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A new multivariate transform and the distribution of a random functional of a Ferguson–Dirichlet process

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Abstract

A new multivariate transformation is given, with various properties, e.g., uniqueness and convergence properties, that are similar to those of the Fourier transformation. The new transformation is particularly useful for distributions that are difficult to deal with by Fourier transformation, such as relatives of the Dirichlet distributions. The new multivariate transformation of the Dirichlet distribution can be expressed in closed form. With this result, we easily show that the marginal of a Dirichlet distribution is still a Dirichlet distribution. We also give a closed form for the filtered-variate Dirichlet distribution. A relation between the new characteristic function and the traditional characteristic function is given. Using this multivariate transformation, we give the distribution, on the region bounded by an ellipse, of a random functional of a Ferguson–Dirichlet process over the boundary.

Keywords: Multivariate *c*-characteristic function; Fourier transformation; Dirichlet distribution; Ferguson–Dirichlet process random functional; Carlson's function; Filtered-variate Dirichlet distribution; Inversion formula

1. Introduction

Jiang (1988) first gave a univariate transformation and showed that it has properties similar to those of the univariate Fourier transformation. This new univariate transformation can be useful when a distribution is difficult to deal with by Fourier

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transformation or usual characteristic function. Examples can be seen in Jiang (1988). Jiang (1991) used it further to study the distribution of a random functional on the unit disk of a Ferguson–Dirichlet process on the boundary. However, its usefulness is restricted to symmetrical distributions or univariate distributions. In this paper, we generalize to a multivariate transformation. In Section 2, a new multivariate transformation, called the multivariate c-transformation, is defined and properties derived. Section 3 demonstrates that our multivariate c-transformation can easily be used to prove properties of the Dirichlet distribution. In Section 4, we show the relation between a random vector and its linear transformation. In Section 5, the relation between the traditional characteristic function and the multivariate c-characteristic function will first be given, and then we will use an example to show one way to determine the distribution function based on its multivariate c-characteristic function. In Section 6, we study, on the region bounded by an ellipse, the distribution of a random functional of a Ferguson–Dirichlet process on the boundary. Conclusions are given in Section 7.

2. The multivariate c-characteristic function and the multivariate c-transformation

First, we define a new multivariate characteristic function called a multivariate c-characteristic function.

Definition 2.1. If $\boldsymbol{u}=(u_1,\ldots,u_L)'$ is a random vector on a subset S of $A=[-a_1,a_1]\times[-a_2,a_2]\times\cdots\times[-a_L,a_L]$, its multivariate c-characteristic function is defined as

$$g(\boldsymbol{t};\boldsymbol{u},c)=\underset{\boldsymbol{u}}{E}[(1-\mathrm{i}\boldsymbol{t}\cdot\boldsymbol{u})^{-c}],\qquad |\boldsymbol{t}|<a^{-1},$$

where c is a positive real number, $a=\sqrt{\sum_{i=1}^{L}a_i^2}$, $\boldsymbol{t}'=(t_1,t_2,\ldots,t_L)$, $|\boldsymbol{t}|=\sqrt{\sum_{i=1}^{L}t_i^2}$, and $\boldsymbol{t}\cdot\boldsymbol{u}$ is the inner product of two vectors (i.e., $\boldsymbol{t}\cdot\boldsymbol{u}=\sum_{i=1}^{L}t_iu_i$). More generally, for any finite measure μ on $\mathbb{R}^L$ ($\mathbb{R}$ is the real line and $\mathbb{R}^L$ is the Lth Cartesian power of $\mathbb{R}$) with support in a subset S of $A=[-a_1,a_1]\times[-a_2,a_2]\times\cdots\times[-a_L,a_L]$, we define the multivariate c-transformation of μ as

$$\hat{\mu}^c(\boldsymbol{t})=\int_S(1-\mathrm{i}\boldsymbol{t}\cdot\boldsymbol{u})^{-c}\,\mathrm{d}\mu(\boldsymbol{u}),\qquad |\boldsymbol{t}|<a^{-1},\ \ c>0,\tag{1}$$

where $a=\sqrt{\sum_{i=1}^{L}a_i^2}$, and again $\boldsymbol{t}$ and $\boldsymbol{u}$ are $L\times 1$ column vectors.

The assumptions that c is positive and μ has a support in S are needed in the above Definition 2.1 for the one-to-one correspondence between $\hat{\mu}^c(\boldsymbol{t})$ and μ in the next lemma.

Lemma 2.2. *For finite measures μ and ν with supports in a subset S of $A=[-a_1,a_1]\times[-a_2,a_2]\times\cdots\times[-a_L,a_L]$ and any positive real number c, if* *we have*

$$\hat{\mu}^c(\boldsymbol{t}) = \hat{\nu}^c(\boldsymbol{t}), \tag{2}$$

for all $|\boldsymbol{t}| < a^{-1}$, *where* $a = \sqrt{\sum_{i=1}^{L} a_i^2}$, *then* $\mu = \nu$.

Proof. Since c is a real number and $|\boldsymbol{t} \cdot \boldsymbol{u}| < 1$, we have

$$(1 - \mathrm{i}\boldsymbol{t} \cdot \boldsymbol{u})^{-c} = \sum_{n=0}^{\infty} \frac{(c,n)}{n!} (\mathrm{i}\boldsymbol{t} \cdot \boldsymbol{u})^n, \tag{3}$$

where Appell's symbol $(c,n) = \Gamma(c+n)/\Gamma(n) = c(c+1)\cdots(c+n-1)$. By Eqs. (1)–(3), we have

$$\sum_{n=0}^{\infty} \frac{(c,n)}{n!} \mathrm{i}^n \int_S (\boldsymbol{t} \cdot \boldsymbol{u})^n \, \mathrm{d}\mu(\boldsymbol{u}) = \sum_{n=0}^{\infty} \frac{(c,n)}{n!} \mathrm{i}^n \int_S (\boldsymbol{t} \cdot \boldsymbol{u})^n \, \mathrm{d}\nu(\boldsymbol{u}) \tag{4}$$

for all $|\boldsymbol{t}| < a^{-1}$. Given an integer k, where $1 \leqslant k \leqslant L$, and let $t_j = 0$, $\forall j \neq k$, then Eq. (4) can be re-expressed as

$$\sum_{n=0}^{\infty} \frac{(c,n)}{n!} \mathrm{i}^n t_k^n \int_S u_k^n \, \mathrm{d}\mu(\boldsymbol{u}) = \sum_{n=0}^{\infty} \frac{(c,n)}{n!} \mathrm{i}^n t_k^n \int_S u_k^n \, \mathrm{d}\nu(\boldsymbol{u}),$$

for all $|t_k| < a^{-1}$. Treating t_k as variable and equating the corresponding coefficients of t_k^n ($(c,n) \neq 0$, for all n, as c is positive) for each n in the two sums, we have

$$\int_S u_k^n \, \mathrm{d}\mu(\boldsymbol{u}) = \int_S u_k^n \, \mathrm{d}\nu(\boldsymbol{u}),$$

where $k = 1, 2, \ldots, L$. Hence we have

$$\int_S P(\boldsymbol{u}) \, \mathrm{d}\mu(\boldsymbol{u}) = \int_S P(\boldsymbol{u}) \, \mathrm{d}\nu(\boldsymbol{u}),$$

where $P(\boldsymbol{u})$ is any polynomial function of $\boldsymbol{u}$, and similarly for any continuous function. This shows that $\mu = \nu$. □

Definition 2.3. A sequence $\{\mu_n,\ n \geqslant 1\}$ of finite measures on the Borel sets of a metric space Ω is said to converge vaguely to a finite measure on the Borel sets of Ω, denoted by $\mu_n \xrightarrow{v} \mu$, if

$$\mu_n(A) \to \mu(A) \quad \text{as } n \to \infty$$

for any Borel set A such that its boundary has measure 0.

We are now ready to give the following important convergence theorem.

Theorem 2.4. *Assume* μ *and* $\mu_1, \mu_2, \ldots$ *are finite measures with supports in S and their corresponding multivariate c-transforms are* $\hat{\mu}^c(\boldsymbol{t}), \hat{\mu}_1^c(\boldsymbol{t}), \hat{\mu}_2^c(\boldsymbol{t}), \ldots$, *respectively, as defined in Eq.* (1). *Then, for a given* $c > 0$, *the following statements are* *equivalent*:

$$\mu_n \xrightarrow{v} \mu, \quad \text{as } n \to \infty, \tag{5}$$

$$\hat{\mu}_n^c(\boldsymbol{t}) \to \hat{\mu}^c(\boldsymbol{t}), \quad \text{as } n \to \infty \text{ for all } |\boldsymbol{t}| < a^{-1}. \tag{6}$$

Proof. (A) First, we prove that (5) implies (6). By Theorem 4.5.1 of Ash (1972), we have

$$\int_S (1 - \mathrm{i}\boldsymbol{t} \cdot \boldsymbol{u})^{-c} \, \mathrm{d}\mu_n(\boldsymbol{u}) \to \int_S (1 - \mathrm{i}\boldsymbol{t} \cdot \boldsymbol{u})^{-c} \, \mathrm{d}\mu(\boldsymbol{u}), \quad \forall\ c > 0 \text{ and } |\boldsymbol{t}| < a^{-1}.$$

Hence, we obtain Eq. (6).

(B) Next, we prove that (6) implies (5). For any finite measure sequence $\{\mu_n\}$, by Theorem 2.5.3 of Ash (1972), there is a subsequence that converges vaguely to a finite measure, say $\mu_{n_k} \xrightarrow{v} \lambda$, as $n_k \to \infty$. By result (A) above, $\hat{\mu}_{n_k}^c(\boldsymbol{t}) \to \hat{\lambda}^c(\boldsymbol{t})$, for all $|\boldsymbol{t}| < a^{-1}$. By Eq. (6), we also have $\hat{\mu}_n^c(\boldsymbol{t}) \to \hat{\mu}^c(\boldsymbol{t})$, for all $|\boldsymbol{t}| < a^{-1}$. Hence, $\hat{\lambda}^c(\boldsymbol{t}) = \hat{\mu}^c(\boldsymbol{t})$, for all $|\boldsymbol{t}| < a^{-1}$. By the uniqueness property of the multivariate c-transformations of Lemma 2.2, we have $\lambda = \mu$. Therefore, we have $\mu_{n_k} \xrightarrow{v} \mu$, as $n_k \to \infty$. If there is another vaguely convergent subsequence of $\{\mu_n\}$ converging to ψ, then, by the same arguments as above, $\psi = \mu$. Therefore, $\mu_n \xrightarrow{v} \mu$. □

The following theorem, which shows how to use the multivariate c-characteristic function to generate moments, can be proved using the interchangeability of integrations and partial derivatives.

Theorem 2.5. *Let* $\boldsymbol{u}$ *be an* $L \times 1$ *random vector in a subset* S *of* $A = [-a_1, a_1] \times [-a_2, a_2] \times \cdots \times [-a_L, a_L]$ *and its corresponding multivariate c-characteristic function is* $g(\boldsymbol{t}; \boldsymbol{u}, c)$ *as defined in Definition* 2.1. *Then the* $\boldsymbol{b}$*th moment of* $\boldsymbol{u}$ *is*

$$E(u_1^{b_1} u_2^{b_2} \cdots u_L^{b_L}) = \frac{1}{(c, b_+)\mathrm{i}^{b_+}} \left. \frac{\partial^{b_+} g(\boldsymbol{t}; \boldsymbol{u}, c)}{\partial t_1^{b_1} \partial t_2^{b_2} \cdots \partial t_L^{b_L}} \right|_{\boldsymbol{t}=\boldsymbol{0}},$$

where $\boldsymbol{b} = (b_1, b_2, \ldots, b_L)'$, $b_+ = \sum_{j=1}^{L} b_j$, $(c, b_+) = c(c+1) \cdots (c + b_+ - 1)$, $\boldsymbol{0} = (0, \ldots, 0)'$, *and the elements of* $\boldsymbol{b}$ *are non-negative integers.*

The following corollary is an immediate consequence of Theorem 2.5.

Corollary 2.6. *Under the assumptions of Theorem* 2.5, *we have*

$$E(u_j) = \frac{1}{c\mathrm{i}} \left. \frac{\partial g(\boldsymbol{t}; \boldsymbol{u}, c)}{\partial t_j} \right|_{\boldsymbol{t}=\boldsymbol{0}}, \quad \text{for } j = 1, 2, \ldots, L,$$

$$E(u_j u_k) = \frac{-1}{c(c+1)} \left. \frac{\partial^2 g(\boldsymbol{t}; \boldsymbol{u}, c)}{\partial t_j \partial t_k} \right|_{\boldsymbol{t}=\boldsymbol{0}}, \quad \text{for } j, k = 1, 2, \ldots, L.$$

3. Applications to properties of the Dirichlet distributions

In this section, we use our multivariate c-characteristic function to show properties of Dirichlet distributions. First, we define the Dirichlet distribution and Carlson's function R (Carlson, 1977).

Definition 3.1. If the random vector $\boldsymbol{u}$ has the density in any $L-1$ of its coordinates,

$$f(\boldsymbol{u};\boldsymbol{b}) = B^{-1}(\boldsymbol{b})\cdot\prod_{i=1}^{L} u_i^{b_i-1},$$

for all $\boldsymbol{u}$ in the probability simplex $\{\boldsymbol{u}\,|\,\text{each } u_i \geqslant 0,\ u_+ = 1\}$, where parameter vector $\boldsymbol{b} = (b_1, b_2, \ldots, b_L)'$, each $b_i > 0$, $u_+ = \sum_{i=1}^{L} u_i$, and

$$B(\boldsymbol{b}) = \frac{\prod_{i=1}^{L}\Gamma(b_i)}{\Gamma(b_+)},$$

then $\boldsymbol{u}$ is said to follow a Dirichlet distribution with parameter $\boldsymbol{b}$. This is denoted by $\boldsymbol{u} \sim D(\boldsymbol{b})$. (Notice that the density function of $\boldsymbol{u}$ depends on only $L-1$ components of $\boldsymbol{u}$ since $\sum_{i=1}^{L} u_i = 1$.)

Definition 3.2. If the random vector $\boldsymbol{u}$ has the Dirichlet distribution with parameter vector $\boldsymbol{b}$, i.e., $\boldsymbol{u} \sim D(\boldsymbol{b})$, define $R_n(\boldsymbol{b}, \boldsymbol{z})$ as the nth moment of the random variable $\theta = \sum_{i=1}^{L} u_i z_i = \boldsymbol{u}\cdot\boldsymbol{z}$, i.e.,

$$R_n(\boldsymbol{b}, \boldsymbol{z}) = \mathop{E}_{\boldsymbol{u}|\boldsymbol{b}} \theta^n = \mathop{E}_{\boldsymbol{u}|\boldsymbol{b}} (\boldsymbol{u}\cdot\boldsymbol{z})^n,$$

where $\boldsymbol{z} = (z_1, z_2, \ldots, z_L)' \in \mathbb{C}^L$ (the Lth Cartesian power of the complex plane $\mathbb{C}$).

Useful references on Carlson's R include Carlson (1977), Dickey (1983), and Jiang et al. (1992).

Lemma 3.3. *If $\mu_{\boldsymbol{b}}$ is a Dirichlet measure with parameter vector $\boldsymbol{b} = (b_1, b_2, \ldots, b_L)'$, then*

$$\hat{\mu}_{\boldsymbol{b}}^{c}(\boldsymbol{t}) = R_{-c}(\boldsymbol{b}; 1-\mathrm{i}t_1, 1-\mathrm{i}t_2, \ldots, 1-\mathrm{i}t_L).$$

Proof.

$$\begin{aligned}\hat{\mu}_{\boldsymbol{b}}^{c}(\boldsymbol{t}) &= \mathop{E}_{\boldsymbol{u}|\boldsymbol{b}} (1-\mathrm{i}\boldsymbol{t}\cdot\boldsymbol{u})^{-c}\\ &= \mathop{E}_{\boldsymbol{u}|\boldsymbol{b}} \left[\sum_{k=1}^{L} u_k(1-\mathrm{i}t_k)\right]^{-c}\\ &= R_{-c}(\boldsymbol{b}; 1-\mathrm{i}t_1, 1-\mathrm{i}t_2, \ldots, 1-\mathrm{i}t_L).\end{aligned}$$

The last equality can be seen by Definition 3.2. □

Corollary 3.4. *If $\mu_{\boldsymbol{b}}$ is a Dirichlet measure with parameter vector $\boldsymbol{b}=(b_1,b_2,\ldots,b_L)'$, and $c=\sum_{k=1}^{L} b_k$, then*

$$\hat{\mu}_{\boldsymbol{b}}^{c}(\boldsymbol{t})=\prod_{k=1}^{L}(1-\mathrm{i}t_k)^{-b_k},$$

where $|\boldsymbol{t}|<1$.

Proof. Carlson (1977, p. 144) shows that

$$R_{-c}(\boldsymbol{b};z_1,z_2,\ldots,z_L)=\prod_{i=1}^{L} z_i^{-b_i},$$

where $\boldsymbol{b}\in\{\boldsymbol{x}=(x_1,\ldots,x_L)'\in\mathbb{C}^L \mid \sum_{i=1}^{L} x_i \neq 0,-1,-2,\ldots\}$, $c=\sum_{i=1}^{L} b_i$, $|\mathbf{1}-\boldsymbol{z}|<1$, and $\mathbf{1}$ is an $L\times 1$ vector with each component 1. Since $|(1,1,\ldots,1)-(1-\mathrm{i}t_1, 1-\mathrm{i}t_2,\ldots,1-\mathrm{i}t_L)|=|\boldsymbol{t}|<1$ and c is positive, by Lemma 3.3, the corollary is proved. □

If $\boldsymbol{u}$ follows a Dirichlet distribution with parameter vector $\boldsymbol{b}$, then, by Corollary 3.4, $g(\boldsymbol{t};\boldsymbol{u},b_+)=\prod_{k=1}^{L}(1-\mathrm{i}t_k)^{-b_k}$, where $|\boldsymbol{t}|<1$. By Corollary 2.6, we have the following moments.

Example 3.1. Let $\boldsymbol{u}=(u_1,u_2,\ldots,u_L)'\sim D(\boldsymbol{b})$ and $c=b_+$. Then, for any j, $j'=1,2,\ldots,L$,

$$E(u_j)=\frac{b_j}{b_+},\quad E(u_j^2)=\frac{b_j(b_j+1)}{b_+(b_++1)},\quad E(u_ju_{j'})=\frac{b_jb_{j'}}{b_+(b_++1)},\quad \text{if } j\neq j'.$$

Now, we can show that any marginal Dirichlet distribution is still a Dirichlet distribution. In particular, we have the following corollary.

Corollary 3.5. *Consider a Dirichlet random vector $\boldsymbol{u}$ with parameter $\boldsymbol{b}$, i.e., $\boldsymbol{u}\sim D(\boldsymbol{b})$. Conformably partition $\boldsymbol{u}=(\boldsymbol{u}^{(1)},\boldsymbol{u}^{(2)})$, $\boldsymbol{b}=(\boldsymbol{b}^{(1)},\boldsymbol{b}^{(2)})$, and $L=L^{(1)}+L^{(2)}$ (i.e., $\boldsymbol{u}^{(1)}$ and $\boldsymbol{b}^{(1)}$ have the same number of components, which is $L^{(1)}$, and $\boldsymbol{u}^{(2)}$ and $\boldsymbol{b}^{(2)}$ have the same number of components, which is $L^{(2)}$). Then the marginal distribution $\tilde{\boldsymbol{u}}=(\boldsymbol{u}^{(1)},u_+^{(2)})$ is still the Dirichlet distribution with parameter vector $\tilde{\boldsymbol{b}}=(\boldsymbol{b}^{(1)},b_+^{(2)})$, where $u_+^{(2)}=\sum_{k=L^{(1)}+1}^{L} u_k$ and $b_+^{(2)}=\sum_{k=L^{(1)}+1}^{L} b_k$. That is, $\tilde{\boldsymbol{u}}\sim D(\tilde{\boldsymbol{b}})$.*

Proof. Let the c-transformation of the distribution of $\boldsymbol{u}$ be $\hat{\mu}_{\boldsymbol{b}}^{c}(\boldsymbol{t})$, where $c=\sum_{i=1}^{L} b_i$. Hence, by Eq. (1) and Corollary 3.4,

$$\hat{\mu}_{\boldsymbol{b}}^{c}(\boldsymbol{t})=\int_S(1-\mathrm{i}\boldsymbol{t}\cdot\boldsymbol{u})^{-c}\,\mathrm{d}\mu(\boldsymbol{u})=\prod_{k=1}^{L}(1-\mathrm{i}t_k)^{-b_k},\quad \text{where } |\boldsymbol{t}|<1.$$

The c-transformation of the marginal distribution $\tilde{\boldsymbol{u}}$ is then $\hat{\mu}_{\boldsymbol{b}}^{c}(t_1,\ldots,t_{L^{(1)}},t_{L^{(1)}+1},t_{L^{(1)}+1},\ldots,t_{L^{(1)}+1})$, which can be expressed as

$$\hat{\mu}_{\boldsymbol{b}}^{c}(t_1,\ldots,t_{L^{(1)}},t_{L^{(1)}+1},t_{L^{(1)}+1},\ldots,t_{L^{(1)}+1})=\left[\prod_{k=1}^{L^{(1)}}(1-\mathrm{i}t_k)^{-b_k}\right](1-\mathrm{i}t_{L^{(1)}+1})^{-b_{\bullet}^{(2)}},$$

where $\sqrt{\sum_{k=1}^{L^{(1)}} t_k^2 + L^{(2)} t_{L^{(1)}+1}^2} < 1$. By Corollary 3.4, the uniqueness property of Lemma 2.2, and the fact that $\sqrt{\sum_{k=1}^{L^{(1)}+1} t_k^2} \leqslant \sqrt{\sum_{k=1}^{L^{(1)}} t_k^2 + L^{(2)} t_{L^{(1)}+1}^2}$, we have that the marginal distribution $\tilde{\boldsymbol{u}}$ follows a Dirichlet distribution with parameter $\tilde{\boldsymbol{b}}$. □

The traditional proof of Corollary 3.5 can be seen in Wilks (1962).

4. Relation between a random vector and its transformation

In the next theorem, we shall give the relation between the multivariate c-characteristic function of a random vector $\boldsymbol{u}$ and that of a linear transformation of $\boldsymbol{u}$.

Theorem 4.1. *Let $\boldsymbol{u}$ be an $L \times 1$ random vector with support S in a subset of $A = [-a_1, a_1] \times \cdots \times [-a_L, a_L]$ and its corresponding multivariate c-characteristic function is $g(\boldsymbol{t}; \boldsymbol{u}, c)$ as defined in Definition 2.1. Then*

$$g(\boldsymbol{s}; G\boldsymbol{u}, c) = g(\boldsymbol{t}; \boldsymbol{u}, c),$$

where G is an $M \times L$ matrix of real numbers, $\boldsymbol{s} = (s_1, s_2, \ldots, s_M)'$ is an $M \times 1$ vector and $\boldsymbol{t} = G'\boldsymbol{s}$.

Proof.

$$\begin{aligned} g(\boldsymbol{s}; G\boldsymbol{u}, c) &= E\left([1 - \mathrm{i}\boldsymbol{s}'(G\boldsymbol{u})]^{-c}\right) \\ &= E([1 - \mathrm{i}(\boldsymbol{s}'G)\boldsymbol{u}]^{-c}) \\ &= g(G'\boldsymbol{s}; \boldsymbol{u}, c) \\ &= g(\boldsymbol{t}; \boldsymbol{u}, c). \qquad \square \end{aligned}$$

Dickey and Jiang (1998) introduced the filtered-variate Dirichlet distribution for Bayesian local smoothness and defined a random probability vector $\boldsymbol{v} = (v_1, v_2, \ldots, v_M)'$, where $v_+ = 1$, to be filtered-variate Dirichlet distributed, $\boldsymbol{v} \sim F_G D(\boldsymbol{b})$, with $\boldsymbol{v} = G\boldsymbol{u}$ for a constant matrix $G_{M \times L}$ and an $L \times 1$ parameter vector $\boldsymbol{b}$, where the $L \times 1$ random vector $\boldsymbol{u}$ has a Dirichlet distribution with parameter $\boldsymbol{b}$. Note, from the fact that $\boldsymbol{u}$ and $\boldsymbol{v}$ are probability vectors, that all entries of G are non-negative and each column sums to unity. The multivariate c-characteristic function of the filtered-variate distribution $F_G D(\boldsymbol{b})$ is given in the next example.

Example 4.1. Let $\boldsymbol{v} = (v_1, \ldots, v_M)' \sim F_G D(\boldsymbol{b})$, where G is an $M \times L$ matrix and $\boldsymbol{b} = (b_1, b_2, \ldots, b_L)'$ is an $L \times 1$ random vector. That is, $\boldsymbol{v} = G\boldsymbol{u}$, where $\boldsymbol{u} \sim D(\boldsymbol{b})$. By Corollary 3.4 and Theorem 4.1, we have

$$g(\boldsymbol{s}; \boldsymbol{v}, c) = g(\boldsymbol{t}; \boldsymbol{u}, c) = \prod_{j=1}^{L} (1 - \mathrm{i}t_j)^{-b_j},$$

where $\boldsymbol{t} = G'\boldsymbol{s}$ and $c = \sum_{j=1}^{L} b_j$. Hence, the multivariate c-characteristic function of the filtered-variate Dirichlet distribution $\boldsymbol{v} \sim F_G D(\boldsymbol{b})$ is

$$g(\boldsymbol{s}; \boldsymbol{v}, c) = \prod_{j=1}^{L} \left[1 - \mathrm{i} \left(\sum_{m=1}^{M} g_{mj} s_m \right) \right]^{-b_j}. \qquad \square$$

The following corollary, which gives the c-characteristic function of a marginal distribution, is a special case of Theorem 4.1.

Corollary 4.2. *Conformably partition* $\boldsymbol{u} = \begin{pmatrix} \boldsymbol{u}_1 \\ \cdots \\ \boldsymbol{u}_2 \end{pmatrix}$ *and* $\boldsymbol{t} = \begin{pmatrix} \boldsymbol{t}_1 \\ \cdots \\ \boldsymbol{t}_2 \end{pmatrix}$. *Then*

$$g(\boldsymbol{t}_1; \boldsymbol{u}_1, c) = g(\boldsymbol{t}_0; \boldsymbol{u}, c), \tag{7}$$

where $\boldsymbol{t}_0 = \begin{pmatrix} \boldsymbol{t}_1 \\ \cdots \\ \boldsymbol{0} \end{pmatrix}$ *has the same dimensionality as that of* $\boldsymbol{t}$.

Proof. Assume $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ are $L_1 \times 1$ and $L_2 \times 1$ vectors, respectively. Let $G = [I \vdots 0]$, where I is an $L_1 \times L_1$ identity matrix and $[0]$ is an $L_1 \times L_2$ zero matrix. By Theorem 4.1 and $\boldsymbol{t}_0 = G'\boldsymbol{t}_1$, identity (7) holds. $\square$

Note that Corollary 4.2 can also be proved directly by the definition of the multivariate c-characteristic function. Now, if we let matrix G in Theorem 4.1 be a vector, the $1 \times L$ matrix $[g_1, g_2, \ldots, g_L]$, we obtain the following corollary.

Corollary 4.3. *Let a random variable v be a linear combination of the random vector* $\boldsymbol{u}$, *i.e.*, $v = \sum_{j=1}^{L} g_j u_j$, *then the univariate c-characteristic function of v can be expressed in terms of the multivariate c-characteristic function of* $\boldsymbol{u}$ *by the relation*

$$g(s; v, c) = g(\boldsymbol{t}; \boldsymbol{u}, c),$$

where $\boldsymbol{t} = (sg_1, sg_2, \ldots, sg_L)'$.

The following example gives the univariate c-characteristic function of a linear combination of a Dirichlet random vector.

Example 4.2. Let $\boldsymbol{u} = (u_1, \ldots, u_L)' \sim D(\boldsymbol{b})$ and $v = \sum_{j=1}^{L} g_j u_j$. Then, by Corollary 3.4 and Corollary 4.3, we have

$$g(s; v, c) = \prod_{j=1}^{L} (1 - \mathrm{i} s g_j)^{-b_j},$$

where $c = \sum_{j=1}^{L} b_j$.

Note that the univariate c-characteristic function of the random variable v in Example 4.2 is the same as that derived by the univariate c-characteristic function directly, see Corollary 3.3 of Jiang (1988).

5. Relation between the multivariate c-characteristic function and traditional characteristic function

We define the mth moment of a linear combination of a random vector $\boldsymbol{u}=(u_1,u_2,\dots,u_L)'$ first.

Definition 5.1. Let $\boldsymbol{u}$ be an $L\times 1$ random vector with support in a subset S of $A=[-a_1,a_1]\times[-a_2,a_2]\times\cdots\times[-a_L,a_L]$, and let $\boldsymbol{t}=(t_1,t_2,\dots,t_L)'$ be a parameter vector with $|\boldsymbol{t}|<a^{-1}$. Then the mth order multinomial function of $\boldsymbol{t}$, denoted by $E_m(\boldsymbol{t})$, is defined as

$$E_m(\boldsymbol{t})=E\left[(\boldsymbol{t}\cdot\boldsymbol{u})^m\right],$$

where $a=\sqrt{\sum_{j=1}^{L}a_j^2}$.

Note that

$$E\left[(\boldsymbol{t}\cdot\boldsymbol{u})^m\right]=\sum\frac{m!}{d_1!d_2!\cdots d_L!}t_1^{d_1}t_2^{d_2}\cdots t_L^{d_L}E[u_1^{d_1}u_2^{d_2}\cdots u_L^{d_L}],$$

where $\{d_1,d_2,\dots,d_L\}$ is any selection of numbers from $0,1,2,\dots,m$ such that $d_1+d_2+\cdots+d_L=m$. Hence, $E_m(\boldsymbol{t})$ is a function of $\boldsymbol{t}$ and depends on $\boldsymbol{t}$ only through the mth power of its components. We shall give the traditional characteristic function and multivariate c-characteristic function in terms of $E_m(\boldsymbol{t})$ in the next theorem, which would also show the relation between these two functions by their expressions.

Theorem 5.2. *Let $\phi_{\boldsymbol{u}}(\boldsymbol{t})$ and $g(\boldsymbol{t};\boldsymbol{u},c)$ be the traditional characteristic function and multivariate c-characteristic function, respectively, for a random vector $\boldsymbol{u}$ with support in a subset S of $A=[-a_1,a_1]\times[-a_2,a_2]\times\cdots\times[-a_L,a_L]$. Then*

$$\text{(A)}\ \phi_{\boldsymbol{u}}(\boldsymbol{t})=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}E_{2k}(\boldsymbol{t})+\mathrm{i}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}E_{2k+1}(\boldsymbol{t}),\quad \forall\ |\boldsymbol{t}|,$$

and

$$\text{(B)}\ g(\boldsymbol{t};\boldsymbol{u},c)=\sum_{k=0}^{\infty}\frac{(-1)^k(c,2k)}{(2k)!}E_{2k}(\boldsymbol{t})+\mathrm{i}\sum_{k=0}^{\infty}\frac{(-1)^k(c,2k+1)}{(2k+1)!}E_{2k+1}(\boldsymbol{t}),\quad \forall\ |\boldsymbol{t}|<a^{-1}, \tag{8}$$

where $a=\sqrt{\sum_{j=1}^{L}a_j^2}$.

Proof. (A) For all $\boldsymbol{t}$,

$$\begin{aligned}\phi_{\boldsymbol{u}}(\boldsymbol{t}) &= E(e^{i\boldsymbol{t}\cdot\boldsymbol{u}})\\ &= E[\cos(\boldsymbol{t}\cdot\boldsymbol{u})] + iE[\sin(\boldsymbol{t}\cdot\boldsymbol{u})]\\ &= E\left[\sum_{k=0}^{\infty}\frac{(-1)^k(\boldsymbol{t}\cdot\boldsymbol{u})^{2k}}{(2k)!}\right] + iE\left[\sum_{k=0}^{\infty}\frac{(-1)^k(\boldsymbol{t}\cdot\boldsymbol{u})^{2k+1}}{(2k+1)!}\right]\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}E_{2k}(\boldsymbol{t}) + i\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}E_{2k+1}(\boldsymbol{t}).\end{aligned}$$

The last identity is obtained by the Fubini theorem.

(B) For all $|\boldsymbol{t}| < a^{-1}$,

$$\begin{aligned}&g(\boldsymbol{t};\boldsymbol{u},c)\\ &\quad= E[(1 - i\boldsymbol{t}\cdot\boldsymbol{u})^{-c}]\\ &\quad= E\left[\sum_{n=0}^{\infty}\frac{(c,n)}{n!}(i\boldsymbol{t}\cdot\boldsymbol{u})^n\right]\\ &\quad= E\left\{\sum_{k=0}^{\infty}\left[\frac{(-1)^k(c,2k)}{(2k)!}(\boldsymbol{t}\cdot\boldsymbol{u})^{2k} + i\,\frac{(-1)^k(c,2k+1)}{(2k+1)!}(\boldsymbol{t}\cdot\boldsymbol{u})^{2k+1}\right]\right\}.\end{aligned}$$

It can be shown, by the ratio test, that the series inside the bracket converges absolutely. By the rearrangement theorem, for all $|\boldsymbol{t}| < a^{-1}$, we have

$$\begin{aligned}&g(\boldsymbol{t};\boldsymbol{u},c)\\ &\quad= E\left\{\sum_{k=0}^{\infty}\left[\frac{(-1)^k(c,2k)}{(2k)!}(\boldsymbol{t}\cdot\boldsymbol{u})^{2k}\right] + i\sum_{k=0}^{\infty}\left[\frac{(-1)^k(c,2k+1)}{(2k+1)!}(\boldsymbol{t}\cdot\boldsymbol{u})^{2k+1}\right]\right\}\\ &\quad= E\left\{\sum_{k=0}^{\infty}\frac{(-1)^k(c,2k)}{(2k)!}(\boldsymbol{t}\cdot\boldsymbol{u})^{2k}\right\} + iE\left\{\sum_{k=0}^{\infty}\frac{(-1)^k(c,2k+1)}{(2k+1)!}(\boldsymbol{t}\cdot\boldsymbol{u})^{2k+1}\right\}\\ &\quad= \sum_{k=0}^{\infty}\frac{(-1)^k(c,2k)}{(2k)!}E_{2k}(\boldsymbol{t}) + i\sum_{k=0}^{\infty}\frac{(-1)^k(c,2k+1)}{(2k+1)!}E_{2k+1}(\boldsymbol{t}). \qquad \square\end{aligned}$$

Notice that the coefficient of the $E_m(\boldsymbol{t})$ term for $\phi_{\boldsymbol{u}}(\boldsymbol{t})$ is the product of the corresponding coefficients for $g(\boldsymbol{t};\boldsymbol{u},c)$ and $(c,m)^{-1}$. Therefore, to determine the traditional characteristic function from the multivariate c-characteristic function, we expand $g(\boldsymbol{t};\boldsymbol{u},c)$ in terms of $E_m(\boldsymbol{t})$ first. That is, for all $|\boldsymbol{t}| < a^{-1}$,

$$g(\boldsymbol{t};\boldsymbol{u},c) = \sum_{k=0}^{\infty} c_{2k}E_{2k}(\boldsymbol{t}) + i\sum_{k=0}^{\infty} c_{2k+1}E_{2k+1}(\boldsymbol{t}), \tag{9}$$

where $E_m(\boldsymbol{t})$ is the mth order multinomial function of $\boldsymbol{t}$, and c_m is a real number and is independent of $\boldsymbol{t}$. We can then express the traditional characteristic function by, for all $\boldsymbol{t}$,

$$\phi_{\boldsymbol{u}}(\boldsymbol{t}) = \sum_{k=0}^{\infty} \frac{c_{2k}}{(c,2k)} E_{2k}(\boldsymbol{t}) + \mathrm{i} \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(c,2k+1)} E_{2k+1}(\boldsymbol{t}). \tag{10}$$

Note that $E_m(\boldsymbol{t})$ exists for all $\boldsymbol{t}$. Although the domain of $E_m(\boldsymbol{t})$ in (9) is $|\boldsymbol{t}| < a^{-1}$, the domain of $E_m(\boldsymbol{t})$ in (10) is all possible real vector values of $\boldsymbol{t}$, since each series in (10) converges for all $\boldsymbol{t}$.

In the following example, we show how to find a traditional characteristic function from its corresponding multivariate c-characteristic function.

Example 5.1. Consider a random vector $\boldsymbol{u} = (u_1, u_2)'$ with support $S = \{(x_1, x_2) : 0 \leqslant x_1^2 + x_2^2 < 1\}$. Given a c-characteristic function

$$g(\boldsymbol{t}; \boldsymbol{u}, c) = \left(\frac{2}{1 + \sqrt{1 + t_1^2 + t_2^2}} \right)^c, \quad |\boldsymbol{t}| < 1, \tag{11}$$

what is the corresponding traditional characteristic function?

The following formula can be seen in p. 101 of Erdélyi et al. (1953), $\forall\, |y| < 1$,

$$\left(\frac{2}{1 + \sqrt{1 - y}} \right)^c = \left(\frac{1}{2} + \frac{\sqrt{1 - y}}{2} \right)^{-c} = \sum_{n=0}^{\infty} \frac{(c/2, n)(c/2 + 1/2, n)}{n!(c+1, n)}\, y^n.$$

Hence, Eq. (11) for $g(\boldsymbol{t}; \boldsymbol{u}, c)$ can be reexpressed as

$$g(\boldsymbol{t}; \boldsymbol{u}, c) = \sum_{n=0}^{\infty} \frac{(c/2, n)(c/2 + 1/2, n)}{n!(c+1, n)} (-1)^n (t_1^2 + t_2^2)^n. \tag{12}$$

By the arguments above, the nth term in Eq. (12) is the term corresponding to $E_{2n}(\boldsymbol{t})$ in Eqs. (9) and (10). Hence, we have

$$\phi_{\boldsymbol{u}}(\boldsymbol{t}) = \sum_{n=0}^{\infty} \frac{(c/2, n)(c/2 + 1/2, n)}{n!(c+1, n)} \frac{1}{(c, 2n)} (-1)^n (t_1^2 + t_2^2)^n.$$

Notice that, by Eqs. (8) and (12), we have

$$\begin{aligned} E_{2n}(\boldsymbol{t}) &= \frac{(2n)!(c/2, n)(c/2 + 1/2, n)}{(c, 2n) n!(c+1, n)} (t_1^2 + t_2^2)^n \\ &= \frac{[n! 2^{2n} (1/2, n)](c/2, n)(c/2 + 1/2, n)}{[2^{2n}(c/2, n)(c/2 + 1/2, n)] n!(c+1, n)} (t_1^2 + t_2^2)^n \\ &= \frac{(1/2, n)}{(c+1, n)} (t_1^2 + t_2^2)^n. \qquad \square \end{aligned}$$

To determine the joint probability density function of this random vector, we may use the well-known inversion formula of the traditional characteristic function. With the above traditional characteristic function $\phi_{\boldsymbol{u}}(\boldsymbol{t})$, it can be shown that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\phi_{\boldsymbol{u}}(\boldsymbol{t})| \mathrm{d}\boldsymbol{t} < \infty$.

The inversion formula to find the corresponding probability density function $f(\boldsymbol{u})$ is, by Roussas (1997, p. 151),

$$f(\boldsymbol{u}) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\boldsymbol{t}\cdot\boldsymbol{u}}\phi_{\boldsymbol{u}}(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}.$$

It can be seen that the joint probability density function of $\boldsymbol{u}$ corresponding to (11) is

$$f(u_1,u_2) = \frac{c}{\pi}(1-u_1^2-u_2^2)^{c-1}, \quad 0 \leqslant u_1^2+u_2^2 < 1.$$

6. Distribution of a random functional of a Ferguson–Dirichlet process over the region bounded by an ellipse

In this section, we use our new multivariate c-characteristic function to study random functionals of a Ferguson–Dirichlet process, on the region bounded by an ellipse. Ferguson (1973) first defined the Ferguson–Dirichlet process. Let μ be a finite non-null measure on (X,A), where A is the σ-field of Borel subsets of Euclidean space X, and let U be a stochastic process indexed by elements of A. We say U is a Ferguson–Dirichlet process with parameter μ, if for every finite measurable partition $\{B_1,B_2,\ldots,B_m\}$ of X (i.e., the B_i are measurable, disjoint, and $\bigcup_{i=1}^m B_i = X$), the random vector $(U(B_1),U(B_2),\ldots,U(B_m))$ has a Dirichlet distribution with parameter $(\mu(B_1),\mu(B_2),\ldots,\mu(B_m))$. Hannum et al. (1981), Yamato (1984), and Jiang (1991), among others, studied the distribution of $\int z\,\mathrm{d}U$. In particular, Jiang (1991) studied the case when X is the unit circle. Here, we shall generalize to an ellipse.

Let $\Theta=[0,2\pi)$, a and b be positive real numbers, U be a Ferguson–Dirichlet process with parameter μ on (Θ,B), B be the σ-field of Borel subsets of Θ, and μ be a non-null (positive) finite measure on (Θ,B). Let $\beta(\theta)=(2\pi/c)\mu[0,\theta)$ with $c=\mu(\Theta)$. Then define a random vector $\boldsymbol{v}=(v_1,v_2)$ as follows:

$$\boldsymbol{v} = \int_{\Theta}(a\cos\beta(\theta), b\sin\beta(\theta))\,\mathrm{d}U(\theta), \tag{13}$$

the random mean of Ferguson–Dirichlet process on the ellipse. Before giving the probability density function of the random vector $\boldsymbol{v}$, we need to obtain its multivariate c-characteristic function.

Lemma 6.1. *The c-characteristic function of* $\boldsymbol{v}$, *as defined in* (13), *has the closed form,*

$$g(\boldsymbol{t};\boldsymbol{v},c) = \left[\frac{2}{1+\sqrt{1+a^2t_1^2+b^2t_2^2}}\right]^c, \quad \text{where } |\boldsymbol{t}| < (a^2+b^2)^{-1/2}.$$

Proof. For any $k \geqslant 2$, define the jth partition of $[0,2\pi)$ as

$$I_{kj} = \left[\frac{2\pi}{k}(j-1), \frac{2\pi}{k}j\right), \quad j=1,2,\ldots,k.$$

Then $(U(I_{k1}), U(I_{k2}), \ldots, U(I_{kk}))$ follows a Dirichlet distribution with parameters $(\mu(I_{k1}), \mu(I_{k2}), \ldots, \mu(I_{kk}))$. Furthermore, let

$$c_{kj} = \cos\left(\beta\left[\frac{2\pi}{k}\left(j-\frac{1}{2}\right)\right]\right), \qquad f_k(\theta) = \sum_{j=1}^{k} c_{kj}\delta_\theta(I_{kj}),$$

$$s_{kj} = \sin\left(\beta\left[\frac{2\pi}{k}\left(j-\frac{1}{2}\right)\right]\right), \qquad g_k(\theta) = \sum_{j=1}^{k} s_{kj}\delta_\theta(I_{kj}),$$

where $\delta_\theta(I_{kj})$ is 1, for $\theta \in I_{kj}$; and is 0, otherwise. Then

$$\lim_{k\to\infty} f_k(\theta) = \cos\beta(\theta), \quad \forall\,\theta \in \Theta,$$

$$\lim_{k\to\infty} g_k(\theta) = \sin\beta(\theta), \quad \forall\,\theta \in \Theta,$$

$$\int_\Theta f_k(\theta)\,\mathrm{d}U(\theta) = \sum_{j=1}^{k} c_{kj}U(I_{kj}), \qquad \int_\Theta g_k(\theta)\,\mathrm{d}U(\theta) = \sum_{j=1}^{k} s_{kj}U(I_{kj}),$$

and

$$\int_\Theta (af_k(\theta), bg_k(\theta))\,\mathrm{d}U(\theta) = \left(a\sum_{j=1}^{k} c_{kj}U(I_{kj}), b\sum_{j=1}^{k} s_{kj}U(I_{kj})\right).$$

Let $\boldsymbol{v}_k = \int_\Theta (af_k(\theta), bg_k(\theta))\,\mathrm{d}U(\theta)$. Then the c-characteristic function of $\boldsymbol{v}_k$, by Definitions 2.1 and 3.2, can be expressed as

$$\begin{aligned} g(\boldsymbol{t}; \boldsymbol{v}_k, c) &= \underset{\boldsymbol{v}_k}{E}(1 - \mathrm{i}t_1v_1 - \mathrm{i}t_2v_2)^{-c} \\ &= E\left[1 - \mathrm{i}t_1a\sum_{j=1}^{k} c_{kj}U(I_{kj}) - \mathrm{i}t_2b\sum_{j=1}^{k} s_{kj}U(I_{kj})\right]^{-c} \\ &= R_{-c}(\mu(I_{k1}), \ldots, \mu(I_{kk}); 1 - \mathrm{i}t_1ac_{k1} - \mathrm{i}t_2bs_{k1}, \ldots, 1 - \mathrm{i}t_1ac_{kk} - \mathrm{i}t_2bs_{kk}), \end{aligned}$$

where $|\boldsymbol{t}| < (a^2 + b^2)^{-1/2}$. By formula (6.6.5) of Carlson (1977), we have

$$\begin{aligned} g(\boldsymbol{t}; \boldsymbol{v}_k, c) &= \prod_{j=1}^{k} (1 - \mathrm{i}t_1ac_{kj} - \mathrm{i}t_2bs_{kj})^{-\mu(I_{kj})} \\ &= \prod_{j=1}^{k} \left(1 - \mathrm{i}t_1a\cos\left(\beta\left[\frac{2\pi}{k}\left(j-\frac{1}{2}\right)\right]\right) - \mathrm{i}t_2b\sin\left(\beta\left[\frac{2\pi}{k}\left(j-\frac{1}{2}\right)\right]\right)\right)^{-\mu(I_{kj})}. \end{aligned}$$

The limit of c-characteristic functions of $\boldsymbol{v}_k$'s, as k approaches ∞, is

$$\lim_{k\to\infty} g(\boldsymbol{t};\boldsymbol{v}_k,c) = \exp\left[\lim_{k\to\infty}\sum_{j=1}^{k} -\mu(I_{kj})\cdot\ln\left(1-\mathrm{i}t_1 a\cos\left(\beta\left[\frac{2\pi}{k}\left(j-\frac{1}{2}\right)\right]\right)\right.\right.$$

$$\left.\left.-\mathrm{i}t_2 b\sin\left(\beta\left[\frac{2\pi}{k}\left(j-\frac{1}{2}\right)\right]\right)\right)\right]$$

$$= \exp\left[-\frac{c}{2\pi}\int_0^{2\pi}\ln(1-\mathrm{i}t_1\, a\cos\alpha - \mathrm{i}t_2 b\sin\alpha)\,\mathrm{d}\alpha\right]$$

$$= \left(\frac{2}{1+\sqrt{1+a^2t_1^2+b^2t_2^2}}\right)^c, \tag{14}$$

where $|\boldsymbol{t}| < (a^2+b^2)^{-1/2}$. Note that the second identity of Eq. (14) is the transition from a Riemann sum to an integral, and the last identity of Eq. (14) is from the result of Lemma A.1 in the appendix. By the Dominated Convergence Theorem, we have

$$\lim_{k\to\infty}\int_{\Theta}(af_k(\theta), bg_k(\theta))\,\mathrm{d}U(\theta) = \int_{\Theta}(a\cos\beta(\theta), b\sin(\theta))\,\mathrm{d}U(\theta). \tag{15}$$

That is, $\lim_{k\to\infty}\boldsymbol{v}_k = \boldsymbol{v}$. From Eqs. (14), (15) and Theorem 2.4, we have the c-characteristic function of $\boldsymbol{v}$ as

$$\left(\frac{2}{1+\sqrt{1+a^2t_1^2+b^2t_2^2}}\right)^c,$$

where $|\boldsymbol{t}| < (a^2+b^2)^{-1/2}$. This completes the proof. □

In the next example, we give the first two moments of the random vector (13).

Example 6.1. Let $\boldsymbol{v}=(v_1,v_2)$ be the random functional of a Ferguson–Dirichlet process as defined in (13). Then, by Corollary 2.6 and Lemma 6.1, its first two moments are as follows:

$$E(v_1)=0,\ E(v_2)=0,\ E(v_1^2)=\frac{a^2}{2(c+1)},\ E(v_2^2)=\frac{b^2}{2(c+1)},\ E(v_1v_2)=0.$$

Before knowing the corresponding probability density function, we can use the above method to calculate moments, which is seen to be easy.

In the next lemma, we define a random vector of interest and find its multivariate c-characteristic function.

Lemma 6.2. *Let a random vector* $\boldsymbol{w}=(w_1,w_2)'$ *have the probability density function*

$$f(\boldsymbol{w}) = \frac{c}{\pi ab}\left(1-\frac{w_1^2}{a^2}-\frac{w_2^2}{b^2}\right)^{c-1},$$

where a, b, and c are positive real numbers and $0 \leqslant w_1^2/a^2 + w_2^2/b^2 < 1$. *Then the c-characteristic function of* $\boldsymbol{w}$ *is*

$$g(\boldsymbol{t}; \boldsymbol{w}, c) = \left(\frac{2}{1 + \sqrt{1 + a^2 t_1^2 + b^2 t_2^2}} \right)^c, \quad \text{where } |\boldsymbol{t}| < (a^2 + b^2)^{-1/2}.$$

Proof. Without loss of generality, we assume $a \geqslant b$. The c-characteristic function of $\boldsymbol{w}$ can be expressed as

$$g(\boldsymbol{t}; \boldsymbol{w}, c) = \underset{\boldsymbol{w}}{E}[(1 - \mathrm{i}\boldsymbol{t} \cdot \boldsymbol{w})^{-c}]$$

$$= \int_{-b}^{b} \int_{-a\sqrt{1-w_2^2/b^2}}^{a\sqrt{1-w_2^2/b^2}} (1 - \mathrm{i}\boldsymbol{t} \cdot \boldsymbol{w})^{-c} \cdot \frac{c}{\pi ab} \left(1 - \frac{w_1^2}{a^2} - \frac{w_2^2}{b^2} \right)^{c-1} \mathrm{d}w_1 \, \mathrm{d}w_2,$$

where $|\boldsymbol{t}| < (a^2 + b^2)^{-1/2}$. Define $e = \sqrt{a^2 - b^2}/a$. Then e is the eccentricity of the ellipse and $b = a\sqrt{1 - e^2}$. Now, letting $w_1 = r \cos \theta$ and $w_2 = r\sqrt{1 - e^2} \sin \theta$, we have

$$g(\boldsymbol{t}; \boldsymbol{w}, c)$$

$$= \frac{c}{\pi a^2} \int_0^a r \left(1 - \frac{r^2}{a^2} \right)^{c-1} \int_0^{2\pi} \left[1 - \mathrm{i}r \left(t_1 \cos \theta + t_2 \sqrt{1 - e^2} \sin \theta \right) \right]^{-c} \mathrm{d}\theta \, \mathrm{d}r,$$

where $|\boldsymbol{t}| < (a^2 + b^2)^{-1/2}$. By the Cauchy inequality, we have $a^2 t_1^2 + a^2 t_2^2 (1 - e^2) = a^2 t_1^2 + b^2 t_2^2 \leqslant |\boldsymbol{t}|^2 \cdot (a^2 + b^2) < 1$. Hence, $\sqrt{t_1^2 + t_2^2(1 - e^2)} < a^{-1}$. By Lemma A.2 in the appendix, letting $s_1 = t_1$ and $s_2 = t_2\sqrt{1 - e^2}$, we have

$$g(\boldsymbol{t}; \boldsymbol{w}, c) = \left(\frac{2}{1 + \sqrt{1 + a^2 t_1^2 + b^2 t_2^2}} \right)^c,$$

where $|\boldsymbol{t}| < (a^2 + b^2)^{-1/2}$. This completes the proof. □

Finally, by Lemmas 2.2, 6.1, and 6.2, we are ready to give the distribution of $\boldsymbol{v}$, as a density in closed form, in the following theorem.

Theorem 6.3. *The probability density function of* $\boldsymbol{v}$, *which is defined in Eq.* (13), *can be expressed as*

$$f(\boldsymbol{v}) = \frac{c}{\pi ab} \left(1 - \frac{v_1^2}{a^2} - \frac{v_2^2}{b^2} \right)^{c-1}, \tag{16}$$

where $0 \leqslant v_1^2/a^2 + v_2^2/b^2 < 1$.

Theorem 6.3 says that the random functional of a Ferguson–Dirichlet process, on the boundary of an ellipse, has a probability density function (16). For the special case, when $a = b = 1$, we have that $\int_{\Theta} \mathrm{e}^{\mathrm{i}\beta(\theta)} \, \mathrm{d}U(\theta)$ follows a distribution with probability density function $f(\boldsymbol{v}) = (c/\pi)(1 - v_1^2 - v_2^2)^{c-1}$ on the unit disk, which is the same result given by Theorem 2 of Jiang (1991). The latter includes the elegant case of the uniform distribution on the disk for the random mean of the Ferguson–Dirichlet process with parameter measure the uniform probability distribution on the circle.

7. Conclusions

Our new multivariate c-transformation (or c-characteristic function) has properties similar to the Fourier transformation (or usual characteristic function). It provides an alternative approach to transformation or characteristic problems for finite measures with compact support. The new multivariate characteristic functions can be useful when problems are difficult to deal with by traditional characteristic functions.

Although the result on the property of Dirichlet distributions in Corollary 3.5 is not new, the proof is very easy with our new transformation. The filtered-variate Dirichlet distribution is important in areas such as Bayesian local smoothness. One way to deal with the filtered-variate Dirichlet distribution is to use the multivariate c-characteristic function through our relation in Theorem 4.1. It is possible that we may need to find the distribution having a known multivariate c-characteristic function. One approach is, first, to find the traditional characteristic function through the indirect approach in Theorem 5.2. We then use the inversion formula of the traditional characteristic function to find the distribution. An interesting future research topic would be to find the distribution directly from the multivariate c-characteristic function. The new result, on the distribution of a random functional of a Ferguson–Dirichlet process, on the region bounded by an ellipse, then generalizes that given by Jiang (1991).

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Appendix

To prove the lemmas in the appendix, we need the following six equations. The first equation is from Gröbner and Hofreiter (1973, p. 105),

$$\int_0^{2\pi} (a\cos\alpha + b\sin\alpha)^n \,\mathrm{d}\alpha = \begin{cases} \dfrac{(1/2, n/2)2(a^2+b^2)^{n/2}\cdot\pi}{(n/2)!}, & n \text{ is even,} \\ 0, & n \text{ is odd,} \end{cases} \tag{A.1}$$

where a and b are real numbers. The second equation is from Gradshteyn and Ryzhik (1980, p. 45),

$$\ln\left(\frac{1+\sqrt{1+x^2}}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot(2n-1)!}{2^{2n}\cdot(n!)^2} x^{2n}, \quad \text{for all } x^2 \leqslant 1, \tag{A.2}$$

where x is a real number. The third equation about the definition of ${}_2F_1$ hypergeometric series is from Carlson (1977, p. 14),

$${}_2F_1(a,b;c;z) = \sum_{m=0}^{\infty} \frac{(a,m)(b,m)}{(c,m)} \frac{z^m}{m!}, \quad |z| < 1, \tag{A.3}$$

where a, b, c, and z are complex numbers and $c \neq 0, -1, -2, \ldots$. The fourth equation about the special case of ${}_2F_1$ series is from Erdélyi et al. (1953, p. 101),

$$ {}_2F_1\left(b-\frac{1}{2}, b; 2b; z\right) = \left(\frac{1}{2}+\frac{\sqrt{1-z}}{2}\right)^{1-2b}, \quad |z|<1. \tag{A.4}$$

The next two equations about Appell's notations can be shown easily.

$$\left(\frac{1}{2}, n\right) = \frac{(2n)!}{2^{2n}\cdot n!}, \tag{A.5}$$

$$(c, 2n) = 2^{2n}\cdot\left(\frac{c}{2}, n\right)\left(\frac{c}{2}+\frac{1}{2}, n\right). \tag{A.6}$$

The lemmas about special definite integrals follow.

Lemma A.1.

$$\int_0^{2\pi} \ln(1-\mathrm{i}t_1 a\cos\alpha - \mathrm{i}t_2 b\sin\alpha)\,\mathrm{d}\alpha = 2\pi\ln\frac{1+\sqrt{1+a^2t_1^2+b^2t_2^2}}{2}, \tag{A.7}$$

where $|\boldsymbol{t}| < (a^2+b^2)^{-1/2}$.

Proof. Since $\ln(1-y) = -\sum_{n=1}^{\infty} y^n/n$, for all $|y|<1$, the integrand of the left-hand side of Eq. (A.7) can be expressed as

$$-\sum_{n=1}^{\infty}\frac{\mathrm{i}^n}{n}(t_1 a\cos\alpha + t_2 b\sin\alpha)^n.$$

The left-hand side of Eq. (A.7), denoted by L, is integrable since it is a definite integral and its integrand is bounded. Hence, integration and summation are exchangeable and we have

$$L = -\sum_{n=1}^{\infty}\frac{\mathrm{i}^n}{n}\int_0^{2\pi}(t_1 a\cos\alpha + t_2 b\sin\alpha)^n\,\mathrm{d}\alpha.$$

By Eqs. (A.1) and (A.5), L can be expressed as

$$L = -2\pi\sum_{n=1}^{\infty}\frac{\mathrm{i}^{2n}(1/2, n)}{2n\cdot n!}(t_1^2a^2+t_2^2b^2)^n$$

$$= 2\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n-1)!}{2^{2n}\cdot(n!)^2}(t_1^2a^2+t_2^2b^2)^n.$$

Finally, by Eq. (A.2),

$$L = 2\pi\ln\frac{1+\sqrt{1+a^2t_1^2+b^2t_2^2}}{2}.$$

This completes the proof. □

Lemma A.2.

$$\int_0^a r\left(1-\frac{r^2}{a^2}\right)^{c-1}\int_0^{2\pi}[1-\mathrm{i}r\,(s_1\cos\theta+s_2\sin\theta)]^{-c}\,\mathrm{d}\theta\,\mathrm{d}r$$

$$=\frac{\pi a^2}{c}\left(\frac{2}{1+\sqrt{1+a^2s_1^2+a^2s_2^2}}\right)^c,$$

where $a>0$ *and* $\sqrt{s_1^2+s_2^2}<1/a$.

Proof. Let $0\leqslant r\leqslant a$ and $\sqrt{s_1^2+s_2^2}<1/a$, then

$$\int_0^{2\pi}[1-\mathrm{i}r(s_1\cos\theta+s_2\sin\theta)]^{-c}\,\mathrm{d}\theta$$

$$=\int_0^{2\pi}\sum_{n=0}^{\infty}\frac{(c,n)}{n!}\,\mathrm{i}^n r^n(s_1\cos\theta+s_2\sin\theta)^n\,\mathrm{d}\theta$$

$$=\sum_{n=0}^{\infty}\frac{(c,n)}{n!}\,\mathrm{i}^n r^n\int_0^{2\pi}(s_1\cos\theta+s_2\sin\theta)^n\,\mathrm{d}\theta$$

$$=\sum_{n=0}^{\infty}\frac{(c,2n)}{(2n)!}\,\mathrm{i}^{2n}r^{2n}\frac{(1/2,n)\cdot 2(s_1^2+s_2^2)^n\cdot\pi}{n!}.$$

Note that the first identity follows by the fact that $|r(s_1\cos\theta+s_2\sin\theta)|<1$ and Eq. (3). The second identity is true since the integrand is bounded and it is a definite integral. The third identity follows from Eq. (A.1). Define

$$L=\int_0^a r\left(1-\frac{r^2}{a^2}\right)^{c-1}\int_0^{2\pi}[1-\mathrm{i}r(s_1\cos\theta+s_2\sin\theta)]^{-c}\,\mathrm{d}\theta\,\mathrm{d}r,$$

then L is also integrable. Hence,

$$L=\int_0^a r\left(1-\frac{r^2}{a^2}\right)^{c-1}\sum_{n=0}^{\infty}\frac{(c,2n)}{(2n)!}(-1)^n r^{2n}\frac{(1/2,n)\cdot 2(s_1^2+s_2^2)^n\pi}{n!}\,\mathrm{d}r$$

$$=2\pi\sum_{n=0}^{\infty}\frac{(-1)^n(c,2n)(1/2,n)(s_1^2+s_2^2)^n}{(2n)!\cdot n!}\int_0^a r^{2n+1}\left(1-\frac{r^2}{a^2}\right)^{c-1}\mathrm{d}r.$$

Now,

$$\int_0^a r^{2n+1}\left(1-\frac{r^2}{a^2}\right)^{c-1}\mathrm{d}r=\frac{a^{2n+2}}{2}\int_0^a\left(\frac{r^2}{a^2}\right)^n\left(1-\frac{r^2}{a^2}\right)^{c-1}\mathrm{d}\left(\frac{r^2}{a^2}\right)$$

$$=\frac{a^{2n+2}}{2}B(n+1,c)$$

$$=\frac{a^{2n+2}}{2}\cdot\frac{n!}{(c,n+1)}.$$

By Eqs. (A.5) and (A.6), we have,

$$L=\frac{\pi a^2}{c}\sum_{n=0}^{\infty}\frac{(c/2,n)(c/2+1/2,n)}{(c+1,n)}\frac{(-a^2s_1^2-a^2s_2^2)^n}{n!},\quad \text{where } \sqrt{s_1^2+s_2^2}<\frac{1}{a}.$$

The infinite series in the right-hand side of the above equation is a ${}_2F_1$ hypergeometric series, see Eq. (A.3). That is,

$$L=\frac{\pi a^2}{c}{}_2F_1\left(\frac{c}{2},\frac{c}{2}+\frac{1}{2};c+1;-a^2s_1^2-a^2s_2^2\right),\quad \text{where } \sqrt{s_1^2+s_2^2}<\frac{1}{a}.$$

By Eq. (A.4), the above ${}_2F_1$ series has a closed form, i.e., $2^c/(1+\sqrt{1+a^2s_1^2+a^2s_2^2})^c$. Hence,

$$L=\frac{\pi a^2}{c}\left(\frac{2}{1+\sqrt{1+a^2s_1^2+a^2s_2^2}}\right)^c,\quad \text{where } a>0 \text{ and } \sqrt{s_1^2+s_2^2}<\frac{1}{a}.\qquad \square$$

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