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A Stochastic Directional Convexity Result and Its Application in Comparison of Queues

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Abstract. Second order properties of queues are important in design and analysis of service systems. In this paper we show that the blocking probability of $M/M/C/N$ queue is increasing directionally convex in $(\lambda, -\mu)$, where λ is arrival rate and μ is service rate. To illustrate the usefulness of this result we consider a heterogeneous queueing system with non-stationary arrival and service processes. The arrival and service rates alternate between two levels (λ_1, μ_1) and (λ_2, μ_2), spending an exponentially distributed amount of time with rate $c\alpha_i$ in level i, $i = 1, 2$. When the system is in state i, the arrival rate is λ_i and the service rate is μ_i. Applying the increasing directional convexity result we show that the blocking probability is decreasing in c, extending a result of Fond and Ross [7] for the case $C = N = 1$.

Keywords: heterogeneous queues, monotonicity, blocking probability, stochastic directional convexity, increasing convex ordering

AMS subject classification:

1. Introduction

A function $f(x, y) : R^2 \to R$, is called supermodular if for any $x_1 \geqslant x_2$ and $y_1 \geqslant y_2$,

$$f(x_1, y_1) + f(x_2, y_2) \geqslant f(x_1, y_2) + f(x_2, y_1).$$

If, in addition, $f(x, y)$ is convex in each component, then it is called directionally convex. If f is differentiable then it is directionally convex if and only if

$$\frac{\partial^2 f(x, y)}{\partial x^2} \geqslant 0, \qquad \frac{\partial^2 f(x, y)}{\partial x \partial y} \geqslant 0, \qquad \frac{\partial^2 f(x, y)}{\partial y^2} \geqslant 0.$$

The function is called increasing directionally convex if f is also increasing in each of its components. Directional convexity is a very important notion and it plays a role

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in stochastic convexity, multivariate statistics, reliability, insurance and risk analysis (see [10,16]). In this paper we consider an $M/M/C/N$ queue with arrival rate λ and service rate μ, and show that its blocking probability is increasing directionally convex in $(\lambda, -\mu)$. However, the blocking probability is not jointly convex in $(\lambda, -\mu)$. To illustrate the usefulness of this result we consider a heterogeneous queueing system with C servers and buffer size N. The system oscillates between two levels denoted by 1 and 2. When the system is in level i ($i = 1, 2$), it is subject to a Poisson arrival process of rate λ_i and exponential service times with rate μ_i. The time interval during which the system functions in level i is also exponential with rate $c\alpha_i$. When $C = N = 1$, this system has been studied by Fond and Ross [7] and they showed that the proportion of lost customers is a decreasing function of c. Applying the increasing directional convexity result we are able to extend their result to arbitrary C and N. In fact, we show that a stronger result holds: the number of blocked customers by time t is decreasing in c in stochastic convex ordering for any $t \geqslant 0$. In particular, when $N = \infty$, the result is reduced to that the number of customers in system at any time t is decreasing in c in stochastic convex ordering.

It is generally believed that a system with more regular arrival and service processes tends to perform better than the one with less regular arrival and service processes. Since this problem, known as Ross's conjecture, was raised by Ross [15], there has been an extensive stream of research devoted to it. See, for example [1,3,7,8,11–14], among others. The parameter c in our model regulates the rate at which the system switches between the two levels. As c increases, the switching rate increases and intuitively the arrival and service processes become more regular. When c goes infinity, the arrival process converges to a Poisson process and the service times become exponential, thus our model is reduced to an $M/M/C/N$. On the other hand when c is zero, the system becomes an $M/M/C/N$ queue with arrival and service rates being λ_i, μ_i with probability π_i, $i = 1, 2$, where π_i is the steady state probability of the system being in level i. The monotonicity result as well as the upper and lower bounds for this queueing system can be used to develop approximation formulas for the system performance.

This paper is organized as follows. In the next section we establish the increasing directional convexity result for $M/M/C/N$ queue, and in section 3 we apply this result to stochastic comparison of queues. Some tedious mathematical derivation is provided in the appendix. Throughout this paper, we use "increasing" and "decreasing" in the nonstrict sense, i.e., they represent "nondecreasing" and "non-increasing", respectively.

2. Directional stochastic convexity result

To present our result, we need several definitions from [3]. A random vector $\boldsymbol{X}(\boldsymbol{\theta}) = (X_1(\boldsymbol{\theta}), \ldots, X_n(\boldsymbol{\theta}))$ parameterized by a vector $\boldsymbol{\theta}$ is *stochastically increasing and directionally convex for each partial sum in sample path sense* (SIDCX-PS(sp)) *in* $\boldsymbol{\theta}$ if for every choice of vector $\boldsymbol{\theta}^{(i)}$, $i = 1, 2, 3, 4$, satisfying conditions (where "C" represents "Condition"):

(C1) $\boldsymbol{\theta}^{(i)} \leqslant \boldsymbol{\theta}^{(4)}$, $i = 1, 2, 3$,

(C2) $\boldsymbol{\theta}^{(2)} + \boldsymbol{\theta}^{(3)} = \boldsymbol{\theta}^{(1)} + \boldsymbol{\theta}^{(4)}$,

where inequality is componentwise, there exist four random vectors $\widehat{\boldsymbol{X}}^{(i)}$, $i = 1, 2, 3, 4$, defined on a common probability space such that

(C3) $\widehat{\boldsymbol{X}}^{(i)} \stackrel{\text{st}}{=} \boldsymbol{X}(\boldsymbol{\theta}^{(i)})$, $i = 1, 2, 3, 4$,

(C4) $\widehat{\boldsymbol{X}}^{(i)} \leqslant \widehat{\boldsymbol{X}}^{(4)}$ a.s., $i = 1, 2, 3$,

(C5) $\sum_{\ell=1}^{j} \widehat{X}_\ell^{(1)} + \sum_{\ell=1}^{j} \widehat{X}_\ell^{(4)} \geqslant \sum_{\ell=1}^{j} \widehat{X}_\ell^{(2)} + \sum_{\ell=1}^{j} \widehat{X}_\ell^{(3)}$ a.s., $j = 1, \ldots, n$.

For brevity we write $\boldsymbol{X}(\boldsymbol{\theta}) \in$ SIDCX-PS(sp). In particular, when $n = 1$, $\boldsymbol{X}(\boldsymbol{\theta})$ is a random variable and is written as $X(\boldsymbol{\theta})$, then $X(\boldsymbol{\theta})$ is said to be *stochastically increasing and directionally convex* in $\boldsymbol{\theta}$ in *sample path* sense, denoted by $X(\boldsymbol{\theta}) \in$ SIDCX(sp). A multi-dimensional stochastic process $\boldsymbol{X}_t(\boldsymbol{\theta}) = (X_{1,t}(\boldsymbol{\theta}), \ldots, X_{n,t}(\boldsymbol{\theta}))$ parameterized by $\boldsymbol{\theta}$ is said to be SIDCX-PS(sp) in $\boldsymbol{\theta}$ if $\boldsymbol{X}_t(\boldsymbol{\theta})$ is SIDCX-PS(sp) for all $t > 0$, given that $\boldsymbol{X}_0(\boldsymbol{\theta})$ is SIDCX-PS(sp). If $\boldsymbol{X}_t(\boldsymbol{\theta})$ is SIDCX-PS(sp) in $\boldsymbol{\theta}$, then it is easily verified that $E[f(\sum_{\ell=1}^{j} X_{\ell,t}(\boldsymbol{\theta}))]$ is increasing directionally convex in $\boldsymbol{\theta}$ for any $1 \leqslant j \leqslant n$ and any $t \geqslant 0$, and for all increasing convex function f.

The following lemma will be needed in proving our result.

Lemma 1. For any nonnegative numbers μ_i and x_i, $i = 1, 2, 3, 4$, and positive number C satisfying $\mu_1 \geqslant \mu_i \geqslant \mu_4$ and $x_1 \leqslant x_i \leqslant x_4$ such that $\mu_1 + \mu_4 = \mu_2 + \mu_3$ and $x_1 + x_4 = x_2 + x_3$, the following inequality holds:

$$\mu_1 \min\{x_1, C\} + \mu_4 \min\{x_4, C\} \leqslant \mu_2 \min\{x_2, C\} + \mu_3 \min\{x_3, C\}.$$

Proof. Define a function $f(x, u) = u \min\{x, C\}$ on $u \leqslant 0$ and $x \geqslant 0$. We first show that f is directionally convex. It follows from the concavity of $\min\{x, C\}$ in x and $u \leqslant 0$ that f is convex in x. Also f clearly convex in u. Therefore to prove f is directionally convex it suffices to show that f is supermodular. To that end consider any $u_1 \leqslant u_2 \leqslant 0$ and $0 \leqslant x_1 \leqslant x_2$. It is easily seen that

$$f(x_1, u_1) + f(x_2, u_2) \geqslant f(x_1, u_2) + f(x_2, u_1)$$

is equivalent to

$$(u_1 - u_2)\big(\min\{x_1, C\} - \min\{x_2, C\}\big) \geqslant 0,$$

which is obviously satisfied. Thus, f is supermodular. This shows that f is directionally convex.

It is well known (see, e.g., [16]) that $f(x, u)$ is directionally convex if and only if for any given (x_i, u_i), $i = 1, 2, 3, 4$, satisfying

$$\begin{aligned} (x_i, u_i) &\leqslant (x_4, u_4), \quad i = 1, 2, 3, \\ (x_1, u_1) + (x_4, u_4) &= (x_2, u_2) + (x_3, u_3), \end{aligned}$$

we have

$$f(x_1, u_1) + f(x_4, u_4) \geqslant f(x_2, u_2) + f(x_3, u_3). \tag{1}$$

Letting $\mu_i = -u_i$ ($i = 1, 2, 3, 4$) and substituting them into (1) completes the proof. □

We now consider an $M/M/C/N$ queueing system with C servers and buffer size N (including those in service), which may be infinity. The arrival process is Poisson with rate λ, and the service rate is μ. Let B_t be the number of blocked customers of this system by time t, and Q_t the number of customers in the system at time t. The state of the system is defined as $\boldsymbol{X}_t = (B_t, Q_t)$, $t \geqslant 0$.

Theorem 1. The process $\boldsymbol{X}_t$ is increasing directionally convex in $\boldsymbol{\theta} = (\lambda, -\mu)$ for each partial sum in sample path sense, i.e., $\boldsymbol{X}_t \in$ SIDCX-PS(sp) in $(\lambda, -\mu)$. In particular, the number of blocked customers $B_t \in$ SIDCX(sp) in $\boldsymbol{\theta} = (\lambda, -\mu)$.

The following result follows immediately from theorem 1.

Corollary 1. The blocking probability P_{B} of $M/M/C/N$ queue, defined by

$$P_{\mathrm{B}} = \lim_{t\to\infty} \frac{B_t}{N_t} = \frac{1}{\lambda} \lim_{t\to\infty} \frac{EB_t}{t},$$

is increasing directionally convex in $(\lambda, -\mu)$. The average number of customers in $M/M/C$ queue is also increasing directionally convex in $(\lambda, -\mu)$.

Proof of theorem 1. We write $\boldsymbol{X}_t$ as $\boldsymbol{X}_t(\lambda, \mu)$ to emphasize the dependence on the arrival and service rates. We need to show that, for any $(\lambda_1, -\mu_1)$, $(\lambda_2, -\mu_2)$, $(\lambda_3, -\mu_3)$, and $(\lambda_4, -\mu_4)$ satisfying

$$(\lambda_1, -\mu_1) \leqslant (\lambda_i, -\mu_i) \leqslant (\lambda_4, -\mu_4), \quad i = 1, 2, 3, 4,$$

and

$$(\lambda_1, -\mu_1) + (\lambda_4, -\mu_4) = (\lambda_2, -\mu_2) + (\lambda_3, -\mu_3),$$

and that for any initial conditions $\widehat{\boldsymbol{X}}_0^{(i)} = (\widehat{B}_0^{(i)}, \widehat{Q}_0^{(i)})$, $i = 1, 2, 3, 4$, satisfying

(C3B) $B_0(\lambda_i, \mu_i) \overset{\mathrm{st}}{=} \widehat{B}_0^{(i)}, \quad i = 1, 2, 3, 4,$

(C3Q) $Q_0(\lambda_i, \mu_i) \overset{\mathrm{st}}{=} \widehat{Q}_0^{(i)}, \quad i = 1, 2, 3, 4,$

(C4B) $\widehat{B}_0^{(i)} \leqslant \widehat{B}_0^{(4)}$ a.s., $\quad i = 1, 2, 3,$

(C4Q) $\widehat{Q}_0^{(i)} \leqslant \widehat{Q}_0^{(4)}$ a.s., $\quad i = 1, 2, 3,$

(C5B) $\widehat{B}_0^{(1)} + \widehat{B}_0^{(4)} \geqslant \widehat{B}_0^{(2)} + \widehat{B}_0^{(3)}$ a.s.,

(C5Q) $\widehat{B}_0^{(1)} + \widehat{Q}_0^{(1)} + \widehat{B}_0^{(4)} + \widehat{Q}_0^{(4)} \geqslant \widehat{B}_0^{(2)} + \widehat{Q}_0^{(2)} + \widehat{B}_0^{(3)} + \widehat{Q}_0^{(3)}$ a.s., we can construct four processes $\widehat{X}_t^{(i)} = (\widehat{B}_t^{(i)}, \widehat{Q}_t^{(i)})$, $i = 1, 2, 3, 4$, such that for all $t \geqslant 0$,

$$
\begin{array}{ll}
\text{(C3B)} & B_t(\lambda_i, \mu_i) \stackrel{\text{st}}{=} \widehat{B}_t^{(i)}, \quad i = 1, 2, 3, 4, \\
\text{(C3Q)} & Q_t(\lambda_i, \mu_i) \stackrel{\text{st}}{=} \widehat{Q}_t^{(i)}, \quad i = 1, 2, 3, 4, \\
\text{(C4B)} & \widehat{B}_t^{(i)} \leqslant \widehat{B}_t^{(4)} \quad \text{a.s.}, \quad i = 1, 2, 3, \\
\text{(C4Q)} & \widehat{Q}_t^{(i)} \leqslant \widehat{Q}_t^{(4)} \quad \text{a.s.}, \quad i = 1, 2, 3, \\
\text{(C5B)} & \widehat{B}_t^{(1)} + \widehat{B}_t^{(4)} \geqslant \widehat{B}_t^{(2)} + \widehat{B}_t^{(3)} \quad \text{a.s.}, \\
\text{(C5Q)} & \widehat{B}_t^{(1)} + \widehat{Q}_t^{(1)} + \widehat{B}_t^{(4)} + \widehat{Q}_t^{(4)} \geqslant \widehat{B}_t^{(2)} + \widehat{Q}_t^{(2)} + \widehat{B}_t^{(3)} + \widehat{Q}_t^{(3)} \quad \text{a.s.}
\end{array}
$$

Note that under these assumptions we have $\lambda_1 \leqslant \lambda_i \leqslant \lambda_4$ and $\mu_1 \geqslant \mu_i \geqslant \mu_4$ for $i = 1, 2, 3, 4$.

We use the standard coupling and uniformization technique [9] to prove this result. Two types of events could occur in each of the four systems under consideration: arrivals, and departures, with respective rates λ_i and $\min\{Q_t(\lambda_i, \mu_i), C\}\mu_i$, $i = 1, 2, 3, 4$. Since $\lambda_4 \geqslant \lambda_i$ and $\mu_1 \geqslant \mu_i$, $i = 1, 2, 3, 4$, we can construct all these events by generating a Poisson process with rate $\Delta = \lambda_4 + 2C\mu_1$. Without loss of generality, we assume $\Delta = 1$. Let $\{\tau_n,\ n \geqslant 1\}$ be the arrival epochs of this Poisson process. Further we generate a sequence of independent random variables $\{U_n,\ n \geqslant 1\}$ that are uniformly distributed on [0,1]. Note that between τ_n and τ_{n+1}, the state of each of the four queueing systems does not change. Hence we only need to show that $\widehat{X}_t^{(i)}$ can be constructed such that (C3B) to (C5Q) are satisfied at the time epochs τ_n, $n = 0, 1, \ldots$. For notational simplicity, in what follows we write $\widehat{X}_{\tau_n}$, $\widehat{B}_{\tau_n}^{(i)}$, and $\widehat{Q}_{\tau_n}^{(i)}$ as $\widehat{X}_n$, $\widehat{B}_n^{(i)}$, and $\widehat{Q}_n^{(i)}$. Thus, we need to construct the processes $\widehat{X}_n^{(i)} = (\widehat{B}_n^{(i)}, \widehat{Q}_n^{(i)})$ such that for all $n \geqslant 0$, the following are satisfied:

$$
\begin{array}{lll}
\text{(C3B)} & B_{\tau_n}(\lambda_i, \mu_i) \stackrel{\text{st}}{=} \widehat{B}_n^{(i)}, & i = 1, 2, 3, 4, \\
\text{(C3Q)} & Q_{\tau_n}(\lambda_i, \mu_i) \stackrel{\text{st}}{=} \widehat{Q}_n^{(i)}, & i = 1, 2, 3, 4, \\
\text{(C4B)} & \widehat{B}_n^{(i)} \leqslant \widehat{B}_n^{(4)}, & i = 1, 2, 3, \\
\text{(C4Q)} & \widehat{Q}_n^{(i)} \leqslant \widehat{Q}_n^{(4)}, & i = 1, 2, 3, \\
\text{(C5B)} & \widehat{B}_n^{(1)} + \widehat{B}_n^{(4)} \geqslant \widehat{B}_n^{(2)} + \widehat{B}_n^{(3)}, & \\
\text{(C5Q)} & \widehat{B}_n^{(1)} + \widehat{Q}_n^{(1)} + \widehat{B}_n^{(4)} + \widehat{Q}_n^{(4)} \geqslant \widehat{B}_n^{(2)} + \widehat{Q}_n^{(2)} + \widehat{B}_n^{(3)} + \widehat{Q}_n^{(3)}. &
\end{array}
$$

The stochastic processes $\widehat{X}_n^{(i)}$, $i = 1, 2, 3, 4$, for the four queueing systems are constructed by induction. When $n = 0$, $\widehat{X}_0^{(i)}$ is just the initial condition, that satisfy (C3B) to (C5Q). Assume that $\widehat{X}_n^{(i)}$, $i = 1, \ldots, 4$, have been constructed such that (C3B) to (C5Q) are satisfied, we need to construct $\widehat{X}_{n+1}^{(i)}$ such that (C3B) to (C5Q) are also satisfied. We consider two cases.

First we consider the case $0 \leqslant U_n \leqslant \lambda_4$. In this case only arrivals to each system may occur. The state of $\widehat{X}_{n+1}^{(i)}$ is constructed as follows:

$$\widehat{B}_{n+1}^{(i)} = \widehat{B}_n^{(i)} + \beta_{n+1}^{(i)} \mathbb{1}[\widehat{Q}_n^{(i)} = N], \quad i = 1, 2, 3, 4,$$
$$\widehat{Q}_{n+1}^{(i)} = \widehat{Q}_n^{(i)} + \beta_{n+1}^{(i)} \mathbb{1}[\widehat{Q}_n^{(i)} < N], \quad i = 1, 2, 3, 4,$$

where $\mathbb{1}[A]$ is the indicator function and

$$\beta_{n+1}^{(1)} = \mathbb{1}[\lambda_4 - \lambda_3 \leqslant U_{n+1} < \lambda_2],$$
$$\beta_{n+1}^{(2)} = \mathbb{1}[0 \leqslant U_{n+1} < \lambda_2],$$
$$\beta_{n+1}^{(3)} = \mathbb{1}[\lambda_4 - \lambda_3 \leqslant U_{n+1} < \lambda_4],$$
$$\beta_{n+1}^{(4)} = \mathbb{1}[0 \leqslant U_{n+1} < \lambda_4].$$

We can prove that the conditions from (C3B) to (C5Q) remain satisfied at τ_{n+1}. However since its proof is similar to that of [3], its details are omitted.

We then consider the case $\lambda_4 \leqslant U_n \leqslant 1$. In this case only departures may occur from the four queueing systems. Construct $\widehat{X}_{n+1}^{(i)}$ as follows:

$$\widehat{Q}_{n+1}^{(i)} = \widehat{Q}_n^{(i)} - \beta_{n+1}^{(i)}, \quad i = 1, 2, 3, 4, \tag{2}$$
$$\widehat{B}_{n+1}^{(i)} = \widehat{B}_n^{(i)}, \quad i = 1, 2, 3, 4. \tag{3}$$

The construction of $\beta_{n+1}^{(i)}$ depends on the state of the systems at time τ_n. For convenience, we write

$$a_i = \mu_i \min\{\widehat{Q}_n^{(i)}, C\}, \quad i = 1, 2, 3, 4.$$

Two cases are discussed below. If

$$\widehat{Q}_n^{(1)} + \widehat{Q}_n^{(4)} \leqslant \widehat{Q}_n^{(2)} + \widehat{Q}_n^{(3)}, \tag{4}$$

then $\beta_{n+1}^{(i)}$ is given by

$$\beta_{n+1}^{(i)} = \mathbb{1}[\lambda_4 + C\mu_1 \leqslant U_{n+1} < \lambda_4 + C\mu_1 + a_i], \quad i = 2, 4, \tag{5}$$
$$\beta_{n+1}^{(1)} = \mathbb{1}[\lambda_4 + C\mu_1 + a_2 - a_1 \leqslant U_{n+1} < \lambda_4 + C\mu_1 + a_2], \tag{6}$$
$$\beta_{n+1}^{(3)} = \mathbb{1}[\lambda_4 + C\mu_1 + a_4 - a_3 \leqslant U_{n+1} < \lambda_4 + C\mu_1 + a_4]. \tag{7}$$

If

$$\widehat{Q}_n^{(1)} + \widehat{Q}_n^{(4)} > \widehat{Q}_n^{(2)} + \widehat{Q}_n^{(3)}, \tag{8}$$

then $\beta_{n+1}^{(i)}$ are defined the same as (5) and (7) for $i = 2, 3, 4$, but $\beta_{n+1}^{(1)}$ is defined as

$$\beta_{n+1}^{(1)} = \mathbb{1}[\lambda_4 + C\mu_1 + a_4 - a_1 \leqslant U_{n+1} < \lambda_4 + C\mu_1 + a_4]. \tag{9}$$

That (C3B) and (C3Q) are satisfied follows easily from the construction process. To prove (C4B)–(C5Q), we first assume (4). Since τ_{n+1} is a potential departure epoch,

$\widehat{B}_n^{(i)} = \widehat{B}_{n+1}^{(i)}$ and (C4B) follows from the induction hypothesis. We then prove (C4Q). To prove $\widehat{Q}_{n+1}^{(3)} \leqslant \widehat{Q}_{n+1}^{(4)}$, we consider two cases.

(a) If $\beta_{n+1}^{(3)} \geqslant \beta_{n+1}^{(4)}$, then necessarily we have

$$\widehat{Q}_{n+1}^{(3)} = \widehat{Q}_n^{(3)} - \beta_{n+1}^{(3)} \leqslant \widehat{Q}_n^{(4)} - \beta_{n+1}^{(4)} = \widehat{Q}_{n+1}^{(4)}.$$

(b) If $\beta_{n+1}^{(3)} < \beta_{n+1}^{(4)}$, then $\beta_{n+1}^{(3)} = 0$ and $\beta_{n+1}^{(4)} = 1$, implying

$$U_{n+1} \in [\lambda_4 + C\mu_1, \lambda_4 + C\mu_1 + a_4)\,,$$
$$U_{n+1} \notin [\lambda_4 + C\mu_1 + a_4 - a_3, \lambda_4 + C\mu_1 + a_4)\,.$$

This shows that we must have $a_4 > a_3$. Hence using the fact $\mu_4 \leqslant \mu_i$ we obtain

$$\min\{\widehat{Q}_n^{(4)}, C\} > \frac{\mu_3}{\mu_4} \min\{\widehat{Q}_n^{(3)}, C\} \geqslant \min\{\widehat{Q}_n^{(3)}, C\},$$

implying $\widehat{Q}_n^{(4)} > \widehat{Q}_n^{(3)}$ and consequently

$$\widehat{Q}_{n+1}^{(4)} = \widehat{Q}_n^{(4)} - 1 \geqslant \widehat{Q}_n^{(3)} = \widehat{Q}_{n+1}^{(3)}.$$

We next prove $\widehat{Q}_{n+1}^{(2)} \leqslant \widehat{Q}_{n+1}^{(4)}$. Again consider two cases.

(a) If $\beta_{n+1}^{(2)} \geqslant \beta_{n+1}^{(4)}$, then $\widehat{Q}_n^{(2)} - \beta_{n+1}^{(2)} \leqslant \widehat{Q}_n^{(4)} - \beta_{n+1}^{(4)}$ and $\widehat{Q}_{n+1}^{(2)} \leqslant \widehat{Q}_{n+1}^{(4)}$.

(b) If $\beta_{n+1}^{(2)} < \beta_{n+1}^{(4)}$, then $\beta_{n+1}^{(2)} = 0$ and $\beta_{n+1}^{(4)} = 1$, implying $a_4 > a_2$, i.e.,

$$\mu_4 \min\{\widehat{Q}_n^{(4)}, C\} > \mu_2 \min\{\widehat{Q}_n^{(2)}, C\}.$$

Equivalently, it produces

$$\min\{\widehat{Q}_n^{(4)}, C\} > \frac{\mu_2}{\mu_4} \min\{\widehat{Q}_n^{(2)}, C\} \geqslant \min\{\widehat{Q}_n^{(2)}, C\}.$$

Therefore we conclude $\widehat{Q}_n^{(4)} > \widehat{Q}_n^{(2)}$ and $\widehat{Q}_n^{(4)} - 1 \geqslant \widehat{Q}_n^{(2)} = \widehat{Q}_{n+1}^{(2)}$, yielding $\widehat{Q}_{n+1}^{(4)} = \widehat{Q}_n^{(4)} - \beta_{n+1}^{(4)} \geqslant \widehat{Q}_{n+1}^{(2)}$.

To prove $\widehat{Q}_{n+1}^{(1)} \leqslant \widehat{Q}_{n+1}^{(4)}$, we first prove $\widehat{Q}_{n+1}^{(1)} \leqslant \widehat{Q}_{n+1}^{(2)}$. Since $\widehat{Q}_n^{(i)} \leqslant \widehat{Q}_n^{(4)}$, it follows from assumption (4) that $\widehat{Q}_n^{(1)} \leqslant \widehat{Q}_n^{(2)}$. Thus it suffices to prove $\beta_{n+1}^{(1)} \geqslant \beta_{n+1}^{(2)}$. Suppose this is not true, i.e., $\beta_{n+1}^{(1)} < \beta_{n+1}^{(2)}$, then we have $\beta_{n+1}^{(1)} = 0$ and $\beta_{n+1}^{(2)} = 1$. It follows from our definition of $\beta_{n+1}^{(1)}$ and $\beta_{n+1}^{(2)}$ in equations (5) and (6) that $a_2 > a_1$; hence using the fact that $\mu_i \leqslant \mu_1$ we obtain

$$\min\{\widehat{Q}_n^{(2)}, C\} > \frac{\mu_1}{\mu_2} \min\{\widehat{Q}_n^{(1)}, C\} \geqslant \min\{\widehat{Q}_n^{(1)}, C\},$$

implying $\widehat{Q}_n^{(2)} > \widehat{Q}_n^{(1)}$. Hence we get the contradiction and thus

$$\widehat{Q}_{n+1}^{(2)} = \widehat{Q}_n^{(2)} - 1 \geqslant \widehat{Q}_n^{(1)} = \widehat{Q}_{n+1}^{(1)}.$$

Therefore

$$\widehat{Q}_{n+1}^{(1)} \leqslant \widehat{Q}_{n+1}^{(2)} \leqslant \widehat{Q}_{n+1}^{(4)}.$$

This proves (C4Q).

Because of (3), (C5B) is satisfied. We then prove (C5Q). It suffices to prove

$$\beta_{n+1}^{(1)} + \beta_{n+1}^{(4)} \leqslant \beta_{n+1}^{(2)} + \beta_{n+1}^{(3)}. \tag{10}$$

The proof of (10) is lengthy and tedious, thus it is provided in the appendix.

We now prove (C4B) to (C5Q) under assumption (8). That is,

$$\widehat{Q}_n^{(1)} + \widehat{Q}_n^{(4)} > \widehat{Q}_n^{(2)} + \widehat{Q}_n^{(3)}.$$

Again, (C4B) and (C5B) are clearly satisfied because τ_{n+1} is a potential departure epoch. For (C4Q), i.e., $\widehat{Q}_{n+1}^{(i)} \leqslant \widehat{Q}_{n+1}^{(4)}$, the proof for $i = 2$ and 3 is identical to that of case (4), so we only need to prove $\widehat{Q}_{n+1}^{(1)} \leqslant \widehat{Q}_{n+1}^{(4)}$.

If $\beta_{n+1}^{(1)} \geqslant \beta_{n+1}^{(4)}$ then the result follows from $\widehat{Q}_n^{(1)} \leqslant \widehat{Q}_n^{(4)}$. If $\beta_{n+1}^{(1)} < \beta_{n+1}^{(4)}$, it deduces from the definition of $\beta_{n+1}^{(1)}$ and $\beta_{n+1}^{(4)}$ in (5) and (9) that $a_4 > a_1$, implying

$$\min\{\widehat{Q}_n^{(4)}, C\} > \frac{\mu_1}{\mu_4} \min\{\widehat{Q}_n^{(1)}, C\} \geqslant \min\{\widehat{Q}_n^{(1)}, C\}.$$

It has $\widehat{Q}_n^{(4)} > \widehat{Q}_n^{(1)}$, leading to $\widehat{Q}_{n+1}^{(4)} \geqslant \widehat{Q}_{n+1}^{(1)}$. This proves (C4Q).

We, finally, prove (C5Q). By the induction hypothesis and assumption (8) we have

$$\begin{aligned} \widehat{B}_n^{(1)} + \widehat{Q}_n^{(1)} + \widehat{B}_n^{(4)} + \widehat{Q}_n^{(4)} &\geqslant \widehat{B}_n^{(2)} + \widehat{Q}_n^{(2)} + \widehat{B}_n^{(3)} + \widehat{Q}_n^{(3)}, \\ \widehat{Q}_n^{(1)} + \widehat{Q}_n^{(4)} &\geqslant \widehat{Q}_n^{(2)} + \widehat{Q}_n^{(3)} + 1. \end{aligned}$$

If $\widehat{Q}_n^{(1)} + \widehat{Q}_n^{(4)} \geqslant \widehat{Q}_n^{(2)} + \widehat{Q}_n^{(3)} + 2$, then it follows from $\widehat{B}_n^{(i)} = \widehat{B}_{n+1}^{(i)}$ and $\widehat{B}_n^{(1)} + \widehat{B}_n^{(4)} \geqslant \widehat{B}_n^{(2)} + \widehat{B}_n^{(3)}$ that

$$\begin{aligned} &\widehat{B}_{n+1}^{(1)} + \widehat{B}_{n+1}^{(4)} + \widehat{Q}_{n+1}^{(1)} + \widehat{Q}_{n+1}^{(4)} \\ &\quad = \widehat{B}_n^{(1)} + \widehat{B}_n^{(4)} + \widehat{Q}_n^{(1)} - \beta_{n+1}^{(1)} + \widehat{Q}_n^{(4)} - \beta_{n+1}^{(4)} \\ &\quad \geqslant \widehat{B}_n^{(2)} + \widehat{B}_n^{(3)} + \widehat{Q}_n^{(2)} + \widehat{Q}_n^{(3)} + 2 - \beta_{n+1}^{(1)} - \beta_{n+1}^{(4)} \\ &\quad \geqslant \widehat{B}_n^{(2)} + \widehat{B}_n^{(3)} + \widehat{Q}_n^{(2)} + \widehat{Q}_n^{(3)} \\ &\quad \geqslant \widehat{B}_n^{(2)} + \widehat{B}_n^{(3)} + \widehat{Q}_n^{(2)} + \widehat{Q}_n^{(3)} - \beta_{n+1}^{(2)} - \beta_{n+1}^{(3)} \\ &\quad = \widehat{B}_{n+1}^{(2)} + \widehat{B}_{n+1}^{(3)} + \widehat{Q}_{n+1}^{(2)} + \widehat{Q}_{n+1}^{(3)}. \end{aligned}$$

In the following, we consider the remaining case

$$\widehat{Q}_n^{(1)} + \widehat{Q}_n^{(4)} = \widehat{Q}_n^{(2)} + \widehat{Q}_n^{(3)} + 1.$$

(a) If $\widehat{Q}_n^{(1)} = 0$, then $\beta_{n+1}^{(1)} = 0$. In this case the following inequality is satisfied:

$$\beta_{n+1}^{(1)} + \beta_{n+1}^{(4)} \leqslant \beta_{n+1}^{(2)} + \beta_{n+1}^{(3)} + 1.$$

Hence

$$\begin{aligned}\widehat{B}^{(1)}_{n+1} &+ \widehat{B}^{(4)}_{n+1} + \widehat{Q}^{(1)}_{n+1} + \widehat{Q}^{(4)}_{n+1}\\ &= \widehat{B}^{(1)}_{n} + \widehat{B}^{(4)}_{n} + \widehat{Q}^{(1)}_{n} + \widehat{Q}^{(4)}_{n} - \beta^{(1)}_{n+1} - \beta^{(4)}_{n+1}\\ &\geqslant \widehat{B}^{(2)}_{n} + \widehat{B}^{(3)}_{n} + \widehat{Q}^{(2)}_{n} + \widehat{Q}^{(3)}_{n} + 1 - \beta^{(2)}_{n+1} - \beta^{(3)}_{n+1} - 1\\ &= \widehat{B}^{(2)}_{n+1} + \widehat{B}^{(3)}_{n+1} + \widehat{Q}^{(2)}_{n+1} + \widehat{Q}^{(3)}_{n+1}.\end{aligned}$$

That is, (C5Q) is satisfied.

(b) If $\widehat{Q}^{(1)}_n > 0$, then it follows from $\widehat{Q}^{(1)}_n - 1 \geqslant 0$, $(\widehat{Q}^{(1)}_n - 1) + \widehat{Q}^{(4)}_n = \widehat{Q}^{(2)}_n + \widehat{Q}^{(3)}_n$ and lemma 1 that

$$\mu_1 \min\{\widehat{Q}^{(1)}_n - 1, C\} + \mu_4 \min\{\widehat{Q}^{(4)}_n, C\} \leqslant \mu_2 \min\{\widehat{Q}^{(2)}_n, C\} + \mu_3 \min\{\widehat{Q}^{(3)}_n, C\}.$$

In particular, we have

$$\mu_4 \min\{\widehat{Q}^{(4)}_n, C\} \leqslant \mu_2 \min\{\widehat{Q}^{(2)}_n, C\} + \mu_3 \min\{\widehat{Q}^{(3)}_n, C\}. \tag{11}$$

If $\beta^{(4)}_{n+1} \leqslant \beta^{(2)}_{n+1} + \beta^{(3)}_{n+1}$, then

$$\beta^{(1)}_{n+1} + \beta^{(4)}_{n+1} \leqslant \beta^{(2)}_{n+1} + \beta^{(3)}_{n+1} + 1,$$

and the earlier argument for the case $\widehat{Q}^{(1)}_n = 0$ can be used to show that (C5Q) holds. If $\beta^{(4)}_{n+1} > \beta^{(2)}_{n+1} + \beta^{(3)}_{n+1}$, then we must have $\beta^{(4)}_{n+1} = 1$ and $\beta^{(2)}_{n+1} = \beta^{(3)}_{n+1} = 0$, implying, by the definition of $\beta^{(2)}_{n+1}$, $\beta^{(3)}_{n+1}$ and $\beta^{(4)}_{n+1}$ in (5) and (7), that

$$\begin{aligned}&\mu_4 \min\{\widehat{Q}^{(4)}_n, C\} > \mu_2 \min\{\widehat{Q}^{(2)}_n, C\}, \quad \text{and}\\ &\mu_4 \min\{\widehat{Q}^{(4)}_n, C\} - \mu_3 \min\{\widehat{Q}^{(3)}_n, C\} > \mu_2 \min\{\widehat{Q}^{(2)}_n, C\}.\end{aligned}$$

But this contradicts with (11), proving (C5Q).

To summarize, we have shown by induction that (C4B) to (C5Q) are satisfied for all $n \geqslant 0$, equivalently for all $t \geqslant 0$. This shows that $\boldsymbol{X}_t \in$ SIDCX-PS(sp) in $(\lambda, -\mu)$. That $B_t \in$ SIDCX(sp) in $(\lambda, -\mu)$ follows from (C3B), (C4B) and (C5B). This completes the proof of theorem 1. □

Remark 1. It should be noted that though the blocking probability $B(\lambda, -\mu)$ in $M/M/C/N$ system is increasing and directionally convex in $(\lambda, -\mu)$, it is not jointly convex in $(\lambda, -\mu)$. To see this, consider the case $C = N = 1$, $B(\lambda, -\mu) = \lambda/(\lambda + \mu)$. Note that

$$\begin{aligned}\frac{\partial B(\lambda, -\mu)}{\partial \lambda} &= \frac{\mu}{(\lambda + \mu)^2},\\ \frac{\partial B(\lambda, -\mu)}{\partial (-\mu)} &= \frac{\lambda}{(\lambda + \mu)^2},\\ \frac{\partial^2 B(\lambda, -\mu)}{\partial \lambda^2} &= \frac{-2\mu}{(\lambda + \mu)^3},\end{aligned}$$

$$\frac{\partial^2 B(\lambda, -\mu)}{\partial(-\mu)^2} = \frac{2\lambda}{(\lambda+\mu)^3},$$
$$\frac{\partial^2 B(\lambda, -\mu)}{\partial\lambda\partial(-\mu)} = \frac{\mu-\lambda}{(\lambda+\mu)^3}.$$

Thus $B(\lambda, -\mu)$ is concave in λ and convex in $-\mu$. The determinant of its Hessen matrix is equal to $-1/(\lambda+\mu)^4$, which is strictly negative. Hence $B(\lambda, -\mu)$ cannot be jointly convex (or concave) in $(\lambda, -\mu)$.

3. Stochastic comparison result

To prove the stochastic comparison result of queues in random environment with two levels, we need another preliminary result Let $0 = S_0 \leqslant S_1 \leqslant S_2 \leqslant \cdots$ be a stochastic increasing sequence. Consider the following queueing system defined in terms of sequence $S_n, n \geqslant 0$: in the interval $(S_n, S_{n+1}]$ the system is $M/M/C/N$ with arrival rate λ_n and the service rate μ_n. Let $\boldsymbol{\theta}_n = (\lambda_n, -\mu_n)$, and $\boldsymbol{\Theta} = (\boldsymbol{\theta}_0, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots)$. We again define $\boldsymbol{X}_t = (B_t, Q_t)$, where B_t is the number of blocked customers by time t and Q_t is the number of customers in the system at time t.

The following result is a property of stochastic process $\boldsymbol{X}_t$ as a functional of the system parameter $\boldsymbol{\Theta}$.

Lemma 2. The stochastic process satisfies $\boldsymbol{X}_t \in$ SIDCX-PS(sp) in $\boldsymbol{\Theta}$.

Proof. Since the sequence $\{S_n, n \geqslant 0\}$ is independent of the arrival process, we may assume that it is deterministic. For any given

$$\boldsymbol{\Theta}^{(1)} = (\boldsymbol{\theta}_0^{(1)}, \boldsymbol{\theta}_1^{(1)}, \boldsymbol{\theta}_2^{(1)}, \ldots),$$
$$\boldsymbol{\Theta}^{(2)} = (\boldsymbol{\theta}_0^{(2)}, \boldsymbol{\theta}_1^{(2)}, \boldsymbol{\theta}_2^{(2)}, \ldots),$$
$$\boldsymbol{\Theta}^{(3)} = (\boldsymbol{\theta}_0^{(3)}, \boldsymbol{\theta}_1^{(3)}, \boldsymbol{\theta}_2^{(3)}, \ldots),$$
$$\boldsymbol{\Theta}^{(4)} = (\boldsymbol{\theta}_0^{(4)}, \boldsymbol{\theta}_1^{(4)}, \boldsymbol{\theta}_2^{(4)}, \ldots),$$

satisfying condition (C1) and (C2), $\boldsymbol{\theta}_n^{(i)} = (\lambda_n^{(i)}, -\mu_n^{(i)})$, $i = 1, 2, 3, 4$, also satisfy (C1) and (C2). Assume that $\widehat{\boldsymbol{X}}_0^{(i)}$, $i = 1, 2, 3, 4$ have been constructed to satisfy (C3B) to (C5Q). For $0 \leqslant t \leqslant S_1$, we can apply theorem 1 with parameters $\boldsymbol{\theta}_0^{(i)}$, $i = 1, 2, 3, 4$ which satisfy (C1) and (C2), to construct the process $\widehat{\boldsymbol{X}}_t^{(i)}$ such that (C3B) to (C5Q) are satisfied for $0 \leqslant t \leqslant S_1$. In particular, $\widehat{\boldsymbol{X}}_{S_1}^{(i)}$, $i = 1, 2, 3, 4$ satisfy (C3B) to (C5Q). Take this as the initial condition on $[S_1, S_2]$, and using the parameter $\boldsymbol{\theta}_1^{(i)} = (\lambda_1^{(i)}, -\mu_1^{(i)})$ which satisfy (C1) and (C2), we again use theorem 1 to conclude that $\widehat{\boldsymbol{X}}_t^{(i)}$ can be constructed on $[S_1, S_2]$ such that (C3B) to (C5Q) are satisfied. This process can be continued to show that the process $\widehat{\boldsymbol{X}}_t^{(i)}$, $i = 1, 2, 3, 4$, can be constructed for all $t \geqslant 0$ and they satisfy (C3B) to (C5Q). This shows that $\boldsymbol{X}_t \in \text{SIDCX} - \text{PS(sp)}$ in $\boldsymbol{\Theta}$ and completes the proof of lemma 2. □

We now turn to the queueing system described in the introduction. There is an underlying Markov chain that alternates between two levels. When the system is in level i, the arrival rate is λ_i and the service rate is μ_i, $i = 1, 2$. When the system is in level 1 (2), it stays there for an exponentially distributed amount of time with rate $c\alpha_1$ ($c\alpha_2$), then it switches to level 2 (1).

To compare the effect of c on the performance of the queueing system, we consider two such queueing systems that are the same except parameter c. The first system has parameter c_1 and the second c_2, where $c_1 \geqslant c_2$. Let $\boldsymbol{X}_t^{(i)} = (B_t^{(i)}, Q_t^{(i)})$ be the state of the system corresponding to c_i. We assume that at time 0 both systems start at level i with probability $\pi_i = \alpha_{3-i}/(\alpha_1 + \alpha_2)$, $i = 1, 2$, i.e., they are in steady state.

The following is the main result of this section.

Theorem 2. For any time t, the number of blocked customers is decreasing in c in increasing convex ordering. That is, when $c_1 \geqslant c_2$, $Ef(B_t^{(1)}) \leqslant Ef(B_t^{(2)})$ for all increasing convex functions f for which the expectations exist.

Proof. Without loss of generality we may assume $1 = c_1 \geqslant c_2$. Generate a Poisson process with rate $\alpha_1 + \alpha_2$, and let $\boldsymbol{S} = (S_0, S_1, S_2, \ldots)$ with $S_0 = 0$ be the arrival epochs of this Poisson process. Further generate a sequence of independent and identically distributed Bernoulli random variables $U_1, U_2, \ldots$ such that

$$P\{U_n = 1\} = c_2 = 1 - P\{U_n = 0\}.$$

Let $V_n, n \geqslant 1$ be a sequence of uniform random variables on [0, 1]. All these random variables as well as the Poisson process are independent of one another.

We construct two queueing systems using the data generated above. The processes we construct are both the type of queueing process describes at the beginning of this section but with the parameters $\boldsymbol{\Theta}$ being random vectors. Furthermore, the two processes will be constructed in such a way that they can be compared, and we will compare them using the subsequence condition (lemma 2.12 of Chang et al. [3]).

Call the system with parameter c_i system i, $i = 1, 2$. Let $B_0^{(i)} = 0$, and $Q_0^{(1)} = Q_0^{(2)}$ be the initial number of customers in the two systems. Both systems start at level i with probability π_i. Let $\boldsymbol{\theta}_0 = (\lambda_1, -\mu_1)$ if the system is at level 1 at time 0 and $\boldsymbol{\theta}_0 = (\lambda_2, -\mu_2)$ otherwise. The arrivals and departures for system 1 are generated according to $\boldsymbol{\theta}_0$. At points $S_1, S_2, \ldots$, the system changes level according to the following: at S_n, $n = 1, 2, \ldots$, it switches to level 1 if $V_n \leqslant \alpha_2/(\alpha_1 + \alpha_2)$, and it switches to level 2 if $V_n > \alpha_2/(\alpha_1 + \alpha_2)$, independently of the past and regardless of the level the system is in before S_n. When the system is in level i and there are n customers in the system, the arrivals and departures are generated at rates λ_i and $\min\{n, C\}\mu_i$, $i = 1, 2$.

We then construct system 2. The same as the construction of system 1, at time 0 the system is constructed according to $\boldsymbol{\theta}_0$. The system changes level at times S_n, $n = 1, 2, \ldots$, according to the following rule: If $U_n = 1$, the system switches to level 1 if $V_n \leqslant \alpha_2/(\alpha_1 + \alpha_2)$, and it switches to level 2 if $V_n > \alpha_2/(\alpha_1 + \alpha_2)$, independently of the past and regardless of the level the system is in just before S_n; if $U_n = 0$, the system stays at the same level as it is in just before S_n. And the same as in system 1, when the system is in level i and there are n customers in the system, the arrivals and departures are generated at rates λ_i and $\min\{n, C\}\mu_i$, $i = 1, 2$.

Now we show that such constructed systems represent the two systems under consideration. Since whenever the system is in level i, the arrivals follow a Poisson process with rate λ_i and service times are exponentially distributed with rate μ_i, we only need to verify that under the construction above, the amount of time system 1 stays in level i before changing to level $3 - i$ $(i = 1, 2)$ is exponentially distributed with rate α_i, and the amount of time system 2 stays in level i before changing to level $3 - i$ $(i = 1, 2)$ is exponentially distributed with rate $c_2\alpha_i$.

First consider system 1. From our construction, the system switches to level 1 at S_n when $V_n \leqslant \alpha_2/(\alpha_1 + \alpha_2)$. This occurs with probability $\alpha_2/(\alpha_1 + \alpha_2)$. Hence whenever the level switches to level 1, it stays there for the amount of time which is the sum of a geometric number $R^{(1)}$ of exponential phases (of rate $\alpha_1 + \alpha_2$), where

$$P\{R^{(1)} = k\} = \left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right)^{k-1}\left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\right), \quad k = 1, 2, \ldots.$$

It is straightforward to verify that the amount of time before the process switches to level 2, which is the sum of $R^{(1)}$ exponential random variables, is exponential with rate α_1. Similar argument shows that whenever the system is switched to level 2, it stays there for an exponentially distributed amount of time with rate α_2. Therefore the system constructed does represent system 1.

For the second system we constructed, whenever the system is at level 1, it switches to level 2 at time S_n only when $U_n = 1$ and $V_n > \alpha_2/(\alpha_1 + \alpha_2)$. The probability for this event is $c_2\alpha_1/(\alpha_1 + \alpha_2)$, therefore the amount of time the process stays in level 1 is the sum of a geometric number $R^{(2)}$ of exponential phases (of rate $\alpha_1 + \alpha_2$), where

$$P\{R^{(2)} = k\} = \left(1 - \frac{c_2\alpha_1}{\alpha_1 + \alpha_2}\right)^{k-1}\left(\frac{c_2\alpha_1}{\alpha_1 + \alpha_2}\right), \quad k = 1, 2, \ldots.$$

Hence the process stays in level 1 for exponentially distributed amount of time with parameter $c_2\alpha_1$. Similar argument shows that whenever the system switches to level 2, it stays there for exponentially distributed amount of time with rate $c_2\alpha_2$. This proves that the second system we constructed also represents the real system.

Let us denote by $\boldsymbol{\theta}_n^{(i)} = (\lambda_n^{(i)}, -\mu_n^{(i)})$ the parameters (arrival and negative service rates) of system i just constructed in the interval $(S_n, S_{n+1}]$. Clearly, $(\lambda_n^{(i)}, -\mu_n^{(i)})$ is either equal to $(\lambda_1, -\mu_1)$ or $(\lambda_2, -\mu_2)$, depending on whether system i is in level 1 or 2. Thus $(\lambda_n^{(i)}, -\mu_n^{(i)})$ are random vectors. From the construction above we have

$$P\{\boldsymbol{\theta}_n^{(i)} = (\lambda_j, -\mu_j)\} = \pi_j, \quad i, j = 1, 2.$$

By lemma 2, for any given $(\lambda_n^{(i)}, -\mu_n^{(i)})$ and for any fixed time t, $\boldsymbol{X}_t^{(i)} \in$ SIDCX-PS(sp) in $(\boldsymbol{\theta}_0^{(i)}, \boldsymbol{\theta}_1^{(i)}, \boldsymbol{\theta}_2^{(i)}, \ldots)$. In particular, $B_t^{(i)} \in$ SIDCX(sp) in $(\boldsymbol{\theta}_0^{(i)}, \boldsymbol{\theta}_1^{(i)}, \boldsymbol{\theta}_2^{(i)}, \ldots)$.

That is, $B_t^{(i)}$ depends on $\boldsymbol{\theta}_0^{(i)}, \boldsymbol{\theta}_1^{(i)}, \boldsymbol{\theta}_2^{(i)}, \ldots$, and this dependency is increasing directionally convex. Let this function be denoted by ϕ, then we can write

$$B^{(i)}(t) = \phi\big(\boldsymbol{\theta}_0^{(i)}, \boldsymbol{\theta}_1^{(i)}, \boldsymbol{\theta}_2^{(i)}, \ldots\big), \quad i = 1, 2.$$

A key observation on the construction of the two systems is that, there exist $0 < n_1 < n_2 < \cdots$, such that

$$\begin{aligned}&\big(\boldsymbol{\theta}_0^{(2)}, \ldots, \boldsymbol{\theta}_{n_1}^{(2)}, \boldsymbol{\theta}_{n_1+1}^{(2)}, \ldots, \boldsymbol{\theta}_{n_2}^{(2)}, \boldsymbol{\theta}_{n_2+1}^{(2)}, \ldots\big)\\ &\quad= \big(\boldsymbol{\theta}_0^{(1)}, \ldots, \boldsymbol{\theta}_0^{(1)}, \boldsymbol{\theta}_{n_1+1}^{(1)}, \ldots, \boldsymbol{\theta}_{n_1+1}^{(1)}, \boldsymbol{\theta}_{n_2+1}^{(1)}, \ldots\big).\end{aligned}$$

That is, our construction satisfies the subsequent condition of Chang et al. [3], and

$$\begin{aligned}B_t^{(1)} &= \phi\big(\boldsymbol{\theta}_0^{(1)}, \ldots, \boldsymbol{\theta}_{n_1}^{(1)}, \boldsymbol{\theta}_{n_1+1}^{(1)}, \ldots, \boldsymbol{\theta}_{n_2}^{(1)}, \boldsymbol{\theta}_{n_2+1}^{(1)}, \ldots\big),\\ B_t^{(2)} &= \phi\big(\boldsymbol{\theta}_0^{(1)}, \ldots, \boldsymbol{\theta}_0^{(1)}, \boldsymbol{\theta}_{n_1+1}^{(1)}, \ldots, \boldsymbol{\theta}_{n_1+1}^{(1)}, \boldsymbol{\theta}_{n_2+1}^{(1)}, \ldots\big).\end{aligned}$$

Therefore, using the fact that ϕ increasing directionally convex and f increasing convex imply $f(\phi)$ increasing directionally convex and Lorenz's inequality (see [2, lemma 2]), we obtain

$$Ef\big(B_t^{(1)}\big) \leqslant Ef\big(B_t^{(2)}\big)$$

for all f increasing convex. This completes the proof of theorem 2. □

Remark 2. The construction process of the two queues using a common sequence $\{S_n, n \geqslant 0\}$ is similar to that used by Chang et al. [2] for a discrete time TDM system.

Corollary 2. Since the blocking probability P_{B} is defined as

$$P_{\mathrm{B}} = \lim_{t\to\infty} \frac{B_t}{N_t} = \frac{1}{\bar{\lambda}} \lim_{t\to\infty} \frac{EB_t}{t},$$

where B_t is the number of blocked customers up to time t, N_t is the number of arrivals by time t, and $\bar{\lambda} = \sum_{i=1}^{2} \pi_i \lambda_i$ is the average arrival rate, it follows from the above result that the blocking probability is decreasing in c.

Corollary 3. When $N = \infty$, the number of customers in system is decreasing in c in increasing convex ordering. The average delay is also decreasing in c.

Proof. The proof of theorem 2 also shows that

$$Ef\big(B_t^{(1)} + Q_t^{(1)}\big) \leqslant Ef\big(B_t^{(2)} + Q_t^{(2)}\big)$$

for all f increasing convex. When $N = \infty$, B_t is identically zero. Hence

$$Ef\big(Q_t^{(1)}\big) \leqslant Ef\big(Q_t^{(2)}\big)$$

for all f increasing convex. That is, Q_t is decreasing in c in increasing convex ordering. The average number of customers in system is

$$\lim_{t\to\infty} \frac{E\int_0^t Q_s \, ds}{t} = \lim_{t\to\infty} \frac{\int_0^t E Q_s \, ds}{t},$$

which is decreasing in c. By Little's formula we obtain that the average delay is also decreasing in c. □

Let $c \to 0$ and $c \to \infty$, we obtain upper and lower bounds for the system under consideration.

Corollary 4. Let P_B be the blocking probability for the system, and $P_B(\lambda, \mu)$ be the blocking probability of the $M/M/C/N$ system with arrival rate λ and service rate μ. Then

$$P_B\left(\sum_{i=1}^{2} \pi_i \lambda_i, \sum_{i=1}^{2} \pi_i \mu_i\right) \leqslant P_B \leqslant \sum_{i=1}^{2} \pi_i P_B(\lambda_i, \mu_i),$$

where $\pi_i = \alpha_{3-i}/(\alpha_1 + \alpha_2)$, $i = 1, 2$. A similar result holds for the average delay when $N = \infty$.

We remark that these several results have been proved when the service process is stationary, that is when the service rate remains the same in different level $\mu_1 = \mu_2$. See, for example, [3,12,13]. For the case $C = N = 1$ and there are more than two levels, the model has been studied by several authors, and the monotonicity results are obtained under some conditions on the transition matrix of the governing Markov chains. See [1,4–6], among others.

Appendix.

In this appendix, we prove (10) under assumption (4). Since $\widehat{Q}_n^{(1)} + \widehat{Q}_n^{(4)} \leqslant \widehat{Q}_n^{(2)} + \widehat{Q}_n^{(3)}$ and $\widehat{Q}_n^{(i)} \leqslant \widehat{Q}_n^{(4)}$, there exists a $\widetilde{Q}_n^{(4)}$, such that $\widetilde{Q}_n^{(4)} \geqslant \widehat{Q}_n^{(4)} \geqslant \widehat{Q}_n^{(i)}$ and $\widehat{Q}_n^{(1)} + \widetilde{Q}_n^{(4)} = \widehat{Q}_n^{(2)} + \widehat{Q}_n^{(3)}$. Applying lemma 1 yields

$$\mu_1 \min\{\widehat{Q}_n^{(1)}, C\} + \mu_4 \min\{\widetilde{Q}_n^{(4)}, C\} \leqslant \mu_2 \min\{\widehat{Q}_n^{(2)}, C\} + \mu_3 \min\{\widehat{Q}_n^{(3)}, C\}.$$

It follows from

$$\mu_4 \min\{\widehat{Q}_n^{(4)}, C\} \leqslant \mu_4 \min\{\widetilde{Q}_n^{(4)}, C\}$$

that

$$a_1 + a_4 \leqslant a_2 + a_3. \tag{A.1}$$

Recall that $a_i = \mu_i \min\{\widehat{Q}_n^{(i)}, C\}$.

To prove (10), we consider several cases separately.

Case 1. If $a_2 \leqslant a_4$, then it follows from (A.1) that

$$a_4 - a_3 \leqslant a_2 - a_1 \leqslant a_2 \leqslant a_4.$$

The following several subcases are based on where 0 is located.

Subcase 1.1.

$$0 \leqslant a_4 - a_3 \leqslant a_2 - a_1 \leqslant a_2 \leqslant a_4.$$

In this subcase we have, if

$$U_{n+1} \in [\lambda_4 + C\mu_1, \lambda_4 + C\mu_1 + a_4 - a_3),$$

then we obtain from (5), (6) and (7) that

$$\beta_{n+1}^{(2)} = \beta_{n+1}^{(4)} = 1, \qquad \beta_{n+1}^{(1)} = \beta_{n+1}^{(3)} = 0;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_4 - a_3, \lambda_4 + C\mu_1 + a_2 - a_1),$$

then we have

$$\beta_{n+1}^{(1)} = 0, \qquad \beta_{n+1}^{(2)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_2 - a_1, \lambda_4 + C\mu_1 + a_2),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_2, \lambda_4 + C\mu_1 + a_4),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = 0, \qquad \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1.$$

If U_{n+1} takes any other value then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 0.$$

Thus (10) is satisfied.

Subcase 1.2.

$$a_4 - a_3 \leqslant 0 \leqslant a_2 - a_1 \leqslant a_2 \leqslant a_4.$$

In this subcase we have, if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_4 - a_3, \lambda_4 + C\mu_1),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = \beta_{n+1}^{(4)} = 0, \qquad \beta_{n+1}^{(3)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1, \lambda_4 + C\mu_1 + a_2 - a_1),$$

then

$$\beta_{n+1}^{(1)} = 0, \qquad \beta_{n+1}^{(2)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_2 - a_1, \lambda_4 + C\mu_1 + a_2),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_2, \lambda_4 + C\mu_1 + a_4),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = 0, \qquad \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1.$$

And $\beta_{n+1}^{(i)}$ are all 0 for other values of U_{n+1}. Thus (10) is satisfied.

Subcase 1.3.

$$a_4 - a_3 \leqslant a_2 - a_1 \leqslant 0 \leqslant a_2 \leqslant a_4.$$

In this subcase we have, if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_4 - a_3, \lambda_4 + C\mu_1 + a_2 - a_1),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = \beta_{n+1}^{(4)} = 0, \qquad \beta_{n+1}^{(3)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_2 - a_1, \lambda_4 + C\mu_1),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(3)} = 1, \qquad \beta_{n+1}^{(2)} = \beta_{n+1}^{(4)} = 0;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1, \lambda_4 + C\mu_1 + a_2),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_2, \lambda_4 + C\mu_1 + a_4),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = 0, \quad \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1.$$

And $\beta_{n+1}^{(i)}$ are all 0 for other values of U_{n+1}. Thus (10) is always satisfied.

We next consider the case that $a_2 > a_4$. In this case it follows from (A.1) that we have two possibilities, that is, either

$$a_4 - a_3 \leqslant a_4 \leqslant a_2 - a_1 \leqslant a_2, \tag{A.2}$$

or

$$a_4 - a_3 \leqslant a_2 - a_1 \leqslant a_4 \leqslant a_2 \tag{A.3}$$

is satisfied. We call these two possibilities case 2 and case 3, respectively.

Case 2. We have (A.2). Several subcases will be considered, again based on where 0 is located.

Subcase 2.1.

$$0 \leqslant a_4 - a_3 \leqslant a_4 \leqslant a_2 - a_1 \leqslant a_2.$$

In this subcase we have, if

$$U_{n+1} \in [\lambda_4 + C\mu_1, \lambda_4 + C\mu_1 + a_4 - a_3),$$

then we have

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(3)} = 0, \qquad \beta_{n+1}^{(2)} = \beta_{n+1}^{(4)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_4 - a_3, \lambda_4 + C\mu_1 + a_4),$$

then we have

$$\beta_{n+1}^{(1)} = 0, \qquad \beta_{n+1}^{(2)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_4, \lambda_4 + C\mu_1 + a_2 - a_1),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 0, \qquad \beta_{n+1}^{(2)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_2 - a_1, \lambda_4 + C\mu_1 + a_2),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = 1, \qquad \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 0.$$

And $\beta_{n+1}^{(i)}$ are all 0 if U_{n+1} takes any other value. Thus (10) is satisfied.

Subcase 2.2.

$$a_4 - a_3 \leqslant 0 \leqslant a_4 \leqslant a_2 - a_1 \leqslant a_2.$$

In this subcase we have, if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_4 - a_3, \lambda_4 + C\mu_1),$$

then we have

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = \beta_{n+1}^{(4)} = 0, \qquad \beta_{n+1}^{(3)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1, \lambda_4 + C\mu_1 + a_4),$$

then we have

$$\beta_{n+1}^{(1)} = 0, \qquad \beta_{n+1}^{(2)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_4, \lambda_4 + C\mu_1 + a_2 - a_1),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 0, \qquad \beta_{n+1}^{(2)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_2 - a_1, \lambda_4 + C\mu_1 + a_2),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = 1, \qquad \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 0.$$

And $\beta_{n+1}^{(i)}$ are all 0 if U_{n+1} takes any other value. Thus (10) is satisfied.

Case 3. We now have (A.3).

Subcase 3.1.

$$0 \leqslant a_4 - a_3 \leqslant a_2 - a_1 \leqslant a_4 \leqslant a_2.$$

In this subcase we have, if

$$U_{n+1} \in [\lambda_4 + C\mu_1, \lambda_4 + C\mu_1 + a_4 - a_3),$$

then we have

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(3)} = 0, \qquad \beta_{n+1}^{(2)} = \beta_{n+1}^{(4)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_4 - a_3, \lambda_4 + C\mu_1 + a_2 - a_1),$$

then we have

$$\beta_{n+1}^{(1)} = 0, \qquad \beta_{n+1}^{(2)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_2 - a_1, \lambda_4 + C\mu_1 + a_4),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_4, \lambda_4 + C\mu_1 + a_2),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = 1, \qquad \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 0.$$

And the $\beta_{n+1}^{(i)}$ are all 0 if U_{n+1} takes any other value. Thus (10) is satisfied.

Subcase 3.2.

$$a_4 - a_3 \leqslant 0 \leqslant a_2 - a_1 \leqslant a_4 \leqslant a_2.$$

In this subcase we have, if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_4 - a_3, \lambda_4 + C\mu_1),$$

then we have

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = \beta_{n+1}^{(4)} = 0, \qquad \beta_{n+1}^{(3)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1, \lambda_4 + C\mu_1 + a_2 - a_1),$$

then we have

$$\beta_{n+1}^{(1)} = 0, \qquad \beta_{n+1}^{(2)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_2 - a_1, \lambda_4 + C\mu_1 + a_4),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_4, \lambda_4 + C\mu_1 + a_2),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = 1, \qquad \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 0.$$

And the $\beta_{n+1}^{(i)}$ are all 0 if U_{n+1} takes any other value. Thus (10) is satisfied.

Subcase 3.3.

$$a_4 - a_3 \leqslant a_2 - a_1 \leqslant 0 \leqslant a_4 \leqslant a_2.$$

In this subcase we have, if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_4 - a_3, \lambda_4 + C\mu_1 + a_2 - a_1),$$

then we have

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = \beta_{n+1}^{(4)} = 0, \qquad \beta_{n+1}^{(3)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_2 - a_1, \lambda_4 + C\mu_1),$$

then we have

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(3)} = 1, \qquad \beta_{n+1}^{(2)} = \beta_{n+1}^{(4)} = 0;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1, \lambda_4 + C\mu_1 + a_4),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 1;$$

if

$$U_{n+1} \in [\lambda_4 + C\mu_1 + a_4, \lambda_4 + C\mu_1 + a_2),$$

then

$$\beta_{n+1}^{(1)} = \beta_{n+1}^{(2)} = 1, \qquad \beta_{n+1}^{(3)} = \beta_{n+1}^{(4)} = 0.$$

And the $\beta_{n+1}^{(i)}$ are all 0 if U_{n+1} takes any other value. Thus (10) is satisfied.

Summarizing, we have shown that in all the possible cases, relationship (10) is satisfied.

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