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ORIGINAL ARTICLE

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Adaptive sampling interval cause-selecting control charts

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Abstract This article considers the statistical adaptive process control for two dependent process steps. We construct an adaptive sampling interval Z_X control chart to monitor the quality variable produced by the first process step, and use the adaptive sampling interval Z_e control chart to monitor the specific quality variable produced by the second process step. By using the proposed adaptive sampling interval control charts, we can quickly detect and distinguish which process step is out of control. The performance of the proposed adaptive sampling interval control charts is measured by the adjusted average time to signal (AATS), which was derived by a Markov chain approach, for an out-of-control process. An empirical automobile braking system example shows the application and the performance of the proposed adaptive sampling control charts in detecting shifts in process means. Some numerical results obtained demonstrated that the performance of the proposed adaptive sampling cause-selecting control charts outperforms the fixed sampling interval cause-selecting control charts.

Keywords Adaptive sampling interval · Control charts · Dependent process steps · Adjusted average time to signal

Introduction

Control charts are important tools in statistical quality control. They are used to effectively monitor and determine whether a process is in control or out of control. Shewhart [20] first developed the $\overline{X}$ control chart to monitor the process mean. The control chart is easy to implement and, hence, it has been widely used for industrial process control. Usually, Shewhart $\overline{X}$ control charts monitor a process by taking equal samples of size at a fixed sampling interval. Once the sample statistic falls outside of its control limits, it indicates that the process is out of control, and that it should be stopped and repaired. Shewhart $\overline{X}$ control charts provide good performance in detecting large shifts in the process mean. However, small and moderate shifts in process means are expected in reality. There have been several alternatives proposed to improve this problem in recent years. Some of the alternatives include Shewhart $\overline{X}$ control charts combined with cumulative sum (CUSUM) control charts [10], Shewhart $\overline{X}$ control charts with run rules [9], exponentially weighted moving average (EWMA) control charts [11], and Shewhart $\overline{X}$ control charts with adaptive sampling interval or/and sample size [16], [13].

The idea of adaptive sampling interval (ASI) Shewhart $\overline{X}$ control charts is that the sampling interval should be short if the previous sample shows some indication of a shift in process mean, and long if there is no indication of a shift. Reynolds et al. [16] propose Shewhart $\overline{X}$ control charts with an adaptive sampling interval where the time period between samples is varied, depending on the location of the previous sample statistic on the charts. Their paper demonstrates that dual sampling intervals are better, and that the proposed control chart can detect small and moderate shifts in process mean faster than a fixed sampling interval (FSI) Shewhart $\overline{X}$ control chart. An ASI Shewhart $\overline{X}$ control chart with run rules is introduced by Cui and Reynolds [6]. The control chart signals not only a sample statistic falling outside the control limits, but also a run of a specified length occurred in a specified region. It offers better performance than ASI Shewhart $\overline{X}$ control charts without run rules and FSI Shewhart $\overline{X}$ control charts with or without run rules. Reynolds et al. [15] propose the CUSUM control chart with adaptive sampling interval to monitor the shift in process mean by extending the approach Reynolds et al. [16]. Their paper shows that the ASI CUSUM control chart is substantially more efficient than either the ASI Shewhart $\overline{X}$ control chart or the FSI CUSUM control chart for small and moderate shifts in process mean. Runger and Pignatiello [18] propose a one-

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sided control limit for an adaptive ASI Shewhart $\overline{X}$ control chart, and show that the improvement in detecting the process mean is better than a two-sided adaptive ASI Shewhart $\overline{X}$ control chart. It also shows that both one-sided and two-sided adaptive Shewhart $\overline{X}$ control charts detect small and moderate shifts in the process mean faster than an FSI Shewhart $\overline{X}$ control chart. Saccucci et al. [19] introduce the properties and performance of a EWMA control chart with adaptive sampling interval. Their paper demonstrates that the ASI EWMA control chart always outperforms the FSI EWMA control chart, as well as the ASI Shewhart $\overline{X}$ control chart, especially in detecting small and moderate shifts. Amin and Miller [1] propose a robust ASI $\overline{X}$ control chart. The chart is constructed based on a non-normal distributed quality characteristic. Reynolds [14] proposes a varied sampling interval $\overline{X}$ control chart with sampling at fixed times (VSIFT), and shows that the performance of the VSIFT $\overline{X}$ control chart is better than the FSI Shewhart $\overline{X}$ control chart and is relatively easy to set up and implement in practice.

Similar to ASI control charts, some papers consider adaptive sample size (ASS) control charts [2, 7, 13, 24]. Some authors combine adaptive sample size and sampling interval (ASSI) control charts [3, 12, 17]. Costa [5] proposes the adaptive parameter (AP) $\overline{X}$ control chart, and also discusses adaptive parameter (AP) $\overline{X}$ and R control charts [4]. Epprecht et al. [8] proposed adaptive control charts for attributes. A survey of recent developments in the adaptive control charts sees Tagaras [21].

All the above papers only consider a single process step. However, most recent products are produced from several different dependent process steps. In multiple process steps, a Shewhart control chart is often used to monitor the process state of each individual step. If the process steps are independent, then, using a Shewhart control chart to monitor each individual step is meaningful. However, many process steps are not independent, and, thus, the control charts are difficult to interpret the correct process state. An alternative approach is to use a multivariate control chart, such as a Hotelling T^2 control chart, to monitor dependent process steps. The disadvantages of using a Hotelling T^2 control chart are that it assumes that all quality characteristics in the chart are multivariate normal random variables, and there is difficulty in interpretation for which process step is out of control when there is a signal from the control chart. For the two dependent process steps, a powerful and popular approach proposed by Zhang [23] is using cause-selecting control charts. The cause-selecting control chart is constructed for an outgoing quality variable only after the observations have been adjusted for the effect of an incoming quality variable. The advantage of this approach is that, once there is a signal, it is easy to determine if the second step of the process is out of control. Wade and Woodall [22] review and analyze the cause-selecting control chart and examine the relationship between the cause-selecting control chart and the Hotelling T^2 control chart. In their opinion, the cause-selecting control chart outperforms the Hotelling T^2 control chart. Yang [25] proposed economic control charts for two dependent process steps. Yang [26] considers a statistical process control for two dependent process steps with a failure mechanism. Yang and Chen [28] propose two dependent processes control for two failure mechanisms from an economic viewpoint. Yang [27] addresses dependent processes control for over-adjusted process means. Yang and Yang [29] study the effects of imprecise measurement on controlling two dependent process steps for autocorrelated observations. However, using adaptive control charts to monitor and distinguish which process step is out of control has not been addressed.

The purpose of this paper is to study the ASI Z_X and Z_e control charts to monitor the process means on the dependent first process step and the second process step, respectively. Following Costa [3], the shifts in the process means do not occur at the beginning, but instead at some random time in the future, and the occurrence time of the shift is assumed to be an exponentially distributed random variable. The performance of the ASI Z_X and Z_e control charts is measured by the Adjusted average time to signal (AATS). An empirical example is given to illustrate the application and performance of the proposed ASI Z_X and Z_e control charts. Finally, some numerical comparison results show that the performance of the proposed ASI charts outperforms the FSI Z_X and Z_e control charts.

Process description

Throughout this article, a two-step process is assumed. Denote X to be the quality measurement of interest for the first step and Y to be the quality measurement of interest for the second step. The two steps of the process are dependent, and the second step is affected by the first step. Suppose that the values of X cannot be observed in the first step, but can be measured at the end of the second step. Hence, a sample of size 1 is taken at a varied sampling interval, and the paired observations (X_i, Y_i) are measured at the end of the second step. It is assumed that the samples are mutually independent. Figure 1 shows a two-step process and the interested quality variables (X, Y) for the process.

The quality variable X is assumed to follow a normal distribution with mean μ_X and a constant standard deviation σ_X when the first step is in control. The quality variable Y is affected by X and the relationship between Y and X is expressed as:

$$Y_i|X_i = f(X_i) + \varepsilon_i, \quad i = 1, 2, 3, \ldots, n. \tag{1}$$

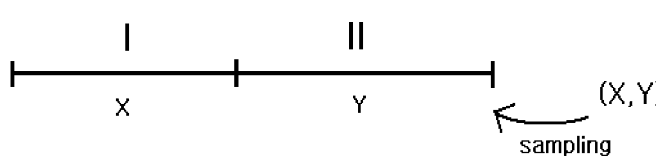

Fig. 1 A two-step process

The variable ε_i is the random error and $\varepsilon_i \sim \mathrm{NID}(0, \sigma_\varepsilon^2)$ when the second step is in control. To monitor the two dependent process steps effectively, two control charts are constructed to control the first step and the second step respectively. To monitor the first step, the individual X control chart is set up based on the in control distribution of X. To monitor the second step, the specific quality of the second step is specified by adjusting the effect of X on Y; that is, the specific quality is presented by the cause-selecting values, $e_i = Y_i|X_i - \widehat{Y}_i|X_i$. The cause-selecting control chart is set up based on the in-control distribution of cause-selecting values. Hence, $X \sim \mathrm{N}(\mu_X, \sigma_X^2)$ and $e \sim \mathrm{N}(0, \sigma_\varepsilon^2)$ when the first step and the second step are both in control.

We assume the process is in control at the beginning, and two assignable causes, say AC1 and AC2, may occur at the process randomly. AC1 may only occur at the first step and influences a shift in the distribution of X to $X \sim \mathrm{N}(\mu_X + \delta_1\sigma_X, \sigma_X^2)$. AC2 may only occur at the second step and influences a shift in the distribution of e to $e \sim \mathrm{N}(\delta_2\sigma_\varepsilon, \sigma_\varepsilon^2)$. When both of AC1 and AC2 occur, the process mean of X shifts from μ_X to $\mu_X + \delta_1\sigma_X$, and the process mean of e shifts from 0 to $\delta_2\sigma_\varepsilon$. The assignable cause occurs according to an exponential distribution with parameter λ_i, i=1, 2. Table 1 shows possible distributions of X and e.

Principles of the ASI Z_X and Z_e control charts for two dependent process steps

To control and diagnose the two dependent process steps effectively, the adaptive sampling interval X and cause-selecting control charts are constructed based on the in-control X and e distributions. The sampling interval is varied and it depends on the locations of the previous sample statistics. Usually, there are three lines (upper control limit, central limit, and lower control limit) for a control chart. Here, we add a warning threshold on each of the proposed control charts. Hence, there are two symmetric warning lines between the control limits. Consequently, the structures of the ASI X and cause-selecting control charts are as shown in Fig. 2.

To monitor the two dependent process steps, a sample of size 1 is taken at the end of the second step, the paired observations (X_i, Y_i) are measured, and the relationship between Y and X is determined. Then, the sample statistics (X_i, e_i) are calculated and plotted on the ASI X control chart and cause-selecting control chart, respectively. If the sample statistic falls between the warning lines, it is reasonable to wait more time to take the next sample because there is no evidence that the process needs adjustment. If the sample statistic falls close to the control limits but not outside them, it is reasonable to wait less time to take the next sample because the process can be in need of adjustment. When at least one sample statistic falls outside of the control limits, we stop monitoring the process and start to repair the assignable cause.

Table 1 Possible distributions of X and e

X		e	
Does AC1 occur?	Mean of X	Does AC2 occur?	Mean of e
No	μ_X	No	0
Yes	$\mu_X+\delta_1\sigma_X$	No	0
No	μ_X	Yes	$\delta_2\sigma_\varepsilon$
Yes	$\mu_X+\delta_1\sigma_X$	Yes	$\delta_2\sigma_\varepsilon$

$$
\begin{array}{ll}
\mathrm{UCL_X} = \mu_X + k_X\sigma_X & \mathrm{UCL}_e = k_e\sigma_\varepsilon \\
\mathrm{UWL_X} = \mu_X + w_X\sigma_X & \mathrm{UWL}_e = w_e\sigma_\varepsilon \\
\mathrm{CL_X} = \mu_X & \mathrm{CL}_e = 0 \\
\mathrm{LWL_X} = \mu_X - w_X\sigma_X & \mathrm{LWL}_e = -w_e\sigma_\varepsilon \\
\mathrm{LCL_X} = \mu_X - k_X\sigma_X & \mathrm{LCL}_e = -k_e\sigma_\varepsilon \\
\text{ASI X control chart} & \text{ASI cause - selecting chart}
\end{array}
$$

Fig. 2 The ASI X and cause-selecting control charts

To easily use the proposed charts for process engineers, we standardize sample statistics X and e as follows. Let the following:

$$Z_X = \frac{X - \mu_X}{\sigma_X}, \tag{2}$$

where $Z_X \sim \mathrm{N}(0, 1)$ when the first step is in control. Let the following:

$$Z_e = \frac{e}{\sigma_\varepsilon}, \tag{3}$$

where $Z_e \sim N(0, 1)$ when the second step is in control.

Based on the in-control distributions of Z_X and Z_e, the structures of the ASI Z_X and Z_e control charts are as shown in Fig. 3.

Before using the proposed ASI Z_X and Z_e control charts, we determine three sampling intervals, t_1, t_2, and t_3, where $t_3 > t_2 > t_1$. If both sample statistics (Z_X, Z_e) fall between the warning limits, it is reasonable to choose a longer sampling interval, t_3, to take the next sample, as no evidence that the process needs adjustment. If both sample statistics fall outside the warning limits but inside the control limits, it is reasonable to choose the shorter sampling interval, t_1, to take the next sample, as the process can be in need of adjustment. If one of the sample statistics falls within the

$$
\begin{array}{ll}
\mathrm{UCL}_{z_x} = k_X & \mathrm{UCL}_{z_e} = k_e \\
UWL_{z_x} = w_X & \mathrm{UWL}_{z_e} = w_e \\
\mathrm{CL}_{z_x} = 0 & \mathrm{CL}_{z_e} = 0 \\
\mathrm{LWL}_{z_x} = -w_X & \mathrm{LWL}_{z_e} = -w_e \\
\mathrm{LCL}_{z_x} = -k_X & \mathrm{LCL}_{z_e} = -k_e \\
\text{ASI } Z_X \text{ chart} & \text{ASI } Z_e \text{ chart}
\end{array}
$$

Fig. 3 The ASI Z_X and Z_e control charts warning limit and the other falls outside the warning limit but inside the control limit, it is reasonable to choose a middle sampling interval, t_2. Once one sample statistic falls outside of its control limits, then the sampling stops and the process restarts. The first sampling interval is randomly chosen from t_1, t_2, and t_3, and the following sampling interval is dependent upon the locations of the previous sample statistics.

To express the relationship between the sampling interval, t_i, and the location of the sample statistics on the proposed charts, the ASI Z_X and Z_e control charts are divided into the following four regions:

$$\begin{aligned}
I_{X1} &= [-w_X,\ w_X] & I_{e1} &= [-w_e,\ w_e]\\
I_{X2} &= (-k_X,\ -w_X) \cup (w_X,\ k_X) & I_{e2} &= (-k_e,\ -w_e) \cup (w_e,\ k_e)\\
I_{X3} &= (-k_X,\ k_X) & I_{e3} &= (-k_e,\ k_e)\\
I_{X4} &= (-\infty,\ -k_X] \cup [k_X,\ \infty) & I_{e4} &= (-\infty,\ -k_e] \cup [k_e,\ \infty)
\end{aligned}$$

According to the adaptive scheme, if the previous sample statistics Z_X falls in region I_{X1} and Z_e falls in region I_{e1}, then we choose the longer sampling interval t_3. If one of the previous sample statistics falls in region I_{X1} or I_{e1}, and the other falls in region I_{X2} or I_{e2}, then we choose the sampling interval t_2. If both of the previous sample statistics fall in region I_{X2} and I_{e2}, then we choose the shorter sampling interval t_1. If at least one of the sample statistics falls outside of the control limits, then we stop sampling. Their relationship is formulated as in Eq. 4:

$$t_k = \begin{cases} t_3 & \text{if } Z_X \in I_{X1} \cap Z_e \in I_{e1} \\ t_2 & \text{if } Z_X \in I_{X1} \cap Z_e \in I_{e2} \\ t_2 & \text{if } Z_X \in I_{X2} \cap Z_e \in I_{e1} \\ t_1 & \text{if } Z_X \in I_{X2} \cap Z_e \in I_{e2} \end{cases}, \boldsymbol{k} = 1, 2, 3, \ldots. \tag{4}$$

No matter if the signal is a true or a false on the proposed Z_X control chart, it indicates that AC1 may have occurred at the first step. Then, the process is stopped for repairs until the process is back in control, similar to the proposed Z_e control chart.

Determination of the warning limits on the ASI Z_X and Z_e control charts

The design parameters of the proposed ASI Z_X and Z_e control charts are the sampling intervals (t_1, t_2, t_3), the control limits (k_X, k_e), and the warning limits (w_X, w_e). The three sampling intervals are always determined by the process engineers, and the control limits are always fixed. This leaves the warning limits to be determined. In order to compare the performance of the proposed ASI Z_X and Z_e control charts and the FSI Z_X and Z_e control charts after the process shifts, we should let the ASI Z_X and Z_e control charts and FSI Z_X and Z_e control charts have the same expected in-control process time. Hence, under the in-control process, we let the expected sampling time interval $E(t_i)$ of the proposed ASI control charts be equal to the sampling time interval of the FSI Z_X and Z_e control charts.

That is:

$$\begin{aligned} &E[t_k|\delta_1 = 0,\ \delta_2 = 0] \\ &= t_0 P(\text{plotted points within control limits}|\delta_1 = 0, \delta_2 = 0), \end{aligned} \tag{5}$$

where t_0 is the sampling time interval of the FSI Z_X and Z_e control charts. We have to note that the only difference between the FSI and ASI Z_X and Z_e control charts is that the former has no warning limits. Using the constraint in Eq. 5, we can derive the warning limits of the proposed ASI Z_X and Z_e control charts. We assume that, once false alarm occurs, then the process stops sampling and restarts. Hence, the constraint can be expressed as follows:

$$\begin{aligned}
&t_3 \times P(Z_X \in I_{X1}|\delta_1 = 0,\ \delta_2 = 0) \\
&\quad \times P(Z_e \in I_{e1}|\delta_1 = 0,\ \delta_2 = 0) + \\
&\quad t_2 \times P(Z_X \in I_{X1}|\delta_1 = 0,\ \delta_2 = 0) \\
&\quad \times P(Z_e \in I_{e2}|\delta_1 = 0,\ \delta_2 = 0) + \\
&\quad t_2 \times P(Z_X \in I_{X2}|\delta_1 = 0,\ \delta_2 = 0) \\
&\quad \times P(Z_e \in I_{e1}|\delta_1 = 0,\ \delta_2 = 0) \\
&\quad t_1 \times P(Z_X \in I_{X2}|\delta_1 = 0,\ \delta_2 = 0) \\
&\quad \times P(Z_e \in I_{e2}|\delta_1 = 0,\ \delta_2 = 0) \\
&\quad \times P(\text{at least one false alarm}) \\
&\quad = t_0 \times P(-k_x < Z_{X,\, i-1} < k_x|\delta_1 = 0,\ \delta_2 = 0) \\
&\quad \times P(-k_e < Z_e < k_e|\delta_1 = 0,\ \delta_2 = 0) \\
&\quad \times P(\text{at least one false alarm}).
\end{aligned} \tag{6}$$

Simplifying Eq. 6 gives:

$$\begin{aligned}
&4\Phi(w_X)\Phi(w_e)[t_3 - 2t_2 + t_1] \\
&\quad + 2\Phi(w_X)[-t_3 + 2t_2\Phi(k_e) + t_2 - 2t_1\Phi(k_e)] \\
&\quad + 2\Phi(w_e)[-t_3 + 2t_2\Phi(k_X) + t_2 - 2t_1\Phi(k_X)] \\
&= t_0(2\Phi(k_X) - 1)(2\Phi(k_e) - 1) - t_3 + 2t_2\Phi(k_e) \\
&\quad + 2t_2\Phi(k_X) - 4t_1\Phi(k_X)\Phi(k_e)
\end{aligned} \tag{7}$$

where $\Phi(\cdot)$ denotes the standard normal cumulative function.

Let $k_X = k_e = k$, Eq. 7 is simplified as:

$$\begin{aligned}
&4\Phi(w_X)\Phi(w_e)[t_3 - 2t_2 + t_1] + 2[\Phi(w_X) + \Phi(w_e)] \\
&\quad \times [-t_3 + 2t_2\Phi(k) + t_2 - 2t_1\Phi(k)] \\
&\quad = t_0(2\Phi(k) - 1)^2 - t_3 + 4t_2\Phi(k) - 4t_1(\Phi(k))^2
\end{aligned} \tag{8}$$

We can solve Eq. 8 by letting $w_X = w_e = w$, and it reduces the complication of implementing the proposed ASI Z_X and Z_e control charts.

Consequently:

$$\begin{aligned}&4\Phi(w)^2[t_3-2t_2+t_1]\\&\quad+4\Phi(w)[-t_3+2t_2\Phi(k)+t_2-2t_1\Phi(k)]\\&\quad=t_0(2\Phi(k)-1)^2-t_3+4t_2\Phi(k)-4t_1(\Phi(k))^2.\end{aligned} \tag{9}$$

It follows that:

$$w=\Phi^{-1}\left(\frac{-4B\pm\sqrt{16B^2-16AC}}{8A}\right), \tag{10}$$

where:

$$\begin{aligned}A&=t_3-2t_2+t_1\\B&=-t_3+2t_2\Phi(k)+t_2-2t_1\Phi(k)\\C&=-\left[t_0\left(2\Phi(k)-1\right)^2-t_3+4t_2\Phi(k)-4t_1\left(\Phi(k)\right)^2\right]\end{aligned}$$

Performance measurement for the ASI Z_X and Z_e control charts

The statistical efficiency of a control chart is measured by the speed of detecting process mean shifts. The average run length (ARL) is often used to measure the speed of any FSI control charts. However, the sampling interval of the proposed Z_X and Z_e control charts in this article is variable, not fixed, so the detecting speed cannot be measured by ARL. Instead, the adjusted average time to signal (AATS) is used to measure the performance of the proposed ASI Z_X and Z_e control charts. The AATS is the average time from any one of the two process steps' means shift until a signal and all occurred assignable causes are removed. A smaller AATS is desirable to detect the out-of-control process faster and avoid the losses of defective products.

The average time of a process cycle (ATC) is the average time from the start of production until the signal after the process is checked and adjusted.

Denote T_i to be the occurrence time of ACi, which follows an exponential distribution with parameter λ_i, i=1, 2, where T_1 and T_2 are independent.

Denote T to be the occurrence time of the first assignable cause, that is:

$$T=\min(T_1,\ T_2),$$

where T~exp(λ_1+λ_2). Hence, the relationship among the AATS, ATC, and the expected T is expressed as:

$$AATS=ATC-E(T)=ATC-\frac{1}{\lambda_1+\lambda_2} \tag{11}$$

Before obtaining the AATS of the proposed ASI Z_X and Z_e control charts, we have to derive the ATC. The Markov chain approach is used to compute the ATC, due to the no-memory property of the exponential distribution. According to the locations of the sample statistics, 17 possible process states are defined. Table 2 shows the 17 process states. When at least one chart signals, we stop monitoring the process no matter whether the signal is true or false. Hence, the 17 states can be classified into absorbing and transient states. The absorbing states are reached when at least one sample statistic falls out of the control limits; that is, in region I_{X4} or I_{e4}. The transient states are reached when both sample statistics fall within the control limits. The states from 1 to 16 are defined as the transient states, and state 17 is defined as the absorbing state.

Table 2 The 17 process states

State	Does AC1 occur?	The location of sample statistic Z_X	Is an alarm in the first step?	Does AC2 occur?	The location of sample statistic Z_e	Is an alarm in the second step?	Transient state or absorbing state?
1	No	I_{X1}	No alarm	No	I_{e1}	No alarm	Transient state
2	No	I_{X1}		Yes	I_{e1}		
3	Yes	I_{X1}		No	I_{e1}		
4	Yes	I_{X1}		Yes	I_{e1}		
5	No	I_{X2}		No	I_{e1}		
6	No	I_{X2}		Yes	I_{e1}		
7	Yes	I_{X2}		No	I_{e1}		
8	Yes	I_{X2}		Yes	I_{e1}		
9	No	I_{X1}		No	I_{e2}		
10	No	I_{X1}		Yes	I_{e2}		
11	Yes	I_{X1}		No	I_{e2}		
12	Yes	I_{X1}		Yes	I_{e2}		
13	No	I_{X2}		No	I_{e2}		
14	No	I_{X2}		Yes	I_{e2}		
15	Yes	I_{X2}		No	I_{e2}		
16	Yes	I_{X2}		Yes	I_{e2}		
17	At least one False signal			or True signal			Absorbing state

Denote $\mathbf{P}$ to be the transition probability matrix, and $\mathbf{P}$ is a square matrix of order 17. Denote $P_{i,\ j}(t_k)$ to be the transition probability from prior state i to current state j with sampling interval t_k, where t_k is determined by the prior state i, i=1, 2,..., 17, and k=1, 2, 3. For example, the transition probability from state 1 to state 4 with sampling interval t_3 is calculated as:

$$\begin{aligned} p_{1,\,4}(t_3) &= P[|Z_X| < w_X|\delta_1] \times P[|Z_e| < w_e|\delta_2] \\ &\quad \times \left(1 - e^{-\lambda_1 t_3}\right) \times \left(1 - e^{-\lambda_2 t_3}\right) \\ &= (\Phi(w_X - \delta_1) - \Phi(-w_X - \delta_1)) \\ &\quad \times (\Phi(w_e - \delta_2) - \Phi(-w_e - \delta_2)) \\ &\quad \times \left(1 - e^{-\lambda_1 t_3}\right) \times \left(1 - e^{-\lambda_2 t_3}\right) \end{aligned}$$

The calculation of all transition probabilities is shown in the Appendix.

Since we consider that the shift in the process mean does not occur at the beginning, hence, the in-control process state would be on 1, 5, 9, or 13. The states 1, 5, 9, and 13 are transient states, but state 17 is an absorbing state. Based on the definition of ATC, the ATC is calculated as the sum of (the expected time to reach an absorbing state from the transient state i)×(the corresponding starting probability for the transient state i), i=1, 5, 9, 13; that is:

$$ATC = b_1M_1 + b_5M_5 + b_9M_9 + b_{13}M_{13}, \tag{12}$$

where b_i is the starting probability for the transient state i, i=1, 5, 9, 13 and M_i is the expected time to reach any absorbing state from the transient state i, i=1, 5, 9, 13.

The M_i, i=1, 2, 3,..., 16, can be derived using the property of Markov chain; that is:

$$(M_1,\ M_2,\ M_3, \ldots,\ M_{16})' = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{t}, \tag{13}$$

where $\mathbf{I}$ is the identity matrix of order 16; $\mathbf{Q}$ is the transition probability matrix, where each element represents the transition probability, $P_{i,\ j}(t_k)$, from transient state i to transient state j, i=1,..., 16, j=1,..., 16; and $\mathbf{t}'$=(t_3, t_3, t_3, t_3, t_2, t_2, t_2, t_2, t_2, t_2, t_2, t_2, t_1, t_1, t_1, t_1) is the vector of the sampling intervals from state 1 to state 16.

Consequently, the AATS is obtained using Eq. 11.

An example

In this section, we illustrate how to construct and use the proposed ASI Z_X and Z_e control charts through an empirical automobile braking system example. The data of this example are the measurements of a component part from the automobile braking system. The variables X=ROLLWT was measured for the first step and Y=BAKEWT was measured for the second step. For every sampling, a sample of size 1 is taken at the end of the second step, and a pair of observations (X, Y) is measured. Two machines are operated in the two dependent process steps. The time (T_1) until machine 1 is out of control follows an exponential distribution with parameter λ_1=0.03, and the time (T_2) until machine 2 is out of control follows an exponential distribution with parameter λ_2=0.04. The variables T_1 and T_2 are independent. From the process history, the means of the process may be shifted when two machines are out of control. The out-of-control machine 1 only influences the mean of X on the first process step and the standard deviation is unaffected. The out-of-control machine 2 only influences the mean of Y on the second process step and the standard deviation is unaffected. We collect 45 pairs of observations (X, Y), from the in-control process, to establish their statistical relationship.

From the results of data analysis, both variables X and Y follow bivariate normal distributions. Variable Y is affected by variable X, and their relationship is expressed by a simple linear regression model. Using the least square method for the 45 observations, the fitted regression model is:

$$\widehat{Y}|X = 93.2 + 0.513X\ . \tag{14}$$

From the model assumptions checking, the fitted regression model is appropriated. So, we use Eq. 14 to calculate the estimated values $\left(\widehat{Y}|X\right)$ and the residuals (e). The estimated means and standard deviations of variables X and e are $\widehat{\mu}_X = 210.25$, $\widehat{\sigma}_X = 1.19$, $\widehat{\mu}_e = 0$, and $\widehat{\sigma}_e = 0.88$. Hence, when both steps are in control, X~N(210.25, 1.19^2) and e~N (0, 0.88^2). According to the historical data, the shift scale of the process mean caused by the out-of-control machine 1 is $\delta_1\sigma_X$, where δ_1=0.5, and the shift scale of the process mean caused by the out-of-control machine 2 is $\delta_2\sigma_e$, where δ_2=0.75. Hence, the distribution of X becomes X~N(210.25 +0.5×1.19, 1.19^2) for the out-of-control first process step, and the distribution of e becomes e~N(0.75×0.88, 0.88^2) for the out-of-control second process step.

For three different adaptive sampling intervals, the process engineers determine t_1=0.01 h, t_2=0.5 h, and t_3=1.15 h; thus, the warning limit is calculated to be 1.58 by using Eq. 10. Consequently, the structures of the proposed ASI Z_X and Z_e control charts are as follows:

$$\begin{array}{ll} UCL_{Z_X} = 3 & UCL_{Z_e} = 3 \\ UWL_{Z_X} = 1.58 & UWL_{Z_e} = 1.58 \\ CL_{Z_X} = 0 & CL_{Z_e} = 0 \\ LWL_{Z_X} = -1.58 & LWL_{Z_e} = -1.58 \\ LCL_{Z_X} = -3 & LCL_{Z_e} = -3 \end{array} \tag{15}$$

To detect the changes in the process means faster, the constructed ASI Z_X and Z_e control charts are used. The first sampling interval is randomly chosen to be 0.5 h. After 0.5 h, the first sample is taken from the end of the second process step, and the paired observation of (X, Y) is (209, 201). Hence, the calculated values of (X, e) is (209, 0.583) by using Eqs. 2 and 14. Using Eqs. 2 and 3, the values of Z_X and Z_e are as follows:

$$Z_X = \frac{209 - 210.25}{1.19} = -1.05 \quad (16)$$

$$Z_e = \frac{0.583}{0.88} = 0.66 \quad (17)$$

Both sample statistics (−1.05, 0.66) are within the warning limits, so we adopt the longer sample interval t_3=1.15 h. After 1.15 h, the second sample is taken, and the paired observation of (X, Y) is (208, 202). Hence, the sample statistics (Z_X, Z_e) of the second sample are (−1.89, 2.38). Both of these statistics fall outside the warning limits, but inside the control limits, so the shorter sampling interval t_1=0.01 h is adopted; that is, the next sample is taken 0.01 h after the second sample. Once at least one of the sample statistics falls outside the control limits, then the process is stopped and the out-of-control machine 1 or/and machine 2 needs to be repaired.

The AATS is used to measure the performance of the proposed ASI Z_X and Z_e control charts. There are 17 possible process states for the empirical example, which are the same as the process states presented in Table 2. The ATC is calculated to be 62.80 h by using Eq. 12. Hence, the AATS is 52.51 h by using Eq. 11. It means that the average time to detect the out-of-control process is 52.51 h.

For comparing the AATS of the ASI Z_X and Z_e control charts with that of the FSI Z_X and Z_e control charts, we also calculate the AATS of the FSI Z_X and Z_e control charts. The approach to obtain the ATC and AATS of the FSI Z_X and Z_e control charts is similar to that of the ASI Z_X and Z_e control charts, except that there are no warning limits on the FSI Z_X and Z_e control charts. After calculation, the AATS of the FSI Z_X and Z_e control charts is 54.91 h. It shows that the proposed ASI Z_X and Z_e control charts can detect the shift of the process means faster than the FSI Z_X and Z_e control charts by about 2 h. Hence, the proposed ASI Z_X and Z_e control charts offer better performance than the FSI Z_X and Z_e control charts.

Comparisons with FSI Z_X and Z_e control charts

Following the example in Sect. 6, we survey the performance of the proposed charts and the FSI charts for various combinations of shift scales (δ_1, δ_2) of the process means and parameters (λ_1, λ_2). According to the historical data, the shift scales (δ_1, δ_2) of the process means caused by the out-of-control machines 1 and 2 range from 0.5 to 1.5, and the parameters (λ_1, λ_2) range from 0.03 to 0.05, respectively. The AATS under various combinations of (δ_1, δ_2) and (λ_1, λ_2) are calculated and illustrated in Table 3 and Table 4, respectively.

The numerical results in Tables 3 and 4 show that, when detecting small and moderate shifts in process means, the performance of the proposed ASI Z_X and Z_e control charts is much better than the FSI Z_X and Z_e control charts. Using ASI Z_X and Z_e control charts instead of FSI Z_X and Z_e control charts can reduce the AATS to about 0.5 h to 5.5 h. In addition, the proposed ASI Z_X and Z_e control charts have better performance in detecting small shifts ($\delta_1 \leq 1$ and $\delta_2 \leq 1$) than moderate shifts ($1 < \delta_1 \leq 1.5$ or $1 < \delta_2 \leq 1.5$). Under small shifts ($\delta_1 \leq 1$ and $\delta_2 \leq 1$) in the process means, the ASI Z_X and Z_e control charts can decrease the AATS to about 3.5 h to 5.5 h, compared to the FSI Z_X and Z_e control charts. Under moderate shifts ($1 < \delta_1 \leq 1.5$ or $1 < \delta_2 \leq 1.5$) in process means, the ASI Z_X and Z_e control charts can decrease the AATS to about 0.3 h to 3.8 h, compared to the FSI Z_X and Z_e control charts. Furthermore, the large values of λ_1 and λ_2 lead to a smaller AATS than the small values of λ_1 and λ_2.

One of the advantages of Shewhart control charts is their speed in detecting large shifts in the process mean. However, it is not efficient to use Shewhart control charts when detecting small and moderate shifts in process means. The proposed ASI Z_X and Z_e control charts may improve this problem for two dependent process steps. Based on the numerical results, the performance of the proposed ASI Z_X and Z_e control charts always outperform the FSI Z_X and Z_e control charts.

Conclusions

In this article, the adaptive sampling interval (ASI) Z_X and Z_e cause-selecting control charts are proposed to effectively detect and distinguish which one of two dependent process steps is out of control.

The adjusted average time to signal (AATS) is calculated to measure the performance of the proposed ASI Z_X and Z_e control charts by the Markov chain approach. An empirical automobile braking system example illustrates the application of the proposed ASI Z_X and Z_e control charts and shows that the performance of the ASI Z_X and Z_e control charts is better than the FSI Z_X and Z_e control charts. From some numerical results, it is demonstrated that the proposed ASI Z_X and Z_e control charts outperform the FSI Z_X and Z_e control charts. It implicates that a process engineer can identify an out-of-control process step quickly when using the proposed ASI Z_X and Z_e control charts, and it also leads a reduction in the losses of defective products.

Several important extensions of the developed model can be expanded. It is straightforward to extend the proposed model to study adaptive sample size (ASS), adaptive sample size and sampling interval (ASSI) cause-selecting control charts, or other control charts, like adaptive cause-selecting control charts in the presence of correlation, exponentially weighted moving average (EWMA), or cumulative sum (CUSUM) cause-selecting control charts. One particularly interesting research area for future research involves the

Table 3 Comparison of the AATS of the ASI and FSI Z_X and Z_e control charts when λ_1=0.03 and λ_2=0.04

			δ_1	0.5					0.75					1					1.25					1.5				
t_1	t_2	t_3	w δ_2	0.5	0.75	1	1.25	1.5	0.5	0.75	1	1.25	1.5	0.5	0.75	1	1.25	1.5	0.5	0.75	1	1.25	1.5	0.5	0.75	1	1.25	1.5
0.01	0.1	1.15	1.7887	74.6527	51.7056	34.2193	23.0980	16.6449	53.6831	40.7301	28.9235	20.3199	14.8787	37.6643	30.6475	23.1768	16.9352	12.5834	27.4356	23.2154	18.2509	13.6685	10.2052	21.4728	18.4556	14.7246	11.0883	8.2014
0.01	0.5	1.15	1.5754	75.3654	52.5110	35.0544	23.8975	17.3649	54.4715	41.5253	29.7116	21.0710	15.5610	38.4694	31.4171	23.9189	17.6440	13.2374	28.2000	23.9372	18.9458	14.3425	10.8422	22.1592	19.1079	15.3622	11.7226	8.8186
0.09	0.1	1.15	1.7874	74.6722	51.7294	34.2430	23.1185	16.6614	53.7069	40.7588	28.9523	20.3451	14.8990	37.6879	30.6765	23.2073	16.9630	12.6063	27.4562	23.2411	18.2792	13.6955	10.2284	21.4894	18.4765	14.7484	11.1121	8.2226
0.09	0.5	1.15	1.5718	75.4029	52.5563	35.0983	23.9346	17.3936	54.5164	41.5779	29.7632	21.1151	15.5955	38.5130	31.4689	23.9721	17.6915	13.2757	28.2370	23.9821	18.9941	14.3879	10.8805	22.1880	19.1434	15.4020	11.7617	8.8531
0.01	0.1	1.5	1.2917	73.5385	50.4976	33.4215	22.9524	17.1516	52.5264	39.6559	28.2447	20.2256	15.3834	36.9346	29.9955	22.7865	16.9726	13.1086	27.3462	23.1394	18.2764	13.9347	10.8161	22.0105	18.9481	15.1966	11.6497	8.9795
0.01	0.5	1.5	1.1095	74.9397	52.0051	34.8842	24.2579	18.2485	54.0040	41.0976	29.6016	21.4430	16.4228	38.3483	31.3240	24.0292	18.1104	14.1087	28.5989	24.3167	19.3992	15.0005	11.7903	23.0625	19.9537	16.1881	12.6314	9.9126
0.09	0.1	1.5	1.2878	73.6228	50.5955	33.5122	23.0248	17.2033	52.6231	39.7631	28.3450	20.3071	15.4428	37.0240	30.0960	22.8848	17.0561	13.1720	27.4177	23.2216	18.3610	14.0109	10.8771	22.0618	19.0089	15.2625	11.7127	9.0324
0.09	0.5	1.5	1.1018	75.0603	52.1407	35.0074	24.3526	18.3117	54.1378	41.2434	29.7353	21.5482	16.4952	38.4693	31.4573	24.1572	18.2165	14.1856	28.6921	24.4223	19.5066	15.0954	11.8633	23.1249	20.0278	16.2683	12.7071	9.9741
FSI control charts				77.7839	55.5624	38.0965	26.4924	19.3611	55.5624	43.6206	32.2752	23.5336	17.6068	38.0965	32.2752	25.7009	19.7653	15.2274	30.0017	25.8075	20.7921	16.0022	12.1730	23.4402	20.4703	16.7788	13.0804	9.9750
Difference 1				**3.79**	**4.41**	**3.95**	**2.84**	**2.07**	**4.28**	**4.37**	**3.80**	**2.76**	**2.08**	**3.76**	**3.72**	**3.31**	**2.56**	**2.06**	**2.66**	**2.67**	**2.54**	**2.33**	**1.97**	**1.97**	**2.01**	**2.05**	**1.99**	**1.77**
Difference 2				**1.93**	**2.35**	**2.27**	**1.44**	**0.41**	**2.29**	**2.44**	**2.29**	**1.44**	**0.46**	**2.18**	**2.25**	**1.94**	**1.28**	**0.46**	**1.31**	**1.39**	**1.29**	**0.91**	**0.31**	**0.32**	**0.44**	**0.51**	**0.37**	**0.00**

Difference 1=AATS of FSI control chart−max. AATS of ASI control chart
Difference 2=AATS of FSI control chart−min. AATS of ASI control chart

Table 4 Comparison of the AATS of the ASI and FSI Z_X and Z_e control charts when λ_1=0.05 and λ_2=0.05

			δ_1	0.5					0.75					1					1.25					1.5				
t_1	t_2	t_3	w δ_2	0.5	0.75	1	1.25	1.5	0.5	0.75	1	1.25	1.5	0.5	0.75	1	1.25	1.5	0.5	0.75	1	1.25	1.5	0.5	0.75	1	1.25	1.5
0.01	0.1	1.15	1.7887	74.6350	51.9107	34.5376	23.4292	16.9429	51.9107	39.8411	28.7234	20.4930	15.1955	34.5376	28.7234	22.3663	16.8552	12.8601	23.4292	20.4930	16.8552	13.2597	10.3462	16.9429	15.1955	12.8601	10.3462	8.1408
0.01	0.5	1.15	1.5754	75.3748	52.7372	35.3853	24.2351	17.6661	52.7372	40.6619	29.5199	21.2404	15.8695	35.3853	29.5200	23.1114	17.5485	13.4922	24.2351	21.2404	17.5485	13.9092	10.9509	17.6661	15.8695	13.4922	10.9509	8.7206
0.09	0.1	1.15	1.7874	74.6567	51.9371	34.5640	23.4523	16.9616	51.9371	39.8732	28.7561	20.5221	15.2190	34.5640	28.7561	22.4012	16.8877	12.8873	23.4523	20.5221	16.8877	13.2914	10.3739	16.9616	15.2191	12.8873	10.3739	8.1662
0.09	0.5	1.15	1.5718	75.4170	52.7876	35.4346	24.2772	17.6992	52.7876	40.7209	29.5786	21.2913	15.9099	35.4346	29.5786	23.1724	17.6040	13.5377	24.2772	21.2913	17.6040	13.9624	10.9967	17.6992	15.9099	13.5377	10.9967	8.7619
0.01	0.1	1.5	1.2917	72.7925	50.0009	33.0790	22.6582	16.8475	50.0009	38.0553	27.3907	19.7887	15.1138	33.0790	27.3907	21.3589	16.3168	12.8237	22.6582	19.7887	16.3168	12.9931	10.4202	16.8475	15.1138	12.8237	10.4202	8.4074
0.01	0.5	1.5	1.1095	74.2286	51.5284	34.5424	23.9508	17.9256	51.5284	39.5195	28.7355	20.9718	16.1128	34.5424	28.7355	22.5722	17.3956	13.7592	23.9508	20.9718	17.3956	13.9818	11.3138	17.9256	16.1128	13.7592	11.3138	9.2532
0.09	0.1	1.5	1.2878	72.8904	50.1130	33.1835	22.7432	16.9104	50.1130	38.1791	27.5076	19.8853	15.1863	33.1835	27.5076	21.4741	16.4164	12.9014	22.7431	19.8853	16.4164	13.0841	10.4950	16.9104	15.1863	12.9013	10.4950	8.4724
0.09	0.5	1.5	1.1018	74.3708	51.6861	34.6868	24.0651	18.0065	51.6861	39.6903	28.8934	21.0992	16.2049	34.6868	28.8934	22.7248	17.5248	13.8569	24.0651	21.0992	17.5248	14.0979	11.4070	18.0065	16.2049	13.8569	11.4070	9.3324
FSI control charts				77.3323	54.9085	37.3681	25.7887	18.7195	56.8085	44.0212	32.0490	22.9898	16.9562	40.6935	33.7139	26.1005	19.4987	14.6413	26.4924	23.5336	19.7653	15.8982	12.5911	19.3611	17.6068	15.2274	12.5911	10.1543
Difference 1				**4.99**	**5.56**	**5.02**	**3.83**	**2.51**	**5.56**	**5.57**	**4.88**	**3.74**	**2.49**	**5.02**	**4.88**	**4.34**	**3.45**	**2.40**	**3.83**	**3.74**	**3.45**	**2.91**	**2.24**	**2.51**	**2.49**	**2.40**	**2.24**	**2.01**
Difference 2				**2.37**	**2.77**	**2.66**	**2.22**	**1.35**	**2.77**	**2.90**	**2.70**	**2.24**	**1.40**	**2.66**	**2.70**	**2.53**	**2.16**	**1.37**	**2.22**	**2.24**	**2.16**	**1.80**	**1.18**	**1.35**	**1.40**	**1.37**	**1.18**	**0.82**

Difference 1=AATS of FSI control chart−max. AATS of ASI control chart
Difference 2=AATS of FSI control chart−min. AATS of ASI control chart

adaptive economic statistical modeling of dependent process steps subject to multiple assignable causes.

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Appendix

The calculation of the transition probabilities are as follows:

$P_{ij}(t_k) \quad i = 1, \ldots, 17, \; j = 1, \ldots, 17, \; k = 1, 2, 3$

$$P_{1,1}(t_3) = P[|Z_X| < w_X] \times P[|Z_e| < w_e] \times e^{-\lambda_1 t_3} \times e^{-\lambda_2 t_3}$$
$$P_{1,2}(t_3) = P[|Z_X| < w_X] \times P[|Z_e| < w_e|\delta_2] \times e^{-\lambda_1 t_3} \times (1 - e^{-\lambda_2 t_3})$$
$$P_{1,3}(t_3) = P[|Z_X| < w_X|\delta_1] \times P[|Z_e \,| < w_e] \times \left(1 - e^{-\lambda_1 t_3}\right) \times e^{-\lambda_2 t_3}$$
$$P_{1,4}(t_3) = P[|Z_X| < w_X|\delta_1] \times P[|Z_e| < w_e|\delta_2] \times \left(1 - e^{-\lambda_1 t_3}\right) \times \left(1 - e^{-\lambda_2 t_3}\right)$$
$$P_{1,5}(t_3) = P[-k_X < Z_X \leq -w_X \cup w_X \leq Z_X < k_X] \times P[|Z_e| < w_e] \times e^{-\lambda_1 t_3} \times e^{-\lambda_2 t_3}$$
$$P_{1,6}(t_3) = P[-k_X < Z_X \leq -w_X \cup w_X \leq Z_X < k_X] \times P[|Z_e| < w_e|\delta_2] \times e^{-\lambda_1 t_3} \times \left(1 - e^{-\lambda_2 t_3}\right)$$
$$P_{1,7}(t_3) = P[-k_X < Z_X \leq -w_X \cup w_X \leq Z_X < k_X|\delta_1] \times P[|Z_e| < w_e] \times \left(1 - e^{-\lambda_1 t_3}\right) \times e^{-\lambda_2 t_3}$$
$$P_{1,8}(t_3) = P[-k_X < Z_X \leq -w_X \cup w_X \leq Z_X < k_X|\delta_1] \times P[|Z_e| < w_e|\delta_2] \times \left(1 - e^{-\lambda_1 t_3}\right) \times \left(1 - e^{-\lambda_2 t_3}\right)$$
$$P_{1,9}(t_3) = P[|Z_X| < w_X] \times P[-k_e < Z_e \leq -w_e \cup w_e \leq Z_e < k_e] \times e^{-\lambda_1 t_3} \times e^{-\lambda_2 t_3}$$
$$P_{1,10}(t_3) = P[|Z_X| < w_X] \times P[-k_e < Z_e \leq -w_e \cup w_e \leq Z_e < k_e|\delta_2] \times e^{-\lambda_1 t_3} \times \left(1 - e^{-\lambda_2 t_3}\right)$$
$$P_{1,11}(t_3) = P[|Z_X| < w_X|\delta_1] \times P[-k_e < Z_e \leq -w_e \cup w_e \leq Z_e < k_e] \times \left(1 - e^{-\lambda_1 t_3}\right) \times e^{-\lambda_2 t_3}$$
$$P_{1,12}(t_3) = P[|Z_X| < w_X|\delta_1] \times P[-k_e < Z_e \leq -w_e \cup w_e \leq Z_e < k_e|\delta_2] \times \left(1 - e^{-\lambda_1 t_3}\right) \times \left(1 - e^{-\lambda_2 t_3}\right)$$
$$P_{1,13}(t_3) = P[-k_X < Z_X \leq -w_X \cup w_X \leq Z_X < k_X] \times P[-k_e < Z_e \leq -w_e \cup w_e \leq Z_e < k_e] \times e^{-\lambda_1 t_3} \times e^{-\lambda_2 t_3}$$
$$P_{1,14}(t_3) = P[-k_X < Z_X \leq -w_X \cup w_X \leq Z_X < k_X] \times P[-k_e < Z_e \leq -w_e \cup w_e \leq Z_e < k_{ee}|\delta_2] \times e^{-\lambda_1 t_1} \times \left(1 - e^{-\lambda_2 t_1}\right)$$
$$P_{1,15}(t_3) = P[-k_X < Z_X \leq -w_X \cup w_X \leq Z_X < k_X|\delta_1] \times P[-k_e < Z_e \leq -w_e \cup w_e \leq Z_e < k_e] \times \left(1 - e^{-\lambda_1 t_3}\right) \times e^{-\lambda_2 t_3}$$
$$P_{1,16}(t_3) = P[-k_X < Z_X \leq -w_X \cup w_X \leq Z_X < k_X|\delta_1] \\ \times P[-k_e < Z_e \leq -w_e \cup w_e \leq Z_e < k_e|\delta_2] \times \left(1 - e^{-\lambda_1 t_3}\right) \times \left(1 - e^{-\lambda_2 t_3}\right)$$
$$P_{1,17}(t_3) = 1 - \sum_{j=1}^{16} P_{1,j}(t_3)$$

$$P_{2,1}(t_3) = 0$$
$$P_{2,2}(t_3) = P[|Z_X| < w_X] \times P[|Z_e| < w_e|\delta_2] \times e^{-\lambda_1 t_3}$$
$$P_{2,3}(t_3) = 0$$
$$P_{2,4}(t_3) = P[|Z_X| < w_X|\delta_1] \times P[|Z_e| < w_e|\delta_2] \times \left(1 - e^{-\lambda_1 t_3}\right)$$
$$P_{2,5}(t_3) = 0$$
$$P_{2,6}(t_3) = P[-k_X < Z_X \leq -w_X \cup w_X \leq Z_X < k_X] \times P[|Z_e| < w_e|\delta_2] \times e^{-\lambda_1 t_3}$$
$$P_{2,7}(t_3) = 0$$
$$P_{2,8}(t_3) = P[-k_X < Z_X \leq -w_X \cup w_X \leq Z_X < k_X|\delta_1] \times P[|Z_e| < w_e|\delta_2] \times \left(1 - e^{-\lambda_1 t_3}\right)$$

$$P_{2,9}(t_3) = 0$$
$$P_{2,10}(t_3) = P[|Z_X| < w_X] \times P[-k_e < Z_e \le -w_e \cup w_e \le Z_e < k_e|\delta_2] \times e^{-\lambda_1 t_3}$$
$$P_{2,11}(t_3) = 0$$
$$P_{2,12}(t_3) = P[|Z_X| < w_X|\delta_1] \times P[-k_e < Z_e \le -w_e \cup w_e \le Z_e < k_e|\delta_2] \times \left(1 - e^{-\lambda_1 t_3}\right)$$
$$P_{2,13}(t_3) = 0$$
$$P_{2,14}(t_3) = P[-k_X < Z_X \le -w_X \cup w_X \le Z_X < k_X] \times P[-k_e < Z_e \le -w_e \cup w_e \le Z_e < k_e|\delta_2] \times e^{-\lambda_1 t_3}$$
$$P_{2,15}(t_3) = 0$$
$$P_{2,16}(t_3) = P[-k_X < Z_X \le -w_X \cup w_X \le Z_X < k_X|\delta_1] \times P[-k_e < Z_e \le -w_e \cup w_e \le Z_e < k_e|\delta_2] \times \left(1 - e^{-\lambda_1 t_3}\right)$$
$$P_{2,17}(t_3) = 1 - \sum_{j=1}^{16} P_{2,j}(t_3)$$

$$P_{3,1}(t_3) = 0$$
$$P_{3,2}(t_3) = 0$$
$$P_{3,3}(t_3) = P[|Z_X| < w_X|\delta_1] \times P[|Z_e| < w_e] \times e^{-\lambda_2 t_3}$$
$$P_{3,4}(t_3) = P[|Z_X| < w_X|\delta_1] \times P[|Z_e| < w_e|\delta_2] \times \left(1 - e^{-\lambda_2 t_3}\right)$$
$$P_{3,5}(t_3) = 0$$
$$P_{3,6}(t_3) = 0$$
$$P_{3,7}(t_3) = P[-k_X < Z_X \le -w_X \cup w_X \le Z_X < k_X|\delta_1] \times P[|Z_e| < w_e] \times e^{-\lambda_2 t_3}$$
$$P_{3,8}(t_3) = P[-k_X < Z_X \le -w_X \cup w_X \le Z_X < k_X|\delta_1] \times P[|Z_e| < w_e|\delta_2] \times \left(1 - e^{-\lambda_2 t_3}\right)$$
$$P_{3,9}(t_3) = 0$$
$$P_{3,10}(t_3) = 0$$
$$P_{3,11}(t_3) = P[|Z_X| < w_X|\delta_1] \times P[-k_e < Z_e \le -w_e \cup w_e \le Z_e < k_e] \times e^{-\lambda_2 t_3}$$
$$P_{3,12}(t_3) = P[|Z_X| < w_X|\delta_1] \times P[-k_e < Z_e \le -w_e \cup w_e \le Z_e < k_e|\delta_2] \times \left(1 - e^{-\lambda_2 t_3}\right)$$
$$P_{3,13}(t_3) = 0$$
$$P_{3,14}(t_3) = 0$$
$$P_{3,15}(t_3) = P[-k_X < Z_X \le -w_X \cup w_X \le Z_X < k_X|\delta_1] \times P[-k_e < Z_e \le -w_e \cup w_e \le Z_e < k_e] \times e^{-\lambda_2 t_3}$$
$$P_{3,16}(t_3) = P[-k_X < Z_X \le -w_X \cup w_X \le Z_X < k_X|\delta_1] \times P[-k_e < Z_e \le -w_e \cup w_e \le Z_e < k_e|\delta_2] \times \left(1 - e^{-\lambda_2 t_3}\right)$$
$$P_{3,17}(t_3) = 1 - \sum_{j=1}^{16} P_{3,j}(t_3)$$

$$P_{4,1}(t_3) = P_{4,2}(t_3) = P_{4,3}(t_3) = 0$$
$$P_{4,4}(t_3) = P(|Z_X| < w_X|\delta_1) \times P(Z_e < w_e|\delta_2)$$
$$P_{4,5}(t_3) = P_{4,6}(t_3) = P_{4,7}(t_3) = 0$$
$$P_{4,8}(t_3) = P(-k_X < Z_X \le w_X \cup w_X \le Z_X < k_X|\delta_1) \times P(Z_e < w_e|\delta_2)$$
$$P_{4,9}(t_3) = P_{4,10}(t_3) = P_{4,11}(t_3) = 0$$
$$P_{4,12}(t_3) = P(|Z_X| < w_X|\delta_1) \times P(-k_e < Z_e \le w_e \cup w_e \le Z_e < k_e|\delta_2)$$
$$P_{4,13}(t_3) = P_{4,14}(t_3) = P_{4,15}(t_3) = 0$$
$$P_{4,16}(t_3) = P(-k_X < Z_X \le w_X \cup w_X \le Z_X < k_X|\delta_1) \times P(-k_e < Z_e \le w_e \cup w_e \le Z_e < k_e|\delta_2)$$
$$P_{4,17}(t_3) = 1 - \sum_{j=1}^{16} P_{4,j}(t_3)$$

The transition probabilities for $P_{5,j}(t_2)$, $P_{6,j}(t_2)$, $P_{7,j}(t_2)$, $P_{8,j}(t_2)$, $P_{9,j}(t_2)$, $P_{10,j}(t_2)$, $P_{11,j}(t_2)$, and $P_{12,j}(t_2)$ are calculated by replacing t_2 on t_3 for $P_{1,j}(t_3)$, $P_{2,j}(t_3)$, $P_{3,j}(t_3)$, and $P_{4,j}(t_3)$, j=1, 2,..., 17:

$$\begin{array}{ll} P_{5,j}(t_2) = P_{1,j}(t_2) & j = 1, 2, \ldots, 17 \\ P_{6,j}(t_2) = P_{2,j}(t_2) & j = 1, 2, \ldots, 17 \\ P_{7,j}(t_2) = P_{3,j}(t_2) & j = 1, 2, \ldots, 17 \\ P_{8,j}(t_2) = P_{4,j}(t_2) & j = 1, 2, \ldots, 17 \\ P_{9,j}(t_2) = P_{1,j}(t_2) & j = 1, 2, \ldots, 17 \\ P_{10,j}(t_2) = P_{2,j}(t_2) & j = 1, 2, \ldots, 17 \\ P_{11,j}(t_2) = P_{3,j}(t_2) & j = 1, 2, \ldots, 17 \\ P_{12,j}(t_2) = P_{4,j}(t_2) & j = 1, 2, \ldots, 17 \end{array}$$

The transition probabilities for $P_{13,j}(t_1)$, $P_{14,j}(t_1)$, $P_{15,j}(t_1)$, and $P_{16,j}(t_1)$ are calculated by replacing t_1 on t_3 for $P_{1,j}(t_3)$, $P_{2,j}(t_3)$, $P_{3,j}(t_3)$, and $P_{4,j}(t_3)$, j=1, 2,..., 17:

$$\begin{array}{ll} P_{13,j}(t_1) = P_{1,j}(t_1) & j = 1, 2, \ldots, 17 \\ P_{14,j}(t_1) = P_{2,j}(t_1) & j = 1, 2, \ldots, 17 \\ P_{15,j}(t_1) = P_{3,j}(t_1) & j = 1, 2, \ldots, 17 \\ P_{16,j}(t_1) = P_{4,j}(t_1) & j = 1, 2, \ldots, 17 \\ P_{17,j} = 0 \quad j \neq 17 & \\ P_{17,17} = 1 & \end{array}$$

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