * P(-k_{\overline{e}} < Z_{\overline{e}} \leqslant w_{\overline{e}} \cup w_{\overline{e}} \leqslant Z_{\overline{e}} < k_{\overline{e}}|\delta_2),\\
&P_{4,13}(t_3,n_1) = P_{4,14}(t_3,n_1) = P_{4,15}(t_3,n_1) = 0,\\
&P_{4,16}(t_3,n_1) = P\left(-k_{\overline{X}} < Z_{\overline{X}} \leqslant w_{\overline{X}} \cup w_{\overline{X}} \leqslant Z_{\overline{X}} < k_{\overline{X}}|\delta_1\right) * P(-k_{\overline{e}} < Z_{\overline{e}} \leqslant w_{\overline{e}} \cup w_{\overline{e}} \leqslant Z_{\overline{e}} < k_{\overline{e}}|\delta_2),\\
&P_{4,17}(t_3,n_1) = 1 - \sum_{m=1}^{16} P_{4,m}(t_3,n_1).
\end{aligned}$$

The transition probabilities for $P_{5,j}(t_2,n_2)$, $P_{6,j}(t_2,n_2)$, $P_{7,j}(t_2,n_2)$, $P_{8,j}(t_2,n_2)$, $P_{9,j}(t_2,n_2)$, $P_{10,j}(t_2,n_2)$, $P_{11,j}(t_2,n_2)$ and $P_{12,j}(t_2,n_2)$ are calculated by replacing t_2 on t_3 and n_2 on n_1 for $P_{1,j}(t_3,n_1)$, $P_{2,j}(t_3,n_1)$, $P_{3,j}(t_3,n_1)$ and $P_{4,j}(t_3,n_1)$, $j = 1,2, \ldots, 17$.

That is,

$$\begin{aligned}
&P_{5,j}(t_2,n_2) = P_{1,j}(t_2,n_2), \quad j = 1,2,\ldots 17,\\
&P_{6,j}(t_2,n_2) = P_{2,j}(t_2,n_2), \quad j = 1,2,\ldots,17,\\
&P_{7,j}(t_2,n_2) = P_{3,j}(t_2,n_2), \quad j = 1,2,\ldots,17,\\
&P_{8,j}(t_2,n_2) = P_{4,j}(t_2,n_2), \quad j = 1,2,\ldots,17,\\
&P_{9,j}(t_2,n_2) = P_{1,j}(t_2,n_2), \quad j = 1,2,\ldots,17,\\
&P_{10,j}(t_2,n_2) = P_{2,j}(t_2,n_2), \quad j = 1,2,\ldots,17,\\
&P_{11,j}(t_2,n_2) = P_{3,j}(t_2,n_2), \quad j = 1,2,\ldots,17,\\
&P_{12,j}(t_2,n_2) = P_{4,j}(t_2,n_2), \quad j = 1,2,\ldots,17.
\end{aligned}$$

The transition probabilities for $P_{13,j}(t_1,n_3)$, $P_{14,j}(t_1,n_3)$, $P_{15,j}(t_1,n_3)$ and $P_{16,j}(t_1,n_3)$ are calculated by replacing t_1 on t_3 and n_3 on n_1 for $P_{1,j}(t_3,n_1)$, $P_{2,j}(t_3,n_1)$, $P_{3,j}(t_3,n_1)$ and $P_{4,j}(t_3,n_1)$, $j = 1,2, \ldots, 17$.

That is,

$$\begin{aligned}
&P_{13,j}(t_1,n_3) = P_{1,j}(t_1,n_3), \quad j = 1,2,\ldots,17,\\
&P_{14,j}(t_1,n_3) = P_{2,j}(t_1,n_3), \quad j = 1,2,\ldots,17,\\
&P_{15,j}(t_1,n_3) = P_{3,j}(t_1,n_3), \quad j = 1,2,\ldots,17,\\
&P_{16,j}(t_1,n_3) = P_{4,j}(t_1,n_3), \quad j = 1,2,\ldots,17,\\
&P_{i,j} = 0, \quad i = 17, j = 1,2,\ldots 7 \quad \text{and} \quad i \neq j,\\
&P_{i,i} = 1, \quad i = 17.
\end{aligned}$$

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