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ANALYSIS ON THE DURATION OF DEEP EARTHQUAKES

Meng-Rong Li and Chuen-Hsin Chang

Abstract. In this paper, we obtain the duration of deep earthquakes coming from the concept of pendulum motion. We consider the well-posedness and periodicity of the solutions for the seismic equation and get representation formula of duration for the equation. We revise the validity of our work by using numerical data gained from Center Weather Bureau in Taipei, Taiwan.

1. Introduction

1.1. Background about Seismology

In this paper, we find the duration of deep earthquakes and would analyze what is the influence factor for the duration. The duration is defined to be the persisting time for a earthquake from the time it occurs to the time it stops.

The seismology of [4] and [6] shows that the types of earthquakes can be classified as shallow, intermediate or deep by the depth of the hypocenter (the depth of shallow earthquakes is not greater than 70 km, and the depth of deep earthquakes is greater than 300 km.), but from NCREE (National Center for Research on Earthquake Engineering) in Taipei, Taiwan, the types of earthquakes are depended on the thickness of crust. From geology the thickness of the Earth's crust of Taiwan is probably around 25 km to 30 km; in actual works on searching data of earthquakes, we use the definition of earthquakes from NCREE.

1.2. Derivation of seismic equation

Considering the deep earthquakes first, because the deep earthquakes are transmitted under the theory of plate tectonics, it is a motion of plate with the periodic

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characteristics. Under this consideration and comparing the motion of deep earthquakes with the motion of pendulum, it can be found that they nearly have the same property, therefore the motion of deep earthquakes can be regarded as the motion of pendulum in the abstract meaning.

When the deep earthquake occurs, the building destroyed by the force sent from the hypocenter, such a phenomenon could be approximately regarded as a torsion offered from the fulcrum by a pendulum, and then causes a motion happened in the pendulum (See Fig. 1). We treat the bob of the pendulum as a building and also the fulcrum as the hypocenter; in the process of moving from the pendulum, there is a tension τ drawn to the fulcrum.

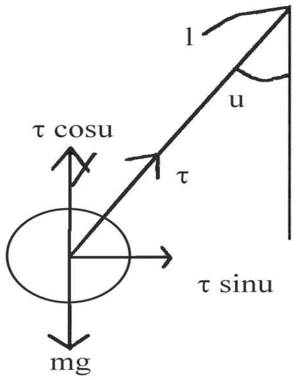

Fig. 1. Motion model of pendulum.

Assume that the length of the pendulum is l, the mass of the bob is m, the mass of the rod is negligible, and let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ be the space variables. Moreover, at time t, we suppose that $u(x,t)$ is the angular displacement of the pendulum, measured from the vertical direction, and g is the acceleration of gravity.

In shallow earthquakes, the influence factors for the motion of earthquakes are not only the motion of plate but also the space variable (that is the interaction between building and soil); those are need to be taken into account. For convenience, let $k(x,u)$ be the soil parameter, where u is the angular displacement of the pendulum we have considered. Then $k(x,u)\nabla_x(l\sin u)$ means the vector fields of seismic wave flowing on the Earth's surface.

Suppose that the flux of the shearing force of earthquakes proportions to the divergence of $k(x,u)\nabla_x(l\sin u)$, that is, the flux equals to $\lambda[\nabla_x\cdot(k(x,u)\nabla_x(l\sin u))]$, where λ is a proportion constant. In the pendulum model, this means the damping force for the motion of pendulum.

Considering the motion upon the bob, dividing the total forces into the horizontal and vertical direction, we can get

$$\left\{\begin{array}{c}\tau\sin u+\lambda\left(\nabla_x\,\cdot\,\left(k\left(x,u\right)\nabla_x\left(l\sin u\right)\right)\right)=m(l\sin u)_{tt},\\ \tau\cos u+mg=f(x,t,u,u_t,u_{x_1},u_{x_2},u_{x_3}).\end{array}\right. \tag{1.1}$$

where $f(x,t,u,u_t,u_{x_1},u_{x_2},u_{x_3})$ is the total force in the vertical direction.

Because of the complicated model on (1.1), the discussion on shallow earthquakes is out of the range in this paper. Starting with the simple structure of deep earthquakes first and considering deep earthquakes, we treat the three dimension problem of a pendulum as a problem on a plane; supposing that no damping force acts on the pendulum, thus the angular displacement of the pendulum becomes $u=u(t)$, that is, we only consider the time variable t, and then the model of the motion of the pendulum becomes

$$\left\{\begin{array}{l}\tau\sin u=m(l\sin u)'',\\ \tau\cos u+mg\le 0,\end{array}\right. \tag{1.2}$$

where the prime means the derivative with respect to time t ($'=\frac{d}{dt}$).

In the second line of (1.2), the inequality $\tau\cos u+mg\le 0$ which means that in deep earthquakes, the position where energy breaks out is far from the Earth's surface, and the seismic wave distributed out has already consumed most energy while getting to the Earth's surface, so the influence on the building is very small; thus the force of the weight of building is dominant among the total force in the vertical direction.

Because the motion of deep earthquakes is based on the theory of plate tectonics, the motion of (1.2) can only be viewed as a horizontal motion, and then (1.2) becomes

$$\left\{\begin{array}{l}\tau\sin u=m(l\sin u)'',\\ \tau\cos u+mg=0,\end{array}\right. \tag{1.3}$$

in the vertical direction. The other reason why inequality can be seen as an equality is that the motion will at least be balanced under the condition of (1.3) for deep earthquakes.

Eliminating τ in (1.3) and combine the two equalities, it follows that

$$u''-(u')^2\tan u+\frac{g}{l}\sin u\sec^2 u=0. \tag{1.4}$$

Generally speaking, provided time $t=0$, the angle $u(0)$ is a state that will suffer the shearing force from earthquakes at the beginning, and then we obtain $u(0)=0$, $u'(0)=\alpha$, namely the initial angular displacement is 0 and the initial angular velocity is α; combining those with (1.4) we can get an initial value problem:

$$\begin{cases} u'' - (u')^2 \tan u + \frac{g}{l} \sin u \sec^2 u = 0, \\ u(0) = 0, \quad u'(0) = \alpha > 0. \end{cases} \tag{1.5}$$

For convenience, we will call (1.5) the *seismic equation* in sequence of this paper.

Back to (1.4), defining $v(t) = u'(t)$, we have

$$\begin{aligned} v\tfrac{dv}{du} - v^2 \tan u + \tfrac{g}{l} \sin u \sec^2 u &= 0, \\ \left(v \cos^2 u\right) dv + \left(-v^2 \sin u \cos u + \tfrac{g}{l} \sin u\right) du &= 0. \end{aligned}$$

We can use the method of integrating factors to convert the above equation into an exact equation and then solve the relation between u and v to get:

$$v^2 \cos^2 u - \frac{2g}{l} \cos u = \alpha^2 - \frac{2g}{l}. \tag{1.6}$$

Since $u'(t) = v(t)$, we get a transformed first order nonlinear differential equation:

$$\begin{cases} (\dfrac{du}{dt})^2 = \dfrac{\alpha^2 - \frac{2g}{l} + \frac{2g}{l} \cos u}{\cos^2 u}, \\ u(0) = 0. \end{cases} \tag{1.7}$$

We treat (1.7) as an equivalent version of seismic equation (1.5). We would mention some properties by studying the first order version of seismic equation.

In section 2 we find the well-posedness of the seismic equation and study which from the phase portrait. We would prove the periodicity of (1.5). In the model of (1.3), we don't consider the damping force, so the periodicity for (1.5) is in the mathematical sense, that is, the solution of (1.5) exists globally, but in fact, the damping force should be taken into account, such as the model (1.1), therefore the periodicity of solution of (1.1) will actually starts from some time, and then stops; thus there is actually a persisting time.

In section 3 we derive the representation formula expressing the duration of the solution of this equation and in section 4 we analyze how the duration been influenced by the depth and peak ground acceleration of deep earthquakes. Finally we quote the data from Center Weather Bureau in Taiwan to check our work.

2. Well-posedness, Asymptotic Behavior and Periodicity of Seismic Equation

2.1. Well-Posedness

We quote the Picard-Lindelöf Theorem from Bellman [1, Theorem 3.1] to obtain the existence and uniqueness of solutions of (1.7).

Lemma 1. *Suppose* u *is the solution for equation* (1.7) *and* β *is the maximal angular displacement* u *of the pendulum, then we have*

$$\cos\beta = \frac{\frac{2g}{l} - \alpha^2}{\frac{2g}{l}}. \tag{2.1}$$

Proof. Because β is the maximal angular displacement of the pendulum, the angular velocity is 0 as the pendulum reaches the angle β; thus $u'(t)|_{u=\beta} = f(\beta) = 0$ and then $\alpha^2 - \frac{2g}{l} + \frac{2g}{l}\cos\beta = 0$, therefore we have (2.1). ■

Define the right hand side of (1.7) to be $f(u)$, then for all $\varepsilon > 0$, if we choose u in the closed subinterval $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}] \subset (-\varepsilon, \varepsilon) \subset E$ (E is the domain of $f'(u)$), we can find $\max\limits_{|u|\le\frac{\varepsilon}{2}} |f(u)|$ and $\max\limits_{|u|\le\frac{\varepsilon}{2}} |f'(u)|$ for t in the finite interval; thus we can get the local existence and uniqueness and the stability of (1.7) by the method in [5].

2.2 Asymptotic behavior

We deal with the asymptotic behavior of solutions of (1.5) by using the method of phase portrait of linearized equation of (1.5).

Back to (1.5), define $u'(t) = v(t) = F(u, v)$, and then

$$v' = v^2 \tan u - \frac{g}{l}\sin u \sec^2 u := G(u, v).$$

Theorem 1. *Suppose that* (u, v) *is the solution to the autonomous system:*

$$\begin{cases} u' = F(u, v) = v, \\ v' = G(u, v) = v^2 \tan u - \frac{g}{l}\sin u \sec^2 u, \end{cases} \tag{2.2}$$

then $(0, 0)$ *is a center of (2.2), and when the initial condition* $(u(0), v(0))$ *close to* $(0, 0)$ *(i.e.* $\alpha \to 0$*), (2.2) has a approximate solution*

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = c_1 \begin{pmatrix} \cos\sqrt{\frac{g}{l}}t \\ \sqrt{\frac{g}{l}}\sin\sqrt{\frac{g}{l}}t \end{pmatrix} + c_2 \begin{pmatrix} \sin\sqrt{\frac{g}{l}}t \\ \sqrt{\frac{g}{l}}\cos\sqrt{\frac{g}{l}}t \end{pmatrix}. \tag{2.3}$$

Because $(0, 0)$ is a center in the phase plane, by (2.3) we obtain that the period of solution u is $2\pi\sqrt{\frac{l}{g}}$, which identifies with the period of solution to the problem of pendulum motion with small amplitude we know classically and this is only the result obtained as $\alpha \to 0$.

2.3. Periodicity

Although in this paper the periodic solutions are a little bit different from the so-called one classically, after having such cognition, we want to prove the existence of periodic solutions for (1.5).

In the $u - v$ phase plane of (2.2), in order to justify that there exists a closed curve in the phase plane of solutions of (1.5), we need to prove that u and v are bounded first. From the boundedness of u and v it follows the global existence of solutions, and if we can prove the phase orbits are symmetric with respect to the v-axis, then the existence of periodic solutions is obtained.

Lemma 2. *The phase orbits of the autonomous system of (2.2) are symmetric with respect to the v-axis.*

Theorem 2. *Suppose u is the solution to the initial value problem (1.5), then we have:*

$$
\begin{array}{lll}
(i) & |u| \le \beta,\ |u'| \le \alpha & \text{for } 0 \le \alpha^2 \le \frac{g}{l}. \\
(ii) & |u| \le \beta,\ |u'| \le \frac{\frac{g}{l}}{\sqrt{\frac{2g}{l} - \alpha^2}} & \text{for } \frac{g}{l} < \alpha^2 < \frac{2g}{l}. \\
(iii) & |u| \text{ unbounded},\ |\alpha| \le |u'| & \text{for } \alpha^2 = \frac{2g}{l}. \\
(iv) & |u| \le \beta,\ |u'| \text{ unbounded} & \text{for } \frac{2g}{l} < \alpha^2 \le \frac{4g}{l}. \\
(v) & |u| \text{ unbounded},\ |u'| \text{ unbounded} & \text{for } \frac{4g}{l} < \alpha^2
\end{array}
$$

and then there exists a periodic solution of (1.5) for $0 \le \alpha^2 \le \frac{g}{l}$.

Proof. Transform (1.6) into the following:

$$(u')^2 = (\alpha^2 - \frac{2g}{l}) \sec^2 u + \frac{2g}{l} \sec u \tag{2.4}$$

regard $(u')^2$ as a quadratic polynomial of variable $\sec u$, then we can get the assertions of this Theorem 2. ∎

Therefore only under the case $0 \le \alpha^2 \le \frac{g}{l}$, we can get the existence of periodic solutions for equation (1.5).

3. Derivation and Estimate of the Duration

3.1. Representation formula

Theorem 3. *Suppose u is the solution of equation (1.7) for $\alpha^2 \leq \frac{g}{l}$, then there exists $T(l,\alpha) := T$ such that $u(t+T) = u(t)$ for all $t > 0$, and we also obtain*

$$T(l,\beta) = T(l,\beta(\alpha)) = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{1 - \frac{\alpha^2}{\frac{2g}{l}}\sin^2\varphi}{\sqrt{1 - \frac{\alpha^2}{\frac{4g}{l}}\sin^2\varphi}}d\varphi. \tag{3,1}$$

Proof. Similar to the derivation of the elliptic integral of the first kind, from (1.6) we have

$$\frac{du}{dt} = \sqrt{\frac{2g}{l}}\frac{\sqrt{\cos u - \cos\beta}}{\cos u}.$$

According to the phenomenon of deep earthquakes, we take the positive sign and thus

$$t = \frac{1}{2}\sqrt{\frac{l}{g}}\int_0^{u(t)} \frac{\cos\phi}{\sqrt{\sin^2\frac{\beta}{2} - \sin^2\frac{\phi}{2}}}d\phi.$$

When the pendulum moves from the lowest $u = 0$ up and then reaches the maximal displacement $u = \beta$, φ goes from 0 to $\frac{\pi}{2}$; thus, from (2.1) we deduce that

$$T(l,\beta) = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{1 - 2\sin^2\frac{\beta}{2}\sin^2\varphi}{\sqrt{1 - \sin^2\frac{\beta}{2}\sin^2\varphi}}d\varphi. \tag{3.2}$$

Since $\sin^2\frac{\beta}{2} = \frac{1-\cos\beta}{2} = \frac{1}{2}(1 - \frac{\frac{2g}{l}-\alpha^2}{\frac{2g}{l}}) = \frac{\alpha^2}{\frac{4g}{l}}$, substitute this into (3.2); thus, we obtain (3.1), and it is the the duration of u. ■

When $\alpha \to 0$,

$$T \to 4\sqrt{\frac{l}{g}}(\frac{\pi}{2}) = 2\pi\sqrt{\frac{l}{g}},$$

which coincides with the period of pendulum with small angular displa-cement.

3.2. Modified formula

From (3.1) we can view the duration $T(l,\beta(\alpha))$ as a function of two variables α and l. In fact, there are no data of α recorded from Center Weather Bureau (CWB) in Taipei, Taiwan, so it is necessary to transform (3.1) to match the data from CWB.

Definition 1. Let the variable A denote the PGA (peak ground acceleration) with unit gal ($\mathrm{cm}/\mathrm{sec}^2$) in each earthquake. That is, A means the maximal horizontal ground acceleration.

A can be related with (3.1) in the following way:

Lemma 3. *From the definition of A, we have*

$$A = \frac{|\alpha| \sqrt{\frac{4g}{l} - \alpha^2}}{\frac{2g}{l} - \alpha^2} g. \tag{3.3}$$

Proof. At each earthquake, the horizontal displacement of the building is $l \sin u$, and using (1.4) we get the acceleration at time t:

$$(l \sin u(t))'' = -l \left(u'(t)\right)^2 \sin u(t) + l\ u''(t) \cos u(t) = -g \tan u(t);$$

thus the PGA (namely A) is

$$A := \max_{|u| \leq \beta < \frac{\pi}{2}} (l \sin u)'' = g \tan \beta. \tag{3.4}$$

By (2.1) we have

$$g \tan \beta = \frac{|\alpha| \sqrt{\frac{4g}{l} - \alpha^2}}{\frac{2g}{l} - \alpha^2} g$$

and so we have (3.3). ∎

From (3.3) we have

$$\alpha^2 = \frac{2g}{l}(1 \pm \frac{1}{\sqrt{(\frac{g}{A})^2 + 1}}). \tag{3.5}$$

Under the restriction $\alpha^2 \leq \frac{g}{l}$, it follows that:

$$\frac{2g}{l}(1 \pm \frac{1}{\sqrt{(\frac{g}{A})^2 + 1}}) \leq \frac{g}{l},$$

and so

$$1 \pm \frac{1}{\sqrt{(\frac{g}{A})^2 + 1}} \leq \frac{1}{2}.$$

Obviously the positive sign in left hand side above can't satisfy the inequality; thus

$$1 - \frac{1}{\sqrt{(\frac{g}{A})^2 + 1}} \le \frac{1}{2}, \quad \text{and then} \quad \frac{A}{g} \le \sqrt{3}.$$

If m is the mass of the building, then the above ratio, $\frac{A}{g} = \frac{mA}{mg}$, results in the following definition:

Definition 2. For the PGA A of a motion, $r = \frac{A}{g}$ is called the rate of receiving force, it means that the building will be forced horizontally with amount of $\frac{A}{g}$ per unit weight through the shearing force.

Although by mathematical derivation it turns out to be the case $r \le \sqrt{3}$, actually, from the table of [2, pp. 1-19] we know that the ratio r is very small (less than 0.1 at least) for deep earthquakes (i.e. $r \ll 1$), so the necessary condition $\alpha^2 \le \frac{g}{l}$ for the existence of periodicity needs a stricter restriction.

Corollary 1. *Substituting (3.5) into (3.1), then*

$$T(l,r) = 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{1 - \left(1 - \frac{1}{\sqrt{r^2+1}}\right) \sin^2 \varphi}{\sqrt{1 - \frac{1}{2}\left(1 - \frac{1}{\sqrt{r^2+1}}\right) \sin^2 \varphi}} d\varphi. \tag{3.6}$$

4. Analysis on the Duration

4.1. Example

From geology, the thickness of the Earth's crust of Taiwan is probably around 25 km to 30 km, so the depth of the numerical data which we adopt also need to be selected in such range. Taking notice that the unit for PGA is gal (cm/sec^2), we will translate its units to be the MKS systems of units in the following.

According to the record of earthquakes given in [3], we want to find whether the actual numerical data are coincide with our formula estimated above. For example, if we adopt the earthquake record $No.10740$ in [3] in vol $51(2)$, its epicenter located at $22.71°N$ $121.37°E$, with depth 27.1 km. At that earthquake, the PGA at the station CHN1 in Tainan, Taiwan with 101 km distance from the epicenter is 6.6 gal, so the distance from this station to the hypocenter can be roughly computed by the Pythagorean Theorem to obtain the value $l = \sqrt{101^2 + (27.1)^2} \approx 104.57$ km. By Definition 10, $r \approx 0.0067347$; from (3.6) we can obtain $T \approx 649.0328688543557 sec$, but by the record of seismographs, no numerical data of records of deep earthquakes will have such a large value.

There is a great error between the theoretical value and the recorded value. The reason may be that:

(1) In the model of (1.3), we hadn't considered damping force.

(2) Because the deep earthquakes are very weak until it passing to the ground, and the seismographs can't technically detect all the actual duration of the earthquakes, so formula (3.6) doesn't have perfect degree with theoretical value and the recorded value.

4.2. Functional properties

Although the values of the two data corresponded above are not the same, the importance is that we can find properties with the numerical data of solutions of (3.6) say that the duration of u is a function of l and r, therefore (3.6) tells us that T is a function of r and l. We want to know whether $T(l,r)$ will be changed with r and l, and if this can be done and the results can match to the numerical data from CWB, then our estimation formula make sense in the characterizing properties of the solution corresponding to its variables.

There are two ways to analyze, considering at first that l is fixed and r changes (i.e. A changes), and then we observe how the duration $T(l,r)$ changes with r. Next, changing the role of r and l and then continuing carrying on the same procedure, we have:

Theorem 4. *Suppose that $T(l,r)$ is the duration of the solution of u to the equation (1.5), then*

(1) *For fixed l and small r, we have*

$$\frac{T(l,r_1)}{T(l,r_2)} = \left(\frac{r_1^2+1}{r_2^2+1}\right)^{-1/4}; \tag{4.1}$$

(2) *For fixed $r>0$, we have*

$$\frac{T(l_1,r)}{T(l_2,r)} = \sqrt{\frac{l_1}{l_2}}; \tag{4.2}$$

(3) *Further, for small $r>0$, we have also*

$$T(l,r) = -\frac{3}{8}\pi + \left(2\sqrt{\frac{l}{g}} + \frac{3}{8}\right)\pi \exp\left(-2\sqrt{\frac{l}{g}}\left(1 - \frac{1}{\sqrt{r^2+1}}\right)\right) \tag{4.3}$$

in the neighborhood of $r=0$.

Proof. To analyze how the duration T changes with r, the method is to keep l fixed and to regard T as a function of one variable r; for convenience, we will find an estimate and an approximated function $T(\cdot, r)$.

(1) For fixed l, (3.6) can be transformed into the following:

$$T(l,r) = 4\sqrt{\frac{l}{g}}\frac{1}{\sqrt[4]{r^2+1}}\int_0^{\frac{\pi}{2}} \frac{\sqrt{r^2+1}\,(1+\cos 2\varphi)+2\sin^2\varphi}{\sqrt{\sqrt{r^2+1}(3+\cos 2\varphi)+2\sin^2\varphi}}d\varphi. \tag{4.4}$$

In actual data, the ratio $r = \frac{A}{g}$ is very small (for deep earthquakes, if the PGA A tens to the value 80 gal, the value $r = \frac{0.8}{9.8} \approx 0.081633$, which is still small compared with 1, but the situation $A = 80$ gal seldom occurs in deep earthquakes by the seismic intensity scales given in CWB, Taiwan), so we can suppose r is small,

$$T(l,r) \approx 2\pi\sqrt{\frac{l}{g}}\frac{1}{\sqrt[4]{r^2+1}} \tag{4.5}$$

and (4.1) follows.

(2) For fixed l, we define

$$T(r) = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} g(r,\varphi)d\varphi,$$

where

$$g(r,\varphi) := \frac{1-\left(1-\frac{1}{\sqrt{r^2+1}}\right)\sin^2\varphi}{\sqrt{1-\frac{1}{2}\left(1-\frac{1}{\sqrt{r^2+1}}\right)\sin^2\varphi}}.$$

It can be easily observed that $\frac{\partial g(r,\varphi)}{\partial r}$ is continuous, so we have

$$\frac{\partial T(l,r)}{\partial r} = 4\sqrt{\tfrac{l}{g}}\int_0^{\frac{\pi}{2}} \tfrac{\partial g(r,\varphi)}{\partial r}d\varphi \tag{4.6}$$

$$= -2\sqrt{\frac{l}{g}}r\left(r^2+1\right)^{-\frac{3}{2}}\int_0^{\frac{\pi}{2}} h(r,\varphi)d\varphi, \tag{4.7}$$

where

$$h(r,\varphi) := \frac{1}{2}\frac{3-\left(1-\frac{1}{\sqrt{r^2+1}}\right)\sin^2\varphi}{\sqrt{\left(1-\frac{1}{2}\left(1-\frac{1}{\sqrt{r^2+1}}\right)\sin^2\varphi\right)^3}}\sin^2\varphi.$$

Making a transform of $h(r, \varphi)$, (4.7) becomes

$$\frac{\partial T(l,r)}{\partial r} = -2\sqrt{\frac{l}{g}}r\left(r^2+1\right)^{-\frac{3}{2}}\left(T + \int_0^{\frac{\pi}{2}} J(r,\varphi)d\varphi\right) \tag{4.8}$$

where

$$J(r,\varphi) = \frac{1}{\sqrt{\left(1-\frac{1}{2}q_1\sin^2\varphi\right)^3}}\left(\left(\frac{1}{2}+\frac{1}{2}q_1+2q_2\right)\sin^2\varphi - q_1q_2\sin^4\varphi\right),$$

and

$$q_1 = 1-\frac{1}{\sqrt{r^2+1}}, \quad q_2 = 1-\frac{1}{2\sqrt{r^2+1}}.$$

Like the same reason before, suppose r is small, then $q_1 \to 0$ and $q_2 \to \frac{1}{2}$; thus (4.8) becomes

$$\frac{\partial T(l,r)}{\partial r} = -2\sqrt{\frac{l}{g}}r(r^2+1)^{-\frac{3}{2}}(T+\frac{3}{8}\pi).$$

Hence, given that r is small, there is a change rate of $T+\frac{3}{8}\pi$ which is a function of r in the form:

$$\frac{1}{(T+\frac{3}{8}\pi)}\frac{d(T+\frac{3}{8}\pi)}{dr} = -2\sqrt{\frac{l}{g}}r(r^2+1)^{-3/2}. \tag{4.9}$$

From (3.6) we know $T(l,0) = 2\pi\sqrt{\frac{l}{g}}$; thus when r is small, solution of (4.9) is

$$T(l,r) = -\frac{3}{8}\pi + \left(2\sqrt{\frac{l}{g}}+\frac{3}{8}\right)\pi\exp\left(-2\sqrt{\frac{l}{g}}\left(1-\frac{1}{\sqrt{r^2+1}}\right)\right)$$

and (4.3) follows.

(3) For fixed r, by (3.6) and (4.1), (4.2) holds. ■

From (4.3), it can be observed that the duration $T(l,r)$ is a decreasing function in r. This result is the same with our past knowledge about deep earthquakes.

Define the right hand side of (4.3) to be $\tilde{T}(l,r)$, we will use the numerical data from CWB to check this approximation of $T(l,r)$ and finding the error $\Delta T := \left|T-\tilde{T}\right|$ in the next subsection.

4.3. Numerical check of data

In order to check Theorem 4, we will quote the numerical data of the earthquakes occurred from 2002 to 2004 nearby around Taiwan from [3]. In the above-mentioned data, for a earthquake, if the two stations have the same l, we can find that the ratios $\frac{T_1}{T_2}$ and $\frac{(r_1^2+1)^{-1/4}}{(r_2^2+1)^{-1/4}}$ are nearly all the same and the error is less than 10^{-6}; thus we can get the result of (4.1) and the duration T is really a function of r and nearly changes in the form (4.1) or (4.5) as r tends to zero. Moreover, from these tables it can be observed that for two stations, the smaller T results from the same l and the bigger r, and the smaller ΔT results from the smaller r, these results from the data coincide with (4.9) (or (4.3)). By using Mathematica we can obtain the graphs of $T(l,r)$ and $\tilde{T}(l,r)$ respectively. The following are the graphs of $T(l,r), \tilde{T}(l,r)$. It can be found that the following graphs and the above numerical data have the same results of Theorem 4, and the graphs are coincide with the above data, providing that the stations have the same r, the ratios $\frac{T_1}{T_2}$ and $\frac{\sqrt{l_1}}{\sqrt{l_2}}$ are nearly all the same and the error is less than 10^{-16}.

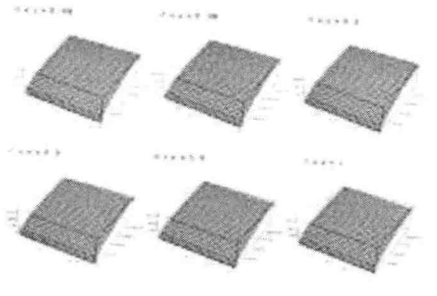

Fig. 2. Graphs of $T(l,r)$ for $0 \le l \le 120km$.

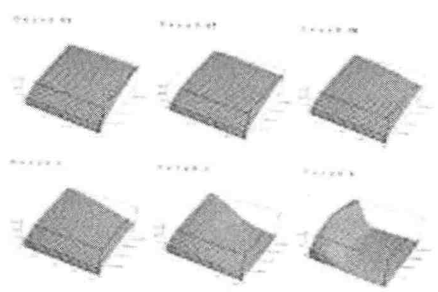

Fig. 3. Graphs of $\tilde{T}(l,r)$ for $0 \le l \le 120\ km$.

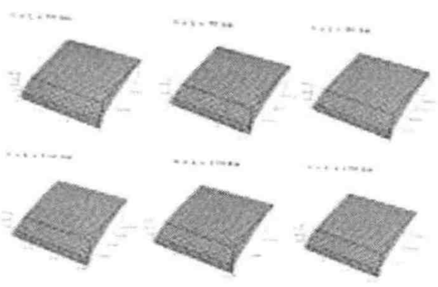

Fig. 4. Graphs of $T(l,r)$ for 0 for ≤ 0.03.

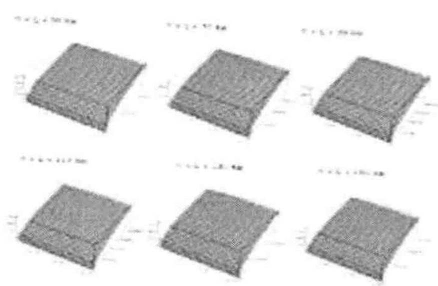

Fig. 5. Graphs of $\tilde{T}(l,r)$ $0 \le r \le r \le 0.03$.

Comparing with the error from fixed l and changing r it can be seen that for the function T in (3.6) which depends on r and l, the influence from l is dominant, and consequently this result and our intuition on deep earthquakes are the same; the reason is that the depth of deep earthquakes consumes the force passing to the ground and then the horizontal shearing force is too small (i.e. r is too small) so that its influence on T is small naturally. The result (4.2) is obtained and the duration T is really a function of l, changes in the form (4.2). Using Mathematica we have the graphs of $T(l, r)$, $\tilde{T}(l, r)$. It can be found that the following graphs are also coincide with the above numerical data.

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REFERENCES

1. R. Bellman, *Stability Theory of Differential Equations*, McGraw-Hill, 1953.
2. C. S. Yeh, *Earthquake Resistant Design Manuals*, Architecture and Building Research Institute, Ministry of the interior, Taiwan, R.O.C., 1998, (in Chinese).
3. Center Weather Bureau, *Seismological Bulletin*, **49(1)** (2002), **51(2)** (2004), Taipei, Taiwan, R.O.C.
4. T. Lay and T. C. Wallace, *Modern Global Seismology*, Academic Press, 1995.
5. L, Perko, *Differential Equations and Dynamical Systems*, Springer, 1991.
6. P. M. Shearer, *Introduction to Seismology*, Cambridge University Press, 1999.
7. Meng-Rong Li, On the Emden-Folwer equation $u'' - |u|^{p-1}u = 0$, *Nonlinear Analysis*, **64** (2006), 1025-1056.
8. Meng-Rong Li, Estimates for the Life-Span of the Solutions for Semilinear Wave Equations 2008, *CPAA*, **7(2)** (2008), 417-432.
9. Meng-Rong Li and Yue-Loong Chang, On a particular Emden-Fowler equation $u'' - u^3 = 0$ with non-positive energy–Mathematical model of enterprise competitiveness and performance, *Applied Mathematics Letters*, **20(9)** (2007), 1011-1014.
10. Renjun Duan, Meng-Rong Li and Tong Yang, Propagation of Singularities in the Solutions to the Boltzmann Equation near Equilibrium, *Mathematical Models and Methods in Applied Sciences* (*M3AS*), **18(17)** (2008), 1093-1114.

Meng-Rong Li and Chuen-Hsin Chang
Department of Mathematical Sciences,
National Chengchi University,
Taipei 116, Taiwan, R.O.C.
E-mail: liwei@math.nccu.edu.tw
juexin2001@yahoo.com.tw