# Economic Adjustment Design For $\overline{X}$ Control Chart: A Markov Chain Approach

**Su-Fen Yang**

Department of Statistics
National Chengchi University
Taipei, 116, Taiwan, ROC
E-mail:YANG@NCCU.EDU.TW
Fax:886-2-29398024
Tel:886-2-29387459

## Abstract

The Markov Chain approach is used to develop an economic adjustment model of a process whose quality can be affected by a single special cause, resulting in changes of the process mean by incorrect adjustment of the process when it is operating according to its capability. The $\overline{X}$ control chart is thus used to signal the special cause. It is demonstrated that the expressions for the expected cycle time and the expected cycle cost are easier to obtain by the proposed approach than by adopting that in Collani, Saniga and Weigang (1994). Furthermore, this approach would be easily extended to derive the expected cycle cost and the expected cycle time for the case of multiple special causes or multiple control charts. A numerical example illustrates the proposed method and its application.

**Keywords** : Control Chart, Special Cause, Adjustment, Markov Chain, Renewal Reward Processes.

## 1. Introduction

Control charts are important tools of Statistical Quality Control(SQC). These charts are used to decide whether a process has achieved a state of statistical control and to maintain current control of a process. The use of control charts as a process monitoring and control tool has received much attention recently.

Deming (1982) explains that there are two kinds of mistakes the production worker can make on the job. These are to over-adjust a process or to under-adjust a process. He goes on to explain that the control chart provides "a rational and economic guide to minimize loss from both mistakes". Precise methods to design control charts that maximize the profit or minimize the cost of a process have been proposed by a number of authors. These methods yield control chart designs known as economic design. Economic design of a control chart was first proposed by Duncan (1956). The pioneering work of Duncan was later extended by others. A review of the literature is available in Montgomery (1980) and Vance (1983). Economic design optimizes the economic model of a production process by considering the costs of under-adjustment along with other costs, but it assumes that the search for a special cause is perfect.

In reality, a common problem in Statistical Process Control (SPC) is process over-adjustment. Processes may be adjusted since the only information available about the process state is that due to sampling. Consequently, a control chart signal outside the control limits is associated with process adjustment. If the signal is a false alarm, the process will be adjusted incorrectly. Woodall (1986) noted the effect of this over-adjustment as being an increase in the variability of the process. The increase in variability and the corresponding loss of quality can be quite marked as noted by Collani, Saniga and Weigand (1994). This problem is common in practice and therefore of importance. Collani, Saniga and Weigand (1994) first proposed economic adjustment design for $\overline{X}$ control chart to monitor a process with two types of mistakes. They assumed that there exists a single special cause that may cause the shift of a process mean. Their model allows for the determination of the design parameters of the $\overline{X}$ control chart that maximizes the profitability of the process or, equivalently, minimizes the process loss from two types of mistakes of over-adjustment and under-adjustment. However, their calculations for the expected cycle time and expected cycle cost / profitability are complicated, and it is not easy to extend their approach to the case of multiple special causes or multiple control charts. In this paper, we consider that the quality of output can be affected by a single special cause, resulting in shifts in the process mean, due to over-adjustment during operation. The $\overline{X}$ control chart is used to signal the special cause, which results in a shift of the process mean. A Markovian chain approach is used. The proposed approach would derive the expected cycle time and the expected cycle cost easier than by adopting that in Collani, Saniga and Weigand (1994). Furthermore, this approach would be easily extended to derive the expected cycle cost and the expected cycle time for the case of multiple special causes or multiple control charts. In next section, the economic adjustment model is derived by using a Markov chain approach. A direct search optimization technique is used to determine the design parameters of the $\overline{X}$ control chart that minimizes the loss of this process. An example illustrating the proposed method is given in the third section. A brief summary is provided in the final section of this paper.

## 2. Economic Adjustment Model: A Markov Chain Approach

A production process may be in control or out of statistical control. If the process is influenced by any special cause, then the process is out of control, otherwise the process is in control. Suppose that there exists a single special cause, say SC, for a production process. We assume that the process mean would be shifted if the SC influences the process. In this analysis, we are using the $\overline{X}$ control chart to signal the need for adjustment in the key dimension of the product. The in-control process can be out-of-control if it is incorrectly adjusted. Specifically, we take a sample of size n units of output every h hours of production time and adjust the process if the sample mean falls outside the control limits of the $\overline{X}$ control chart. Our objective is to derive the economic adjustment model using the Markov chain approach and to find the set of parameters n, h, and k (control coefficient of $\overline{X}$ control chart) such that the average long-term loss of the process is minimized.

The following assumptions are adopted from Collani, Saniga and Weigand (1994). Suppose that the product's quality can be represented by one key dimension, say X. When the process is in control, $X \sim N(\mu, \sigma^2)$. When a special cause SC of poor quality occurs in the process, there is a shift in the distribution of X to $X \sim N(\mu+\delta\sigma, \sigma^2)$ with probability w and to $X \sim N(\mu-\delta\sigma, \sigma^2)$ with probability 1-w, where $\delta>0$ and $\delta \neq 0$, $0<w<1$. The time until the occurrence of the special cause SC is assumed to be exponential with a mean of $1/\lambda$, $\lambda>0$. We also assume that the process cannot correct itself, and the time to sample and plot $\overline{X}$ is small and hence can be neglected in the model. However, it should be noted that the Markov chain approach can be extended to include cases involving the time to sample and plot $\overline{X}$.

An adjustment to the process is performed if the sample mean falls outside the control limits of the $\overline{X}$ control chart, respectively $LCL^{-}$ and $UCL^{-}$, where

$$\text{LCL}_{\bar{X}} = \mu - k\sigma/\sqrt{n}$$
$$\text{UCL}_{\bar{X}} = \mu + k\sigma/\sqrt{n} \quad (1)$$

where k is the number of standard deviation above or below the center line of the $\bar{X}$ control chart. The adjustment can take one of two forms. When the shift is such that $X \sim N(\mu+\delta\sigma, \sigma^2)$, SC is adjusted downward; when the shift is such that $X \sim N(\mu-\delta\sigma, \sigma^2)$, SC is adjusted upward. The decision rule can result in adjustment following a false alarm. A transition from in-control to out-of-control during sampling is assumed impossible.

Before deriving the economic adjustment model using Markov chain approach , we define some variables as follows:

$\alpha_{\bar{X}}$: the probability that the process is incorrectly adjusted (or over-adjusted) when $\bar{X}$ control chart gives a false alarm,

$$\alpha_{\bar{X}} = 1 - P(\text{LCL}_{\bar{X}} \le \bar{X} \le \text{UCL}_{\bar{X}} \mid x \sim N(\mu, \sigma^2)) = 2\Phi(-k)$$

$ARL_0$: the average run length for $\bar{X}$ control chart when the process is in control; $ARL_0 = 1/\alpha_{\bar{X}}$.

$\beta_{\bar{X}}$: the probability that the process is under-adjusted since it is influenced by SC but $\bar{X}$ control chart does not give the true alarm, where

$$\beta_{\bar{X}} = w\,P(\text{LCL}^- \le \bar{X} \le \text{UCL}^- \mid x \sim N(\mu-\delta\sigma, \sigma^2))$$
$$+(1-w)\,P(\text{LCL}_{\bar{X}} \le \bar{X}\,\text{UCL}_{\bar{X}} \mid x \sim N(\mu-\delta\sigma, \sigma^2))$$
$$= \Phi(k - \delta\sqrt{n}).$$

$ARL_1$: the average run length for $\bar{X}$ control chart when the process is influenced by SC; $ARL_1 = 1/(1-\beta_{\bar{X}})$.

$T_f$: the expected time of incorrect adjustment following a false alarm.

$T_{sc}$: the time before the special cause SC occurs in the process, $T_{sc} \sim \exp(\lambda)$.

$T_{sr}$: the expected time to search and repair the special cause.

$C_f$: the expected cost of incorrect adjustment.

$C_0$: the production loss per unit time when the process is in control.

$C_1$: the production loss per unit time when the process is influenced by SC.

$C_{sr}$: the expected cost to search and repair the special cause.

$\tau$: the expected arrival time of the special cause SC given that it occurred in the first sampling and testing time interval h, where $\tau = \dfrac{1-(1+\lambda h)e^{-\lambda h}}{\lambda - \lambda e^{-\lambda h}}$ (see Lorenzn and Vance (1986)).

a: fixed cost per sample and test.

b: variable cost.

To derive the expected cycle time (ET) and the expected cycle cost (EC) using the Markov chain approach, we must study the possible states at the end of every sampling and testing time. Depending on the states of the system, one can compute the transition probabilities and transition costs. Using the properties of the Markov chain, the expected cycle time and the expected cycle cost can be obtained. There are four possible states at the end of every sampling and testing time, as in Table 1.

**Table 1: Definition for Each State**

| State No. | SC occur? | $\overline{X}$ chart signal? | Process Adjustment? |
|---|---|---|---|
| 1 | No | NO | NO |
| 2 | No | Yes | Yes* |
| 3 | Yes | NO | No |
| 4 | Yes | Yes | Yes |

*process over-adjusted

These states can be classified into two types; transient states and absorbing states. State 4 is an absorbing state, the others are transient states. Transition probability from state i to state j in time interval h is described as follows:

$P_{1,1} = \exp(-\lambda h)\ (1-\alpha_{\overline{X}})$ $\qquad P_{1,2} = \exp(-\lambda h)\ (\alpha_{\overline{X}})$

$P_{1,3} = (1-\exp(-\lambda h))\ (\beta_{\overline{X}})$

$P_{1,4} = (1-\exp(-\lambda h))\ (1-\beta_{\overline{X}})$

$P_{2,1} = P_{2,2} = 0 \qquad P_{2,3} = (\beta_{\overline{X}}) \qquad P_{2,4} = (1-\beta_{\overline{X}})$

$P_{3,1} = P_{3,2} = 0 \qquad P_{3,3} = (\beta_{\overline{X}}) \qquad P_{3,4} = (1-\beta_{\overline{X}})$

$P_{4,1} = P_{4,2} = P_{4,3} = 0 \qquad P_{4,4} = 1$

We denote transition probability matrix **P11**=$[P_{i,j}]$, i,j= 1,2,3, **P12**=$[P_{i,j}]$, i= 1,2,3, j=4, zero matrix **0**=$[P_{i,j}]$, $P_{i,j}=0$ for i=4, j=1,2,3, **P22**=$P_{4,4}=1$, and matrix **P** is the combination of sub-matrices **P11, P12, 0,** and **1.** That is

$$\mathbf{P} = \begin{vmatrix} \mathbf{P11} & \mathbf{P12} \\ \mathbf{0} & \mathbf{1} \end{vmatrix} \qquad (2)$$

The cycle time is the time from the process starting in control until a true alarm is detected, corrected and the process is restarted, or equivalently it is the time from transient state 1 to reach an absorbing state. The state variable $Y_t(t=0,h,2h,...)$ is a Markov chain on the state 1,2,3,4 and so the Markov property can be effectively used to find the expected cycle time.

Let random variable Ti be the time up to the absorption state from the transient state i. Then, by using the Markov property and conditioning on the first step, we find

$$\begin{aligned} T_i &\overset{d}{=} h+T_{sr} \quad \text{w.p. } P_{i,4},\ i=1,2,3, \\ &\overset{d}{=} h+T_f+T_2 \quad \text{w.p. } P_{i,2},\ i=1,2, \\ &\overset{d}{=} h+T_j \quad \text{w.p. } P_{i,j},\ i=1,2,3,\ j=1,3. \end{aligned} \qquad (3)$$

($\overset{d}{=}$ means the same distribution as)

In fact $T_2=T_3$ because the process would become out of control following a false alarm.

Equation (3) can be expressed in matrix form

$$\mathbf{M}=h\mathbf{1}+\mathbf{P11}\ \mathbf{M}_{sr1}+\mathbf{P12}*\mathbf{M}_{sr2}+\mathbf{P11M},$$

where * denotes the Hadamard product of two matrices.

So

$$\mathbf{M}=h(\mathbf{I}-\mathbf{P11})^{-1}\mathbf{1}+(\mathbf{I}-\mathbf{P11})^{-1}\mathbf{P11M}^{SR}\mathbf{1}+(\mathbf{I}-\mathbf{P11})^{-1}\mathbf{P12}*\mathbf{M}^{SR}\mathbf{2}, \qquad (4)$$

where **M** is a (3x1) vector, with the expected time up to absorption from transient state i, $i\neq 4$.

**1** is a (3x1) vector, with elements 1,

$\mathbf{M}_{sr1}$ is a (3x1) vector, $\mathbf{M}_{sr1}^T=[\mathbf{0}\ \mathbf{T}_f\ \mathbf{0}]$,

$\mathbf{M}_{sr2}$ is a (3x1) vector, $\mathbf{M}_{sr2}^T=[\mathbf{Tsr}\ \mathbf{Tsr}\ \mathbf{Tsr}]$

**P11** is defined as above.

The expected cycle time is the first element of vector **M**, i.e. $\mathbf{M}_1$ or $E(T_1)$,where

$$E(T_1)=\frac{h+[1-\exp(-\lambda h)](1-\beta_{\bar{X}})T_{sr}+[\exp(-\lambda h)]\alpha_{\bar{X}}T_f+{[(1-\exp(-\lambda h)\beta_{\bar{X}})+\alpha\exp(-\lambda h)][h+(1-\beta_{\bar{X}})T_{sr}]}\Big/{1-\beta_{\bar{X}}}}{1-[\exp(-\lambda h)](1-\alpha_{\bar{X}})} \qquad (5)$$

After we obtain the expected cycle time, we must calculate the expected cycle cost, then the adjustment model can be derived by taking the ratio of the expected cycle cost and expected cycle time.

The derivation of the expected cycle cost uses the Markov property in a similar way to that used for the expected cycle time. Let $C_{i,j}$ be the expected cumulative cost that would be associated with transition from state i to j in time interval h, i,j=1,2,3,4. The calculation of $C_{i,j}$ is illustrated as follows:

$C_{1,1}=(C_0h)+(a+bn)$

$C_{1,2}=(C_0h)+(a+bn)+C_f$

$C_{1,3}=(C_0\tau)+(C_1(h-\tau))+(a+bn))$

$C_{1,4}= C_{1,3}+C_{sr}$

$C_{2,1}=C_{2,2}=0$

$C_{2,3}=(C_1h)+(a+bn)$

$C_{2,4}=(C_1h)+(a+bn)+C_{sr}$

$C_{3,1}=C_{3,2}=0$

$C_{3,3}=(C_1h)+(a+bn)$

$C_{3,4}= (C_1h)+(a+bn)+C_{sr}$

$C_{4,1}= C_{4,2}= C_{4,3}=0 C_{4,4}= C_{sr}$

We denote transition cost matrices $\mathbf{C11}=[C_{i,j}]$, i,j= 1,2,3; $\mathbf{C12}=[C_{i,j}]$, i=1,2,3, j=4; zero matrix $\mathbf{0}=[C_{i,j}]$, $C_{i,j}=0$ for i=4, j=1,2,3; **C22**= C4,4 and matrix **C** is the combination of submatrices **C11, C12, C22,** and **0.** That is

$$\mathbf{C}=\begin{vmatrix}\mathbf{C11} & \mathbf{C12}\\ \mathbf{0} & \mathbf{C22}\end{vmatrix} \qquad (6)$$

The cycle cost is the cumulative cost from the process starting in control until a ture alarm is detected, corrected and the process is restarted, or equivalently it is the cost from transient state 1 to reach an absorbing state.

Let random variable Ci be the cumulative cost up to absorption from transient state I. Then using the Markov property and conditioning on the first step, we find

$$C_i \overset{d}{=} C_{i,4} \quad \text{w.p. } P_{i,4},\ i=1,2,3 \tag{7}$$
$$\overset{d}{=} C_{i,j}+C_j \text{ w.p. } P_{i,j},\ i=1,2,3,\ j=1,2,3.$$

Equation (7) can be expressed in matrix form

**U= P12*C12+P11*C11+ P11U.**

So $\mathbf{U=(I\text{-}P11)^{-1}W1}$, where **W=| P11*C11 P12*C12|**, and **U** is a (3x1) vector with the expected cost up to absorption from transient state i, i≠4. The expected cycle cost is the first element of vector **U**, i.e. $\mathbf{U_1}$ or E(C1), where

$$\mathbf{U}_1=\frac{[a+bn+C_0h\exp(-\lambda h)+(\exp(-\lambda h))\alpha_{\overline{X}}C_f+[C_0\tau+C_1(h-\tau)][1-\exp(-\lambda h)]}{1-[\exp(-\lambda h)](1-\alpha_{\overline{X}})} \tag{8}$$

$$+\frac{[1-\exp(-\lambda h)](1-\beta_{\overline{X}})C_{sr}+[(\exp(-\lambda h))\alpha_{\overline{X}}+(1-\exp(-\lambda h))\beta_{\overline{X}}][\frac{C_1h+a+bn+C_{sr}(1-\beta_{\overline{X}})}{1-\beta_{\overline{X}}}]}{1-[\exp(-\lambda h)](1-\alpha_{\overline{X}})}$$

Applying the property of renewal reward processes by Ross (1993), the objective function (**L**), the expected cost per unit time is derived by taking the ratio of the expected cycle cost ($\mathbf{U_1}$) and the expected cycle time ($\mathbf{M_1}$); $\mathbf{L=U_1/\ M_1}$. The long-term loss is the function of design parameters n, h, k; **L**(n, k, h). Hence, the optimal design parameters of the economic $\overline{X}$ control chart can be determined by minimizing the objective function, that is Min. **L**(n,k,h).

It may be noted that the proposed approach can also be used to derive the identical economic adjustment model obtained by Collani, Saniga and Weigand (1994) if we put the expected time of incorrect adjustment=0, the expected time to search and correct a special cause=0, and transform the loss into benefit in the economic adjustment model. Furthermore, this approach would be easily extended to derive the expected cycle cost and expected cycle time for the case of multiple special causes or multiple control charts.

## 3. Difference Between The Economic Adjustment Model And Economic Model

Economic design optimizes the economic model of a production process that considers the costs of under-adjustment along with other costs, but it assumes that the search for a special cause is perfect. The proposed Markov chain approach can also be used to derive the economic model for a production process monitored by an $\overline{X}$ control chart. For the economic model, there are four possible states at the end of every sampling and testing. States 1,3 and 4 are defined as those in Table 1. But state 2 is defined as that when the process is in control, the sampling and testing result has a false alarm and no correction action is required. Once the process reaches state 2, the process continues and it is like that in state 1. Hence, the transition probabilities from state 2 to any states are equivalent to the transition probabilities from state 1 to any other states; $P_{2,j}=P_{1,j}$, j=1,2,3,4. The transition costs from state 2 to any other states are equivalent to the transition costs from state 1 to any other states; $C_{2,j}=C_{1,j}$, j=1,2,3,4. For the economic adjustment model, it may be noted that the transition probabilities from state 2 to any other states are equivalent to the transition probabilities from state 3 to any other states; $P_{2,j}=P_{3,j}$, j=1,2,3,4. The transition costs from state 2 to any other states are equivalent to the transition costs from state 3 to any other states; $C_{2,j}=C_{3,j}$, j=1,2,3,4. That is, the economic adjustment model would become the economic model if its transition probabilities $P_{2,j}$ and transition costs $C_{2,j}$ are replaced by $P_{1,j}$ and $C_{1,j}$, j=1,2,3,4, respectively.

# 4. A Numerical Example

A simple example is taken to illustrate the proposed method, and its application. To determine the economic adjustment design of the $\overline{X}$ chart, the following set of artificial process and cost parameters is chosen: $\mu=0$, $\sigma=1$, $w_1=w_2=0.5$, $\delta=1.5$, $\lambda=0.01$, $C_0=10$, $C_1=100$, a=0.5, b=0.1, $C_f=10$, $C_{sr}=35$, $T_f=0$, $T_{sr}=0.4$. The algorithm used to obtain the approximate optimal values (n*,h*,k*) of the design values (n,h,k), with constraints 0<k<6, 1<n<=25, 0<h<=8, is a simple grid search method yielding the following result n*=25, h*=1.0, k*=3.0. The control limits of the $\overline{X}$ control chart are set at 3.0 and -3.0. Every 1.0 hour a sample of size 25 is taken from the production process, and the values of $\overline{X}$ are calculated and plotted on the proposed $\overline{X}$ control chart. If the plotted point falls within the control limits of the $\overline{X}$ control chart, then no action is taken and the process continues, and a new sample is taken after 1.0 hour. If the plotted point falls outside the control limits of the $\overline{X}$ control chart, then process is stopped and adjusted. The process is new or over-adjusted after the process is adjusted, and a new sample is taken to monitor the process state every 1.0 hour.

# 5. Summary

The Markov chain approach is proposed to derive a model of a production process whose quality can be affected by the occurrence of a single special cause which causes a shift of process mean by an incorrect adjustment of the process when the process is in control. It is demonstrated that the expressions for the expected cycle time and the expected cycle cost are easier to obtain by the proposed approach than by adopting that in Collani, Saniga and Weigang (1994). Furthermore, this approach would be easily extended to derive the expected cycle cost and expected cycle time for the case of multiple special causes or multiple control charts. An example illustrates the proposed approach and its application. Several important extensions of the developed model can be studied for further research. Generally, it is straightforward to extend the proposed model to study other control charts, such as the np-charts for attributes. A particularly interesting research area concerns the economic statistical modeling of production processes subject to multiple special causes.

# Acknowledgments

Support for this research was provided in part by the National Science Council of the Republic of China, grant No. NSC-86-2115-M-004-015.

# References

1. Collani, E.V., Saniga, E.M. and Weigand (1994), "Economic adjustment designs for $\overline{X}$ control charts," IIE Transactions, Vol. 26, 37-43.
2. Deming, W. E.(1982), Quality, Productivity and Competition Position, M.I.T. Press.
3. Duncan, A. J. (1956), "The economic design of $\overline{X}$ charts used to maintain current control of a process," Journal of American Statistical Association, Vol. 51, 228-242.
4. Lorenzen, T. and Vance, L. (1986), "The Economic Design of Control Charts: A Unified Approach", Technometrics, Vol. 28, No. 1, pp. 3-10.
5. Montgomery, D. C. (1980), "The economic design of control charts: a review and literature survey," Journal of Quality Technology, Vol. 12, 75-78.
6. Ross, S.M. (1993), Introduction to Probability Models, Academic Press, New York.
7. Vance, L. C. (1983), "A bibliography of statistical quality control chart techniques, 1970-1980," Journal of Quality Technology, Vol. 15, 59-62.
8. Woodall, W. H., (1986), "Weakness of the economic design of control charts," Technometrics, Vol. 28, 408-409.