TORSION THEORY AND QUOTIENT RINGS(*)

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Introduction: Any ring in this paper means a ring with identity and any module means unitary right R-module.

In this paper we introduce the construction of the ring of right quotients Q(R) of the ring R with respect to a given torsion theory for the category ModR of modules.

It is well known that any module has an injective hull which is unique up to isomorphism and is divisible.

A torsion theory $(\underline{\underline{T}}, \underline{\underline{F}})$ for ModR is a pair of nonempty classes of modules satisfying the following conditions:

$$\underline{\underline{T}} = \{T \in \text{ModR} \mid \text{Hom}_R(T, H(F)) = o,\ \forall F \in \underline{\underline{F}}\} \text{ and}$$

$$\underline{\underline{F}} = \{F \in \text{ModR} \mid \text{Hom}_R(T, H(F)) = o,\ \forall T \in \underline{\underline{T}}\}.$$

A module M is divisible with respect to a given torsion theory $(\underline{\underline{T}}, \underline{\underline{F}})$ if $H(M)/M \in \underline{\underline{F}}$. A submodule N of M is closed in M with respect to a given torsion theory $(\underline{\underline{T}}, \underline{\underline{F}})$ if $M/N \in \underline{\underline{F}}$, and the closure of N is the intersection of all closed submodules containing N. Every module M has a divisible hull D(M) which is just the closure of M in H(M)

Let A be the full subcategory of ModR consisting of torsion-free divisible modules, then the inclusion functor $U: A \longrightarrow \text{ModR}$ has a left adjoint Q whose object function $Q(M) = D(M/t(M))$ which is exact. Moreover, if U is representable, then Q(R) becomes a ring. This ring is called the ring of right quotients of R with respect to the torsion theory $(\underline{\underline{T}}, \underline{\underline{F}})$.

1. Torsion Theory for Category of R-modules over integral domain R.

In the elementary algebra course one is familiar with the torsion and the torsion-free groups. Let G be an additive abelian group. An element $x \in G$ has finite order if there exists an $n \in N$ such that $n\,x = o$, the identity element of G. It is easily seen that the annihilator of x in Z, $\text{Ann}_Z(x) = \{n \in Z \mid nx = o\}$, is an

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ideal of Z, the ring of integers. Suppose now that the order of x is n, then nx $=o$ and hence $n \in Ann_Z(x)$. Consequently $Ann_Z(x) \neq o$. Conversely, if $Ann_Z(x) \neq o$, then there exists a nonzero integer $m \neq o$ such that $mx=o$. This implies that there is an $n \in N$ which is the order of x. Thus an element $x \in G$ has finite order if and only if its annihilator $Ann_Z(x) \neq o$.

An element $x \in G$ is said to be torsioin if $Ann_Z(x) \neq o$; if x is not a torsion element, it is said to be torsion-free.

Note that any additive abelian group is an Z-module. This concept can be extended to the R-modules over integral domain R.

Definition 1.1. (Levy (1)) Let R be an integral domain, and let M_R be an Rmodule. Then

(i) an element $x \in M$ is said to be a torsion element if $Ann_R(x) = \{r \in R \mid xr=o\} \neq o$; otherwise, x is said to be a torsion-free element.

(ii) M_R is said to be a torsion module if all its elements are torsion.; it is said to be torsion-free module if every nonzero element of it is torsion-free.

From the definition, it is obvious that the zero module is both torsion and torsion-free.

Since R is an integral domain, it has a quotient field Q. Note that any field is a vector space over itself, Thus Q_R is both projective and injective, and hence Q_R is torsion-free and injective as R-module. (2) (3).

We can state the torsion module in the following equivalent form:

Theorem 1.1. The module M_R is torsion if and only if $Hom_R(M_R, Q_R)=o$.

Proof: ($\Longrightarrow$) Suppose that M_R is torsion. For any $f \in Hom_R(M_R, Q_R)$. If $x \in M$, then x is a torsion element and hence $Ann_R(x) \neq o$. Thus there exists a nonzero element $r \in R$ such that $xr=o$. Hence $f(xr)=f(o)=o$. This implies that $f(x)r=o$ in the field Q. Hence $f(x)=o$, $\forall x \in M$. That is, $f=o$ and $Hom_R(M_R, Q_R) =o$.

($\Longleftarrow$) Suppose that $Hom_R(M_R, Q_R)=o$. Assume, on the contrary, that M_R is not torsion, then there is an nonzero element $x \in M$ which is torsion-free. Thus $Ann_R(x) =o$. Let the mapping $f: xR \longrightarrow Q$ be the homomorphism defined by $f(xr)= r$, $\forall r \in R$. Clearly, $f \neq o$. Since Q_R is injective, there is an $f^* \in Hom_R(M_R, Q_R)$ such that $f^*|_{xR}=f$. Hence $f^* \neq o$. Therefore $Hom_R(M_R, Q_R) \neq o$, a contradiction.

Q. E. D.

Proposition 1.1. Le M_R, N_R be modules, and let f: $M_R \longrightarrow N_R$ be R-homomorphism from M_R onto N_R. Suppose that M_R is torsion, then so is N_R.

Proof: For any $g \in Hom_R(N_R, Q_R)$, the composite function gof is an element of $Hom_R(M_R, Q_R)$. Since M_R is torsion, we have $Hom_R(M_R, Q_R) = o$. This implies that gof=o. Thus $\forall x \in N$, there exists an $a \in M$ such that $f(a) = x$. Hence $g(x) = g(f(a)) = gf(a) = o$. That is $g = o$. Hence $Hom_R(N_R, Q_R) = o$. Q. E. D.

We can also state the torsion-free module in the following equivalent form:

Theorem 1.2. A nonzero module M_R is torsion-free if and only if it contains no nonzero torsion submodules.

Proof: ($\Longrightarrow$) Suppose that M_R is torsion-free. Let N_R be any nonzero submodule of M_R, then there exists a nonzero torsion element $x \in N$. Hence $Ann_R(x) \neq o$. On the other hand, since $x \in M$ and M_R is torsion-free, we must have $Ann_R(x) = o$, a contradiction.

($\Longleftarrow$) Suppose that M_R contains no nonzero torsion submodules. Assume that M_R is not torsion-free, then there is a nonzero torsion element $x \in M$, and hence $Ann_R(x) \neq o$. But then the cyclic submodule xR is a nonzero torsion submodule of M_R, a contradiction. Q. E. D.

Proposition 1.2. Any submodule of a torsion-free module is torsion-free.

Proof: Let F be a given torsion-free module and let M_R be submodule of F. Assume that M_R is not torsion-free, then there is a nonzero torsion submodule N_R of M_R. Thus N_R is also a nonzero torsion submodule of F, contradicting the fact that F is torsion-free. Q. E. D.

Theorem 1.3. Let F be a given module. Then F is torsion-free if and only if $Hom_R(T, F) = o$ for all torsion modules T.

Proof: ($\Longleftarrow$) Suppose that $Hom_R(T, F) = o$ for all torsion modules T. Assume that F is not torsion-free, then there is a nonzero torsion submodule N_R of F. Thus the canonical injection j: $N \longrightarrow F$ is an element of $Hom_R(N_R, F) = o$. Hence $j = o$, a contradiction.

($\Longrightarrow$) Suppose that F is torsion-free. Let T be any torsion module. For any $f \in Hom_R(T, F)$. Since T is torsion, the homomorphic image f(T) is also torsion submodule of F. On the other hand, since F is torsion-free, f(T) is a torsion-free submodule of F. Hence $f(T) = o$ and hence $f = o$. Therefore $Hom_R(T, F) = o$

for all torsion modules. Q. E. D.

Theorem 1.4. Let T be a given module. Then T is torsion if and only if $Hom_R(T,F)=o$ for all torsion-free modules F.

Proof: ($\Longrightarrow$) Suppose that T is torsion. Let F be any fixed torsion-free module and let $f\in Hom_R(T,F)$. Since T is torsion, then f(T) is a torsion submodule of F. On the other hand, since F is torsion-free, it follows that f(T) must be torsion-free. Hence $f(T)=o$ and hence $f=o$. Therefore $Hom_R(T,F)=o$. ($\Longleftarrow$) Suppose that $Hom_R(T,F)=o$ for all torsion-free modules F. Note that Q_R is torsion-free. We have $Hom_R(T,Q_R)=o$, and hence T is torsion. Q. E. D.

Let ModR denote the category of all modules over the integral domain R.

Definition 1.1. Let $\underline{\underline{T}}$ and $\underline{\underline{F}}$ be nonempty classes of modules in ModR satisfying the following conditions:

$$\underline{\underline{T}}=\{T\in ModR \mid Hom_R(T,F)=o,\ \forall F\in \underline{\underline{F}}\} \text{ and}$$

$$\underline{\underline{F}}=\{F\in ModR \mid Hom_R(T,F)=o,\ \forall T\in \underline{\underline{T}}\}.$$

Then the pair $(\underline{\underline{T}},\underline{\underline{F}})$ is called a torsion theory for ModR.

Theorem 1.5. Let $\underline{\underline{T}}$ be the class of all torsion modules in ModR, and let $\underline{\underline{F}}$ be the class of all torsion-free modules in Mod R. Then the pair $(\underline{\underline{T}}, \underline{\underline{F}})$ is a torsion theory for ModR. (This is called the usual torsion theory for ModR).

Proof: Clearly $o\in\underline{\underline{T}}$ and $o\in\underline{\underline{F}}$ and hence $\underline{\underline{T}}\neq\phi$, $\underline{\underline{F}}\neq\phi$. The result follows from Theorems 1.3. and 1.4. Q. E. D.

Let the pair $(\underline{\underline{T}},\underline{\underline{F}})$ be a torsion theory for ModR. Then we have the following properties:

(T1). $\underline{\underline{T}}$ is closed under isomorphic images.

Proof: For any $T\in\underline{\underline{T}}$. Let $f: T \longrightarrow T^*$ be an isomorphism from T onto T^*. For any fixed $F\in\underline{\underline{F}}$ and for any $g\in Hom_R(T^*, F)$, then the composite function $g\circ f\in Hom_R(T, F)=o$ because $T\in\underline{\underline{T}}$. This implies that $g\circ f=o$ and hence $g=o$, Therefore $Hom_R(T^*,F)=o$. Hence $T^*\in\underline{\underline{T}}$. Q. E. D.

(T2). $\underline{\underline{T}}$ is closed under factor modules.

Proof: Let $T\in\underline{\underline{T}}$ and let S be any submodule of T. For any fixed $F\in\underline{\underline{F}}$ and $\forall f\in Hom_R(T/S, F)$. If $\pi: T \longrightarrow T/S$ is the canonical epimorphism, then the composite function $f\circ\pi$ is an element of $Hom_R(T, F)=o$ since $T\in\underline{\underline{T}}$. This implies that $f \circ \Pi = o$ and hence $f = o$. Hence $Hom_R(T/S, F) = o$ and $T/S \in \underline{\underline{T}}$.

(T3). $\underline{\underline{T}}$ is closed under extensions.

Proof: Suppose that the sequence $o \longrightarrow M \longrightarrow T \longrightarrow T/M \longrightarrow o$ is exact, where M_R is a submodule of T_R and M_R and T/M are in $\underline{\underline{T}}$. We want to show $T \in \underline{\underline{T}}$. Since the exact sequence $o \longrightarrow M \longrightarrow T \longrightarrow T/M \longrightarrow o$ implies that the sequence $o \longrightarrow Hom_R(T/M,F) \longrightarrow Hom_R(T,F) \longrightarrow Hom_R(M,F)$ is exact. Since M_R and T/M are in $\underline{\underline{T}}$, we have $Hom_R(M,F) = Hom_R(T/M,F) = o$. This implies that $Hom_R(T,F) = o$, $\forall F \in \underline{\underline{F}}$. Hence $T \in \underline{\underline{T}}$. Q. E. D.

(T4). $\underline{\underline{T}}$ is closed under direct sums.

Proof: Let $T = \sum_i \oplus T_i$, where each $T_i \in \underline{\underline{T}}$. We want to show $T \in \underline{\underline{T}}$. Since $Hom_R(T, F) = Hom_R(\sum_i \oplus T_i, F) = \prod_i Hom_R(T_i, F)$, it follows that $Hom_R(T_i, F) = o$, $\forall i$. Hence $Hom_R(T,F) = o$. Hence $T \in \underline{\underline{T}}$.

(F1) $\underline{\underline{F}}$ is closed under isomorphic images.

Proof: Let $f: F \longrightarrow F^*$ be the isomorphism from F onto F^*, and $F \in \underline{\underline{F}}$. Let T be any fixed module in $\underline{\underline{T}}$, and let $g \in Hom_R(T, F^*)$, then the composite function $f^{-1} \circ g \in Hom_R(T,F) = o$. This implies that $f^{-1} \circ g = o$. and hence $g = o$. Hence $Hom_R(T,F^*) = o$ and F^* is in $\underline{\underline{F}}$. Q. E. D.

(F2). $\underline{\underline{F}}$ is closed under submodules.

Proof: Let F be any element of $\underline{\underline{F}}$, and let H be any submodule of F. Since any element of $Hom_R(T, H)$ is an element of $Hom_R(T,F)$, we see that H must be torsion-free, and hence $H \in \underline{\underline{F}}$. Q. E. D.

(F3). $\underline{\underline{F}}$ is closed under extensions.

Proof: Let F be an module and let H be submodule of F such that H and F/H are in $\underline{\underline{F}}$. Assume that F is torsion, then so are H and F/H, a contradiction.

(F4). $\underline{\underline{F}}$ is closed under direct products.

Proof: Let $F = \prod_i F_i$, where each $F_i \in \underline{\underline{F}}$, we want to show $F \in \underline{\underline{F}}$. Let T be any fixed element of $\underline{\underline{T}}$. Note that $Hom_R(T, F) = Hom_R(T, \prod_i F_i) = \prod_i Hom_R(T, F_i)$. Thus if each F_i is in $\underline{\underline{F}}$, then $Hom_R(T, F_i) = o$, $\forall i$, and hence $Hom_R(T, F) = o$. Therefore $F \in \underline{\underline{F}}$. Q. E. D.

Let $\underline{C}$ be any given class of modules in ModR. Then $\underline{C}$ generates a torsion theory for MorR in the following way:

$\underline{F}=\{F\in \text{ModR} \mid \text{Hom}_R(C,F)=o,\ \forall C\in \underline{C}\}$ and

$\underline{T}=\{T\in \text{ModR} \mid \text{Hom}_R(T,F)=o,\ \forall F\in \underline{F}\}$

Clearly, $o\in \underline{T}$ and $o\in \underline{F}$, and hence $\underline{T}\neq\phi$, $\underline{F}\neq\phi$. Moreover, this pair $(\underline{T},\underline{F})$ is a torsion theory for ModR.

Theorem 1.6 Let $\underline{T}$. $\underline{F}$ be nonempty classes ofm odules in ModR. Then $(\underline{T},\underline{F})$ is a torsion theory for ModR if and only if $\underline{T}$ is closed under (T1), (T2), (T3), and (T4).

Proof: ($\Longrightarrow$) This follows from Theorem 1.5. ($\Longleftarrow$) Let $\underline{C}$ be a class of torsion modules satisfying the conditions (T1), (T2), (T3), and (T4). Let the pair $(\underline{T},\ \underline{F})$ be the torsion theory generated by $\underline{C}$. By our construction, it is easily seen that $\underline{C}\subset\underline{T}$. It suffices to prove the reverse inclusion. If $M\in\underline{T}$ is such that $\text{Hom}_R(M,F)=o,\ \forall F\in\underline{F}$. Let T (M) denote the sum of all submodules of M in $\underline{C}$. Let S be the direct sum of all submodules of M in $\underline{C}$. Since $\underline{C}$ is closed under direct sums, it follows that $S\in\underline{C}$. Since T(M) is a homomorphic image of S and $\underline{C}$ is closed under factor modules, we see that $T(M)\in\underline{C}$ and T(M) is a submodule of M. We now show that M=T(M). To prove this, it suffices to show $M/T(M)\in\underline{F}$. If $f\in\text{Hom}_R(C,\ M/T(M))$, $\forall C\in\underline{C}$. Assume that $f\neq o$, then f(C) is a nonzero torsion submodule of M/T(M). Thus $f(C)\in\underline{C}$. Hence there is a submodule K of M such that $T(M)\subset K\subset M$ and $f(C)=K/T(M)$. Since $\underline{C}$ is closed under extensions, we see that $K\in\underline{C}$. But then this contradicts the maximality of T(M). Hence f=o. Therefore $M/T(M)\in\underline{F}$. Q. E. D.

Corollary 1.1. Let $\underline{T}$, $\underline{F}$ be a given torsion theory for ModR, then every module has a largest torsion submodule T(M) and $\underline{T}=\{M\in\text{ModR} \mid T(M)=M\}$.

Theorem 1.7. Let $\underline{T}$, $\underline{F}$ be nonempty classes of modules in ModR, Then $(\underline{T},\underline{F})$ is a torsion theory for ModR if and only if $\underline{F}$ is closed under (F1), (F2), (F3), and (F4).

Proof: One can prove this by duality.

2. Torsion Theories for ModR over a general ring R.

Let ModR denote the category of all right unitary R-modules and let H(M) denote the injective hull of the R-module M.

Lemma 2.1. Let B and C be any two R-modules such that $Hom_R(B, H(C)) = o$, then $Hom_R(B,C)=o$.

Proof: For any $f \in Hom_R(B,C)$, we have $f \in Hom_R(B,H(C))=o$. Q. E. D.
The converse of Lemma 2.1. is not always true, in fact, we have the following lemma.

Lemma 2.2. Let B and C be any two R-modules, then the following statements are equivalent:
(a) $Hom_R(B,H(C))=o$, (b) $Hom_R(B, E(c))=o$ for any essential extension E(c) of C, (c) $Hom_R(S(b),C)=o$ for any submodule S(b) of B.

Proof: (a)$\Longrightarrow$(b): Let E(c) be an arbitary but fixed essential extension of C and let $f \in Hom_R(B,E(c))$. Since $C \subset E(c) \subset H(C)$, it follows that $f \in Hom_R(B,H(C)) = o$. Hence $f=o$ by (a).

(b)$\Longrightarrow$(c): Let S(b) be any submodule of B, and let $f \in Hom_R(S(b),C)$, then the mapping f^*: $B \longrightarrow C$ defined by $f^*(x)=o$, if $x \notin S(b)$ and $f^*(x)=f(x)$, if $x \in S(b)$. Clearly, $f^* \in Hom_R(B,C)$. Since C is an essential extension of itself, it follows that $f^*=o$ and hence $f=o$.

(c)$\Longrightarrow$(a): Let $f \in Hom_R(B,H(C))$ be such that $f \neq o$. Since f(B) is a nonzero submodule of H(C), we get $f(B) \cap C \neq o$. Thus $\exists o \neq b \in B \ni o \neq f(b) = c \in C$. Let $B^* = bR$ be the submodule of B generated by b. Clearly, $f|_{B^*} \in Hom_R(B^*,C)$ and $f|_{B^*} \neq o$ a contradiction. Q. E. D.

It is well known that any module has an injective hull which is unique up to isomorplism.

Definition 2.1. A Lambek torsion theory for ModR is a pair($\underline{\underline{T}}$, $\underline{\underline{F}}$) of nonempty classes of right R-modules satisfying the following conditions:

(1) $\underline{\underline{T}} = \{T \in ModR \mid Hom_R(T, H(F)) = o, \forall F \in \underline{\underline{F}}\}$

(2) $\underline{\underline{F}} = \{F \in ModR \mid Hom_R(T, H(F)) = o, \forall T \in \underline{\underline{T}}\}$

The elements of $\underline{\underline{T}}$ are called torsion modules and the class $\underline{\underline{T}}$ is called torsion class for the torsion theory. The elements of $\underline{\underline{F}}$ are called torsion-free and the class $\underline{\underline{F}}$ is called torsion-free class.

Theorem 2.1. Let $(\underline{T},\underline{F})$ be a given Lambek torsion theory for McdR, then

(a) $Hom_R(T,F)=o$, $\forall T\in\underline{T}$, $\forall F\in\underline{F}$.

(b) $\underline{T}\cap\underline{F}=o$.

Proof: (a) Follows from Lemma 2.1.

(b) Assume that $\underline{T}\cap\underline{F}\neq o$, then $\exists o\neq M\in\underline{T}\cap\underline{F}$.

This implies that $Hom_R(M, M)=o$ by (a). On the other hand, since $M\neq o$, the identity mapping i_M on M is an element of $Hom_R(M, M)\neq o$. Hence $\underline{T}\cap\underline{F}=o$.

Q. E. D.

Let $T_1=(\underline{T},\underline{F})$ and $T_2=(\underline{T}^*,\underline{F}^*)$ be two Lambek torsion theories for ModR. We say that T_1 is smaller than T_2 if $\underline{T}\subset\underline{T}^*$(or $\underline{F}^*\subset\underline{F}$).

Let $\underline{C}$ be a given nonempty class of R-modules, then $\underline{C}$ generates a Lambek torsion theory for ModR in the following way:

let $\underline{F}=\{F\in ModR \mid Hom_R(C,H(F))=o,\forall C\in\underline{C}\}$

$\underline{T}=\{T\in ModR \mid Hom_R(T,H(F))=o,\forall F\in\underline{F}\}$

Clearly the pair $(\underline{T},\underline{F})$ is a Lambek torsion theory for ModR and is called the torsion theory generated by the class $\underline{C}$. This is the smallest torsion theory in which all elements of $\underline{C}$ are torsion modules; that is, $\underline{T}$ is the smallest class of torsion modules containing the given class $\underline{C}$.

If we let $\underline{T}=\{T\in ModR \mid Hom_R(T,H(C))=o,\forall C\in\underline{C}\}$ and

$\underline{F}=\{F\in ModR \mid Hom_R(T,H(F))=o,\forall T\in\underline{T}\}$.

Then the pair $(\underline{T},\underline{F})$ is a Lambek torsion theory for ModR and is the largest Lambek torsion theory in which all elements of $\underline{C}$ are torsion-free.

Theorem 2.2. Let $\underline{T}$, $\underline{F}$ be nonempty classes of modules, then the following statements are equivalent:

(a) The pair $(\underline{T},\underline{F})$ is a Lambek torsion theory for ModR.

(b) $\underline{T}$ is closed under isomorphic images, factor modules, extensions, direct sums and submodules.

(c) $\underline{F}$ is closed under isomorphic images, submodules, direct products, and injective hulls.

(d) There is an idempotent radical t on ModR such that $M\subset N$ implies $M\cap t(N)=t(M)$; moreover, $\underline{T}=\{M\in ModR \mid t(M)=M\}$, $\underline{F}=\{M\in ModR \mid t(M)=o\}$.

Proof:

(a)$\Longrightarrow$(b): i) Let $T\in\underline{T}$ and let T^* be the isomorphic image of T under f. Suppose that $g\in Hom_R(T^*,H(F))$, $F\in\underline{F}$, then $g\circ f\in Hom_R(T,H(F))=o$; that is, $g\circ f=o$. But then $g=o$. Hence $T^*\in\underline{T}$.

ii) Let $\underline{T} \in T$ and let M be any submodule of T. We want to show that $T/M \in \underline{T}$. For any $f \in Hom_R(T/M, H(F))$, $\forall F \in \underline{F}$ is such that $f \neq 0$. Let $g \in Hom_R(T, T/M)$ be the canonical epimorphism, then $f \circ g \neq 0$ because $f \neq 0$, but then $f \circ g \in Hom_R(T, H(F)) = 0$, a contradiction. Hence $f = 0$ and $T/M \in \underline{T}$.

(iii) Let M be a submodule of T such that $M, T/M \in \underline{T}$. Claim: $T \in \underline{T}$. Since the sequence $0 \longrightarrow M \longrightarrow T \longrightarrow T/M \longrightarrow 0$ is exact, we have the sequence $0 \longrightarrow Hom_R(T/M, H(F)) \longrightarrow Hom_R(T, H(F)) \longrightarrow Hom_R(M, H(F))$ is also exact. Since $Hom_R(T/M, H(F)) = Hom_R(M, H(F)) = 0$, $\forall F \in \underline{F}$, we must have $Hom_R(T, H(F)) = 0$. $\forall F \in \underline{F}$. Hence $T \in \underline{T}$.

(iv) Let $T = \sum_i \oplus T_i$, where each $T_i \in \underline{T}$. Since $Hom_R(T, H(F)) =$

$Hom_R(\sum_i \oplus T_i, H(F)) = \prod_i Hom_R(T_i, H(F)) = 0$, it follows that $T \in \underline{T}$.

(v) Let $T \in \underline{T}$, and let M be a submodule of T. We want to show that $M \in \underline{T}$. For any $f \in Hom_R(M, H(F))$. Since $H(F)$ is injective, there exists an $g \in Hom_R(T, H(F)) = 0$ such that $g|_M = f$. But then $f = 0$ since $g = 0$.

(a)$\Longrightarrow$(c): The same as in the proof of (a) implies (b).

(b)$\Longrightarrow$(a): Let $\underline{C}$ be a class of modules which is closed under isomorphic images, factor modules, extensions, direct sums and submodules. Let $(\underline{T}, \underline{F})$ be the Lambek torsion theory generated by the class $\underline{C}$, where

$$\underline{F} = \{F \in ModR \mid Hom_R(C, H(F)) = 0, \forall C \in \underline{C}\},$$
$$\underline{T} = \{T \in ModR \mid Hom_R(T, H(F)) = 0, \forall F \in \underline{F}\}.$$

Claim: $\underline{C} = \underline{T}$. Since $\underline{C} \subset \underline{T}$ by construction, it suffices to show that $\underline{T} \subset \underline{C}$. To do this, let $M \in \underline{T}$ and $Hom_R(M, H(F)) = 0 \, \forall F \in \underline{F}$. Let $t(M)$ be the sum of all submodules of M which are in $\underline{C}$. Let S be the direct sum of all torsion submodules of M in $\underline{C}$, then $S \in \underline{C}$ since $\underline{C}$ is closed under direct sum. Since $t(M)$ is a homomorphic image of S, it follows that $t(M) \in \underline{C}$. Since $t(M)$ is a submodule of M and $t(M) \in \underline{C}$, we have $t(M) \in \underline{T}$ and hence $M/t(M) \in \underline{T}$. To show that $t(M) = M$, it suffices to show that $M/t(M) \in \underline{F}$. For any $C^* \in \underline{C}$ and $f \in Hom_R(C^*, H(M/t(M)))$. Since $\underline{C}$ is closed under isomorphic images and factor modules, it follows that $f(C^*) \in \underline{C}$. Assume that $f \neq 0$, then $f(C^*)$ is a nonzero sumodule of $H(M/t(M))$. Thus $f(C^*) \cap M/t(M) \neq 0$. Hence $\exists$ a module K with $t(M) \subset K \subset M$ such that $f(C^*) = K/t(M) \in \underline{T}$. Since $\underline{C}$ is closed under extensions, we must have $K \in \underline{C}$. But then $K = t(M)$ by construction. This is a contradiction. Hence $f = 0$, that is, $M/t(M) \in \underline{F}$. Hence $M = t(M) \in \underline{C}$ and $\underline{T} = \underline{C}$.

(c)$\Longrightarrow$(a): The same as the proof in (b)$\Longrightarrow$(a).

(b)$\Longrightarrow$(d): As in the proof of (a)$\Longrightarrow$(b) it is easily seen that every module M has a largest submodule t(M) belonging to $\underline{\underline{T}}$,(t(M) is called the torsion submodule of M). Moreover, (i) $M \in \underline{\underline{T}}$ if and only if t(M)=M. In this case, $\underline{\underline{T}}=\{M \in ModR \mid t(M)=M\}$. (ii) $M \in \underline{\underline{F}}$ if and only if t(M)=o. In this case, $\underline{\underline{F}}=\{M \in ModR \mid t(M)=o\}$.

Note that t: ModR $\longrightarrow$ ModR is an objection function.

$M \subset N \Longrightarrow t(M) \subset t(N)$ and $t(M) \subset M$. Hence $t(M) \subset M \cap t(N)$.

Let $x \in M \cap t(N) \Longrightarrow x \in M$ and $x \in t(N) \Longrightarrow xR \subset t(N) \in \underline{\underline{T}}$. Since $\underline{\underline{T}}$ is closed under submodules, we have $xR \in \underline{\underline{T}}$ and $xR \subset M$. Hence $xR \subset t(M)$. This implies that $x \in t(M)$. Hence $M \cap t(N)=t(M)$. Clearly, the object function t has the following properties:

(i) $t(M) \subset M$, (ii) For any $f \in Hom_R(M,N) \Longrightarrow f(t(M)) \subset N \Longrightarrow f(t(M)) \in \underline{\underline{T}}$ because $t(M) \in \underline{\underline{T}}$ and hence $f(t(M)) \subset t(N)$,(iii) Since $t(M) \subset M$ we have $t(t(M))=M \cap t(M) =t(M)$. Consequently t is an idempotent radical on ModR.

(d)$\Longrightarrow$(a): Suppose there exists an idempotent radical t on ModR such that $M \subset N \Longrightarrow M \cap t(N)=t(M)$. Then $\underline{\underline{T}}=\{M \in ModR \mid t(M)=M\}$, $\underline{\underline{F}}=\{M \in ModR \mid t(M)=o\}$. Clearly the pair ($\underline{\underline{T}}$, $\underline{\underline{F}}$) is a torsion theory for ModR. For if $M \in \underline{\underline{T}}$ and $f \in Hom_R(M,H(F))$, $F \in \underline{\underline{F}}$, then t(M)=M and t(F)=o. Since t is an idempotent radical, it follows that $f(M)=f(t(M)) \subset t(H(F))$. Since $F \subset H(F)$, we have $o=t(F) =F \cap t(H(F))$. Since H(F) is an essential extension of F, it follows that t(H(F))=o. This implies that f(M)=o. Hence f=o.

Next for any $F \in \underline{\underline{F}}$, t(F)=o. If $g \in Hom_R(T,H(F))$, $T \in \underline{\underline{T}}$, Since $T \in \underline{\underline{T}}$, t(T) =T. Since t is an idempotent radical, it follows that $g(T)=g(t(T)) \subset t(H(F))$ and $o=t(F)=F \cap t(H(F))$. But then t(H(F))=o and hence g(T)=o. Thus g=o

Q. E. D.

Corollary 2.1. Let ($\underline{\underline{T}}$,$\underline{\underline{F}}$) be a given Lambek torsion theory for ModR, then every module has a unique torsion submodule t(M).

Corollary 2.2. There is a one-one correspondence between torsion theories and torsion radicals (that is, an idempotent t satisfying the condition (d) in Theorem *2.2*).

Remark: Let ($\underline{\underline{T}}$, $\underline{\underline{F}}$) be a given Lambek torsion theory for ModR, then $\underline{\underline{F}}$ is a reflective subcategory of ModR and $\underline{\underline{T}}$ is a coreflective subcategory of ModR. Thus the inclusion functor $\underline{\underline{F}} \longrightarrow$ ModR has a left adjoint ModR $\longrightarrow \underline{\underline{F}}$ which maps M to M/t(M).

3. Filters (or, Pretopology)

Definition 3.1. An idempotent filter $\underline{D}$ on R is a nonempty class of right ideals of R satisfying the following conditions:

(1) $D \in \underline{D}$ and $D \subset K$ (K is a right ideal of R) $\Longrightarrow K \in \underline{D}$.

(2) $D \in \underline{D}$ and $a \in R \Longrightarrow a^{-1}D = \{r \in R \mid ar \in D\} \in \underline{D}$.

(3) If D, $K \in \underline{D}$, then $D \cap K \in \underline{D}$.

Clearly, $R \in \underline{D}$ follows from (1) and $\underline{D} \neq \phi$.

Lemma 3.1. Let $(\underline{T}, \underline{F})$ be a given Lambek torsion theory for ModR and let $\underline{D}$ be the class of all dense right ideals (*) of R, then $\underline{D}$ is an idempotent filter on R.

Proof: Clearly, $R \in \underline{D}$ and hence $\underline{D} \neq \phi$.

i) $\forall D \in \underline{D}$ and $\forall$ right ideal K of R with $D \subset K$. Since $R/K \cong R/D / D/K$ (as modules) and $R/D \in \underline{T}$, and $\underline{T}$ is closed under isomorphic images and factor modules, we have $R/K \in \underline{T}$. Hence $K \in \underline{D}$.

ii) $\forall D \in \underline{D}$ and $\forall a \in R$, then $a^{-1}D$ is a right ideal of R. By i) $D + a^{-1}D \in \underline{D}$ and hence $R/D + a^{-1}D \in \underline{T}$. Since $R/a^{-1}D \cong R/(D+a^{-1}D) / (D+a^{-1}D)/(a^{-1}D)$, we have $R/a^{-1}D \in \underline{T}$. Hence $a^{-1}D \in \underline{D}$.

iii) $\forall D$, $K \in \underline{D}$, then R/D, $R/K \in \underline{T}$. Since $D \cap K \subset D \subset R$, it follows that $R/D \cap K \cong (R/D) / (D/D \cap K) \in \underline{T}$ since $\underline{T}$ is closed under isomorphic images and factor modules. Hence $R/D \cap K \in \underline{T}$. and $D \cap K \in \underline{D}$.

Theorem 3.1. There is a one-one correspondence between torsion theories and idempotent filters.

Proof: By Lemma 3.1. we know that given any Lambek torsion theory $(\underline{T}, \underline{F})$, there correspondence an idempotent filter D of dense right ideals of R relative to $(\underline{T}, \underline{F})$.

Conversely, if $\underline{D}$ is a given idempotent filter. Let

$\underline{T} = \{M \in \text{ModR} \mid \text{Ann}_R(m) \in \underline{D},\ \forall m \in M\}$ and

$\underline{F} = \{F \in \text{ModR} \mid \text{Hom}_R(T, H(F)) = o,\ \forall T \in \underline{T}\}$.

Then the pair $(\underline{T}, \underline{F})$ is a torsion theory for ModR. Q. E. D.

Definition 3.2. Let $(\underline{T}, \underline{F})$ be a given torsion theory for ModR and let $M \in$

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*: c.f. Definition 3.2. (a)

ModR. Then

(a) A submodule N of M is said to be dense in M relative to $(\underline{\underline{T}},\underline{\underline{F}})$ if $M/N\in\underline{\underline{T}}$.
(b) A submodule N of M is said to be closed in M relative to $(\underline{\underline{T}},\underline{\underline{F}})$ if $M/N\in\underline{\underline{F}}$.
(c) The closure of the submodule N in M is the intersection of all closed submodules of M containing N.

Let $(\underline{\underline{T}},\underline{\underline{F}})$ be a given Lambek torsion theory for ModR and let $M\in$ModR. If D(M) denotes the closure of M in H(M), then it is not difficult to verify the following things:

(i) D(M) is an essential extension of M and $M\subset D(M)\subset H(M)$.
(ii) D(M)/M is a torsion module; that is, M is dense in D(M).

Definition 3.3. Let $(\underline{\underline{T}},\underline{\underline{F}})$ be a given Lambek torsion theory for ModR. Then a module $M\in$ModR is said to be divisible relative to $(\underline{\underline{T}},\underline{\underline{F}})$ if $H(M)/M \in\underline{\underline{F}}$.

Theorem 3.2. Let $(\underline{\underline{T}},\underline{\underline{F}})$ be a given Lambek torsion theory for ModR and let $M\in$ModR, then the following statements are equivalent:

(a) M is divisible relative to $(\underline{\underline{T}},\underline{\underline{F}})$.
(b) For any dense right ideal D of R and any $f\in Hom_R(D,M)$, $\exists$ an $m\in M$ such that $f(d)=md$, $\forall d\in D$.
(c) If module N is dense in the module K, then any $f\in Hom_R(N, M)$ can be extended to an element $f^*\in Hom_R(K,M)$.

Proof:

(a)$\Longrightarrow$(b): Suppose that M is divisible, then $H(M)/M\in\underline{\underline{F}}$. Let D be any dense right ideal of R and let $f\in Hom_R(D,M)$, then $f\in Hom_R(D, H(M))$. Since H(M) is injective, R. Baer's theorem assures that an element $x\in H(M)$ such that $f(d)=xd$, $\forall d\in D$.

$$\begin{array}{ccccc} o & \longrightarrow & D & \longrightarrow & R \\ & & f\downarrow & & \\ & & M\subset H(M) & & \end{array}$$

Claim: $x\in M$.

Note that $f(d)=xd\in M$, $\forall d\in D$, we have $D\subset x^{-1}M$. Since D is dense, it follows that $x^{-1}M$ is also dense in R. Hence $R/x^{-1}M\in\underline{\underline{T}}$. Note that $xR+M/M\cong R/x^{-1}M\in\underline{\underline{T}}$. But $H(M)/M\in\underline{\underline{F}}$. This implies that $xR+M=M$ and hence $x\in M$.

(b)$\Longrightarrow$(c): Consider the following diagram: $\forall f\in Hom_R(N,M)$. Let C be the collection of all couples (D, h) where $N\subset D\subset K$ and $h\in Hom_R(D,M)$ such that $h|_N=f$. Define a partial ordering "$\leqslant$" on C as follows:

$$\begin{array}{ccccc} o & \longrightarrow & N & \xrightarrow{\ j\ } & K\text{(exact)} \\ & & f\downarrow & & \\ & & M & & \end{array}$$

$(D_1,h_1)\leqslant(D_2, h_2)$ if and only if $D_1\subset D_2$ and $h_2|_{D_1}=h_1$.

Then $(C, \leqslant)$ is a poset. By Zorn's Lemma, there is a maximal element, say, (K^*, g), in C. We shall show that $K^* = K$. If $K^* \neq K$, then $\exists\, x_o \in K - K^*$. Consider the right ideal $D = x_o^{-1}K^* = \{r \in R \mid x_o r \in K^*\} = \{r \in R \mid (x_o + K^*) r = K^*\} = Ann_R(x_o + K^*)$, $x_o + K^* \in K/K^*$. Since
$K/K^* \cong (K/N)/(N/K^*)$ and $K/N \in \underline{T}$, it follows that $K/K^* \in \underline{T}$. This implies that D is dense in R. Now consider the mapping $f^*: D \longrightarrow M$ defined by $f^*(d) = f(x_o d)$, $\forall d \in D$, is an R-homomorphism. By (b), $\exists\, m \in M$ such that $f(x_o d) = md$, $\forall d \in D$. Thus we may extend g to h: $K^* + x_o R \longrightarrow M$ defined by $h(k + x_o r) = g(k) + mr$ (note that $f(x_o r) = mr, \forall r \in R$ and $x_o r \in K^*$), a contradiction. Hence $K = K^*$.
(c)$\Longrightarrow$(a): Let D(M) be the closure of M in H(M), then $D(M)/M \in \underline{T}$. Thus, by (c) the identity mapping i_M on M can be extended to f: $D(M) \longrightarrow M$. Since D(M) is an essential extension of M, it follows that f must be a monomorphism. Hence the injection from $M \longrightarrow D(M)$ must be onto. Thus M is closed in H(M). Q. E. D.

Corollary 3.1. Any injective module is divisible and hence the injective hull of a module is divisible.

Remark: The closure D(M) of the module M in H(M) is called the divisible hull of M. Note that D(M) is divisible.

Let $(\underline{T}, \underline{F})$ be a given torsion theory for ModR in which all modules are torsion, then $D(M) = \cap \{C \in ModR \mid C$ is closed in H(M) and $M \subset C\} = \cap \{C \in ModR \mid H(M)/C \in F = o$ and $M \subset C\} = H(M)$, the injective hull of M.

4. Construction of Rings and Modules of Quotients.

Given a Lambek torsion theory $(\underline{T}, \underline{F})$ for ModR and given $M \in ModR$. Then the divisible hull of M, D(M), is an object function D from ModR to ModR. It is not, in general, a functor $M \longrightarrow D(M)$ natural in M, but the restriction of D to the category F of torsionfree modules is a functor, in fact, it is the left adjoint of the inclusion functor $\underline{A} \longrightarrow \underline{F}$, where $\underline{A}$ is the category of torsion-free divisible modules. To prove this, let $F \in \underline{F}$. Since $D(F)/F \in \underline{T}$, it follows that F is dense in D(F). Thus any $f \in Hom_R(F, A)$ can be extened to $f^* \in Hom_R(D(F), A)$, $A \in \underline{A}$. Since $D(F)/F \in \underline{T}$ and $A \in \underline{F}$, it follows that $Hom_R(D(F)/F, A) = o$. Hence the extension f^* of f is uniquely determined by f. Since D(F) is divisible and $D(F) \in \underline{F}$ because D(F) is an essential extension of F. and $H(D(F)) = H(F)$.

Hence $Hom_R(T,H(D(F)))=Hom_R(T,H(F))=o$ since $F\in \underline{F}$. Thus $D(F)\in \underline{A}$. Hence $D: F \longrightarrow D(F)$ from $\underline{F}$ to $\underline{A}$ is the left adjoint of the inclusion functor from $\underline{A}$ to $\underline{F}$.

Recall that the inclusion functor from $\underline{F}$ to ModR has a left adjoint. Then the composite of the left adjoints $\underline{A} \dashrightarrow \underline{F}$ and $\underline{F} \dashrightarrow$ ModR has a left adjoint, say Q, whose object function is defined by $Q(M)=D(M/t(M))$.

Remark: Q(M) is called the module of quotients of M, and Q the quotient functor relative to the given Lambek torsion theory $(\underline{T},\underline{F})$.

Another construction of Q is as follows:
Let $\underline{D}$ be the class of the dense right ideals of R. We define a partial ordering on $\underline{D}$ as follows:

$I_1 \leqslant I_2$ if and only if $I_1 \supset I_2$. Then the pair $(D, \leqslant)$ is a poset. If $I_1, I_2 \in \underline{D}$, then the intersection $I_1 \cap I_2$ of I_1 and I_2 is a dense right ideal of R and hence $I_1 \cap I_2 \in \underline{D}$. Moreover, $I_1 \leqslant I_1 \cap I_2$ and $I_2 \leqslant I_1 \cap I_2$. Hence D is a directed set. Let $M \in$ ModR. Then foreach element $D \in \underline{D}$, there corresponds a $Hom_R(D,M) \in$ Mod Z. If $D_1 \leqslant D_2$, we let $f(D_2, D_1)$: $Hom_R(D_1, M) \longrightarrow Hom_R(D_2, M)$ by letting $f(D_2,D_1)(g)=g|D_2$. $\forall g \in Hom_R(D_1,M)$. It is easily seen that $f(D,D)={}^{i}Hom_R(D,M)$, $\forall D \in \underline{D}$ and if $D_1 \leqslant D_2 \leqslant D_3$, then $f(D_3,D_2) o f(D_2,D_1)=f(D_3,D_1)$. Thus $(\{Hom_R(D,M)\}\ D \in \underline{D}, \{f_D\}\ D \in \underline{D})$ forms a direct system. The direct limit $\varinjlim_{D \in \underline{D}} Hom_R(D, M)$ of this system exists and is denoted by L(M); that is, $L(M)=\varinjlim_{D \in \underline{D}} Hom_R(D,M)$. L(M) can be easily made into an R-module. Moreover, $L(M)=\bigcup_{D \in \underline{D}} f_D(Hom_R(D,M))$, where f_D: $Hom_R(D,M) \longrightarrow L(M)$ is the canonical injection.

Theorem 4.1. Let $(\underline{T}, \underline{F})$ be a given torsion theory for ModR and let F be any torsionfree module, then $D(F)=L(F)$.

Proof: For each $D \in \underline{D}$, let f_D: $Hom_R(D,F) \longrightarrow L(F)$ be the canonical injection. Now $\forall f \in Hom_R(D,F) \Longrightarrow f \in Hom_R(D,D(F))$. Since D(F) is divisible and D is dense, it follows that $\exists$ an $x_D(f) \in D(F)$ such that $f(d)=x_D(f)d$, $\forall d \in D \cdots\cdots$ (A).

Then x_D: $Hom_R(D,F) \longrightarrow D(F)$. Since $L(F)=\varinjlim_{D\in \underline{D}} Hom_R(D,F)$, it follows that $\exists$ $x\in Hom_R(L(F), D(F))$ such that $x o f_D = x_D$, $\forall D\in \underline{D}$ …… (B).

Claim: x is an isomorphism.

i) show that x is one-one:

Let $m\in L(F)$ be such that $x(m)=o$. Since $L(F)=\bigcup_{D\in \underline{D}} f_D(Hom_R(D,F))$, there exists a dense right ideal of R, $D\in \underline{D}$ and an $f\in Hom_R(D,F)$ such that $m=f_D(f)$. Thus, by (A), $o=x(m)=x(f_D(f))=x_D(f)$. Hence for any $d\in D$, $f(d)=x_D(f)d=o$. That is, $f=o$ and hence $m=f_D(f)=o$. This shows that x is one-one.

ii) show that x is onto:

For any $n\in D(F)$, since F is dense in D(F), it follows that $\exists$ $D\in \underline{D}$ such that $nD\subset F$. We define the mapping f: $D \longrightarrow F$ by letting $f(d)=nd$, $\forall d\in D$. Then $\forall d\in D$, $x_D(f)d=f(d)=nd \Longrightarrow x_D(f)=n$. From i) and ii) we conclude that $D(F) \cong L(F)$. Q. E. D.

Corollary 4.1. If M is a torsion-free divisible module, then $Q(M)=D(M/t(M))\cong L(M/t(M))$ since M/t(M) is torsionfree.

Theorem 4.2. Let $(\underline{T},\underline{F})$ be a given Lambek torsion theory for ModR, then the category $\underline{A}$ of torsion-free divisible modules is Abelian and the inclusion functor $\underline{A} \longrightarrow$ ModR has a left adjoint Q which is exact.

Proof: For any $M,N \in \underline{A}$, $Hom_R(M,N)$ is an aditive abdelian group. Moreover, the associative laws of homomorphisms hold in $\underline{A}$: $(g+h)f = gf + hf$ and $f(g+h)=fg+fh$. Thus $\underline{A}$ is an additive category. Clearly, the zero module o is in $\underline{A}$. Let $M, N\in \underline{A}$, then the direct sum $M\oplus N$ is torsionfree and $H(M\oplus N)= H(M)\oplus H(N)$, so $H(M\oplus N)/M\oplus N = H(M)\oplus H(N)/M\oplus N = H(M)/_M \oplus H(N)/N$. Since $M, N\in \underline{A}$, it follows that $H(M)/M\in \underline{F}$ and $H(N)/N\in \underline{F}$. Hence $H(M)/M\oplus H(N)/N \in \underline{F}$. Hence $M\oplus N = M\times N\in \underline{A}$. Thus, $\underline{A}$ is closed under finite direct sums (= products). For any $M,N\in \underline{A}$ and $\forall f\in Hom_R(M,N)$, since ker f is a submodule of M and $M\in \underline{F}$, it follows that ker $f\in \underline{F}$. Also if f: $M \longrightarrow N$ is a monomorphism, then ker $f=o$ and hence it is a kernel in $\underline{A}$. The cokernel of f: $M \longrightarrow N$ in $\underline{A}$ is $N \longrightarrow F \longrightarrow Q(F)$, where $N \longrightarrow F$ is the cokernel of f in ModR. Note that a map f: $M \longrightarrow N$ is an epimorphism if and only if its cokernel in MordR is torsion; that is, f(M) is dense in N. Hence f is the cokernel of ker f in $\underline{A}$.

Consequently, $\underline{A}$ is an Abelian category. Note that in an Abelian category $\underline{A}$, the following things concerning the left adjoint Q of the inclusion functor U: A$\longrightarrow$ ModR are equivalent:

(i) Q preserves monomorphisms, (ii) Q is left exact, (iii) Q is exact, (iv) UoQ preserves monomorphisms, (v) UoQ is left exact. Thus it is easily seen that the functor $M \mapsto M/t(M)$ and the functor $F \longrightarrow D(F)$ for $F \in \underline{F}$ preserve monomorphisms. Hence Q is exact. Q. E. D.

Definition 4.1. Let $\underline{A}$ be any category and let U: $\underline{A} \longrightarrow S$, where S is the category of sets. A universal element for U is a pair (s_o, A_o) consisting of an object A_o of $\underline{A}$ and an element $s_o \in U(A_o)$ satisfying the following property:

For any object $A \in \underline{A}$ and for any $s \in U(A)$, there exists a unique mapping $f: A_o \longrightarrow A$ such that $U(f)(s_o) = s$.

If (s_o, A_o) is a universal element for the functor U, then for each object $A \in \underline{A}$, the assignment $f \mapsto U(f)(s_o)$ is a bijection ϕ_A: $\hom(A_o, A) \cong U(A)$ of sets.

Definition 4.2. Let U: $\underline{A} \longrightarrow S$ be any functor from the category $\underline{A}$ to the category of sets. A representation of U is a pair (A_o, ϕ) consisting of an object $A_o \in \underline{A}$ and a family of bijections ϕ_A: $\hom(A_o, A) \cong U(A)$ given by $\phi_A(f) = U(f)(s_o)$ where $s_o = \phi_{A_o}(1A_o)$ natural in A. A functor U with such a representation is said to be representable.

Lemma 4.1. For each functor U: $\underline{A} \longrightarrow S$, the formula $s_o = \phi_{A_o}(1A_o)$ and $\phi_A(f) = U(f)(s_o)$ for $1A_o: A_o \longrightarrow A_o$ the identity and $f: A_o \longrightarrow A$ any morphism, establish a bijection from representations (A_o, ϕ) of U to universal elements (s_o, A_o) for U.

Proof: Let (A_o, ϕ) be a representation of U. Since ϕ is natural, the following diagram commutes for each $f: A_o \longrightarrow A$.

$$\begin{array}{ccc} 1A_o \in \hom(A_o, A_o) & \xrightarrow{\phi_{A_o}} & U(A_o) \\ \big\downarrow f_* & \phi_A & \big\downarrow U(f) \\ f_*(1A_o) = f \in \hom(A_o, A) & \longrightarrow & U(A) \end{array} \qquad \text{where } f_*(g) = f \circ g.$$

Let $s_o = \phi_{A_o}(1A_o)$. Since the above diagram commutes we have $\phi_A(f) = U(f)(s_o)$ But ϕ_A is a bijection, so each element $s \in U(A)$, $\exists$ a $f \in \hom(A_o, A)$ such that $\phi_A(f) = s$, and so $U(f)(s_o) = s$. Thus (s_o, A_o) is a universal element for U. Conversely, let (s_o, A_o) be a universal element for the functor U. For each object $A \in \underline{A}$, define ϕ_A: $\hom(A_o, A) \longrightarrow U(A)$ by $\phi_A(f) = U(f)(s_o)$. Since (s_o, A_o) is a universal element for U, any $s \in U(A)$, $f \in hom(A_o, A)$, $U(f)(s_o) = s$. Hence ϕ_A is a bijection. It is natural in A, for any g: $A \longrightarrow B$ gives $U(g)\phi_A(f) = U(g)U(f)(s_o)$ $= U(g \circ f)(s_o) = \phi_B(g \circ f) = \phi_B g_*(f)$. Hence (A_o, ϕ) is a representation of U.

Q. E. D.

Theorem 4.3. Let $\underline{A}$ be an additive category and U: $\underline{A} \longrightarrow$ ModR be an additive representable functor with representation (A_o, ϕ); then

(a) $U(A_o)$ can be made into a ring, with unity element s_o, where $s_o = \phi_{A_o}(1A_o)$.

(b) For each object $A \in \underline{A}$, U(A) is a right $U(A_o)$-module.

Proof: (a) Since U is representable by (A_o, ϕ), we see that ϕ_{A_o}: $Hom(A_o, A_o) \cong U(A_o)$, a bijection. The proof of the remaining part is just the same as in the proof of the *Lemma 4.1.* except the operations preserving.

Note that (s_o, A_o) is a universal element for U, then for each $s \in U(A_o)$, $\exists$ map s^*: $A_o \longrightarrow A_o \ni U(s^*)(s_o) = s$. Now we define $s_1 s_2 = U(s_1^* s_2^*)(s_o) = U(s_1^*)U(s_2^*)(s_o)$ $= U(s_1^*)(s_2)$. Clearly, $(s_1 s_2)^* = s_1^* s_2^*$ and $s_1 s_o = U(s_1^*)(s_o) = s_1$. Since $U(s_o^*)(s_o) = s_o$, we see that $s_o^* = 1A_o$, and hence

$s_o s_1 = U(s_o^*)(s_1) = s_1$. Hence $U(A_o) \cong Hom_R(A_o, A_o)$ as rings. (b)$\forall a \in U(A)$, $\forall s \in U(A_o)$, we define

$as = U(a^* s^*)(s_o) = U(a^*)\ U(s^*)(s_o) = U(a^*)(s)$. It is easily checked that U(A) becomes a right $U(A_o)$-module. Q. E. D.

Finally, let $(\underline{T}, \underline{F})$ be a given torsion theory for ModR and let $\underline{A}$ be the full subcategory of ModR consisting of all torsion-free divisible modules. It is an additive category.

By *Theorem 4.2.* the inclusion functor U: A $\longrightarrow$ ModR has a left adjoint Q which is exact. Theorem 4.3. assures that Q(R) is a ring and is called the ring of right quotients of R with respect to the given torsion theory.

In particular, if we take the Lambek torsion theory $(\underline{T}, \underline{F})$ to be the largest torsion theory for which R_R is torsion-free, then we obtain the Utumi's maximal ring of right quotients of R.

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