TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 1, January 1995

GENERALIZATIONS OF BROWDER'S DEGREE THEORY

SHOUCHUAN HU AND NIKOLAOS S. PAPAGEORGIOU

ABSTRACT. The starting point of this paper is the recent important work of F. E. Browder, who extended degree theory to operators of monotone type. The degree function of Browder is generalized to maps of the form $T+f+G$, where T is maximal monotone, f is of class $(S)_+$ bounded, and $G(\cdot)$ is an u.s.c. compact multifunction. It is also generalized to maps of the form $f+N_G$, with f of class $(S)_+$ and N_G the Nemitsky operator of a multifunction $G(x, r)$ satisfying various types of sign conditions. Some examples are also included to illustrate the abstract results.

1. INTRODUCTION

The resolution of a large variety of problems in nonlinear analysis depends on the study of equations of the form $Tx = y$, where T is an operator defined on an appropriate space X and $y \in X$. The Leray-Schauder degree has proven to be a very powerful tool in such investigations. The most important property of this degree is, of course, the homotopy invariance property, which forms the basis for the continuation method, which was originally developed by Poincaré and which consists of embedding the problem in a parametrized family of problems and considering its solvability as the parameter varies. Ever since the introduction of the Leray-Schauder degree theory in 1934 (which is an infinite-dimensional extension of Brouwer's degree theory), there have been various extensions and generalizations in different directions. By far the most important of these generalizations is due to F. E. Browder. In a series of important papers [7–12], Browder developed a degree theory, which is a generalization of the Leray-Schauder degree theory, for maps from a bounded open subset of a reflexive Banach space X into its dual X^*. Browder's breakthrough work paved the way for the application of degree-theoretic techniques to large classes of nonlinear partial differential equations.

Browder's degree theory is defined primarily for $(S)_+$ mappings (see §2) and $(S)_+$ mappings with maximal monotone perturbations, which cover a substantially large class of nonlinear partial differential operators. Browder demonstrated that the $(S)_+$ maps are the right class to consider and he proved the

Received by the editors February 2, 1994.

1991 *Mathematics Subject Classification*. Primary 35J60, 47H05, 47H11.

Key words and phrases. Degree function, monotone operator, operator of class $(S)_+$, Nemitsky operator, sign condition, multifunction, approximate selector, normalization, additivity on domain, homotopy invariance, compact embedding.

The paper was written while the first author was on sabbatical at Florida Tech. The hospitality in the Department of Applied Mathematics there is acknowledged.

existence of his degree through ingenious arguments involving Galerkin approximations. Specifically he proved

Theorem A [11]. *Let X be a reflexive Banach space. Then there exists one and only one degree function on the class of maps $T+f$, with T maximal monotone and f bounded and of class $(S)_+$, which satisfies the additivity on domain property, is normalized by J, the duality map of X into X^* corresponding to an equivalent norm on X with respect to which both X and X^* are locally uniformly convex, and is invariant under affine homotopies of the form $(1-t)(T+f)+tf_1$ with T maximal monotone, f and f_1 of class $(S)_+$.*

Remark. In fact, Browder showed that the unique degree function is invariant under a much broader class of homotopies, namely homotopies of the form T_t+f_t, $t\in[0,1]$, where T_t is a pseudomonotone homotopy for T and f_t is a homotopy of class $(S)_+$ for f (cf. Browder [11, Theorem 10]).

When applied to partial differential operators, we can have $X=W_0^{m,p}(Z)$ and $X^*=W^{-m,q}(Z)$ with $1<p,q<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$, $T: D\subseteq X\to 2^{X^*}\backslash\{\varnothing\}$ and $f:\overline{U}\to X^*$ where U is a bounded open set of X. As will be clear from the definitions (cf. §2), a compact perturbation of $(S)_+$ maps is still an $(S)_+$ map. Therefore compact maps from X into X^* are harmless, in the sense that they can always be absorbed in the original $(S)_+$ operator. But if such a compact map originates from a Nemitsky (superposition) operator $N_g(u)(x)=g(x,u(x))$, $x\in Z$, $u\in W_0^{m,p}(Z)$ with $g:Z\times\mathbb{R}\to\mathbb{R}$, it is clear that $g(x,r)$ has to satisfy certain growth conditions. To avoid such restrictions which are not always satisfied in applications, Browder [12] proved the following theorem. Let $Z\subseteq\mathbb{R}^n$ be a domain in $\mathbb{R}^n$ (bounded or unbounded), $X=W_0^{m,p}(Z)$, and $U\subseteq X$ open and bounded. We will say that $g:Z\times\mathbb{R}\to\mathbb{R}$ satisfies the sign condition provided it has the following two properties:

(i) for each fixed $r\in\mathbb{R}$, $x\to g(x,r)$ is measurable, for each fixed $x\in Z$, $r\to g(x,r)$ is continuous, and for each integer $s>0$, there exists a function $h_s\in L^1_{\mathrm{loc}}(Z)$ such that $|g(x,r)|\le h_s(x)$ a.e. on Z, $|r|\le s$.

(ii) For all $(x,r)\in Z\times\mathbb{R}$, we have $g(x,r)r\ge 0$.

Then the theorem of Browder [12, Theorem 7] reads as follows:

Theorem B [12]. *Assume that f is a bounded mapping of class $(S)_+$ of $\overline{U}$ into X^*, and $N_g: X\to X^*$ is the Nemitsky operator corresponding to a function $g(z,r)$ satisfying the sign condition. Assume that $y_0\in X^*$ is a target point such that $y_0\notin(f+N_g)(\partial U)$. Then the degree $d(f+N_g,U,y_0)$ is well defined. Furthermore, this degree function is the unique one satisfying the additivity on domain property, is normalized by the duality map J, and is invariant under permissible homotopies.*

In this paper we present the following generalizations of Browder's degree theory, contained in Theorems A and B above. First we prove that the degree function stipulated by Theorem A can be extended uniquely to the case where f is allowed to have a multivalued compact perturbation (i.e. for operators of the form $T+f+G$ with $G(\cdot)$ being the multivalued compact perturbation). Second, we establish the existence of a unique degree function for maps of the form $f+N_G$, where N_G is the multivalued Nemitsky operator corresponding to a multifunction $G(x,r)$ satisfying a sign condition. This extends Theorem

B, since the function $g : Z \times \mathbb{R} \to \mathbb{R}$ is replaced by a multifunction $G : Z \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\varnothing\}$. These two extensions are presented in §§3 and 4. In §5, we present some examples illustrating the applicability of our abstract results.

2. Preliminaries

Let X be a Banach space and consider the family F of all continuous mappings $f : \overline{U} \to X$, with U a bounded open subset of X (we consider all possible such sets U) and with $(I - f)(\overline{U})$ relatively compact in X, where I is the identity map on X. Let H be the family of continuous homotopies $\{f_t : t \in [0, 1]\}$ in F, with a common domain $\overline{U}$ such that $(I - f_t)(\overline{U}) \subseteq K$ for all $t \in [0, 1]$ and $K \subseteq X$ is compact. Then the Leray-Schauder degree theory states that there is an integer-valued degree function $d(\cdot, \cdot, \cdot)$ on the triples (f, U, y_0), with $f \in F$, $y_0 \in X \backslash f(\partial U)$ such that

(a) *Normalization*: If $y_0 \in U$, then $d(I, U, y_0) = 1$.
(b) *Additivity on domain*: If U_1 and U_2 are disjoint open subsets of U such that $y_0 \notin f(\overline{U} \backslash (U_1 \cup U_2))$, then

$$d(f, U, y_0) = d(f, U_1, y_0) + d(f, U_2, y_0).$$

(c) *Homotopy invariance*: If the homotopy $\{f_t : t \in [0, 1]\}$ belongs in H, $y : [0, 1] \to X$ is continuous, and $y(t) \notin f_t(\partial U)$ for any $t \in [0, 1]$, then $d(f_t, U, y(t))$ is independent of $t \in [0, 1]$.

By a result proved independently by Fuhrer [16] and Amann-Weiss [1], properties (a), (b), and (c) above determine uniquely the Leray-Schauder degree function.

In order to discuss a degree theory for maps from X into X^*, where X is a reflexive Banach space, we need to introduce the type of mappings we will be dealing with.

Definition 1. (i) A map $T : D \subseteq X \to 2^{X^*} \backslash \{\varnothing\}$ is said to be "monotone" if

$$\langle x^* - y^*, x - y \rangle \geq 0$$

for all $[x, x^*], [y, y^*] \in \operatorname{Gr} T$. Here $\operatorname{Gr} T$ denotes the graph of $T(\cdot)$ and $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X).

(ii) We say that $T(\cdot)$ is "maximal monotone" if it is monotone and for any $[u, u^*] \in X \times X^*$ for which $\langle u^* - x^*, u - x \rangle \geq 0$ for all $[x, x^*] \in \operatorname{Gr} T$ we have $[u, u^*] \in \operatorname{Gr} T$.

Following Browder, we will be concentrating on maps of type $(S)_+$ as the primary class to define a degree function. The class $(S)_+$ of operators was first introduced by Browder [3, 4].

Definition 2. Let $B \subseteq X$ and $f : B \to X^*$. We say that $f(\cdot)$ is of class $(S)_+$ if (i) $f(\cdot)$ is demicontinuous (i.e. $x_n \to x$ in B implies $f(x_n) \xrightarrow{w} f(x)$ in X^*) and (ii) if $\{x_n\}_{n \geq 1} \subseteq B$ and $x_n \xrightarrow{w} x$ for some $x \in X$ and $\overline{\lim} \langle f(x_n), x_n - x \rangle \leq 0$, then $x_n \to x$ in X.

Finally let us introduce the kind of multivalued perturbations that we will be considering:

Definition 3. A multifunction $G : B \subseteq X \to 2^{X^*}\backslash\{\varnothing\}$ is said to belong to class (P) if it maps bounded sets to relatively compact sets, for every $x \in B$ $G(x)$ is a closed and convex subset of X^*, and $G(\cdot)$ is u.s.c. in the sense that for every closed set $C \subseteq X^*$ $G^-(C) = \{x \in B : G(x) \cap C \neq \varnothing\}$ is closed in X.

By a well-known renorming theorem due to Troyanski [20], given a reflexive Banach space, we can always renorm it equivalently so that both X and X^* are locally uniformly convex. Thus without loss of generality we may assume from the beginning that both X and X^* are locally uniformly convex. Recall that a locally uniformly convex Banach space has the Kadec-property; i.e. if $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$ in X.

Define $J : X \to X^*$, the duality map, by

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|_*^2\}.$$

Then from Browder [11, Proposition 8], we have the following result:

Lemma 4 [11]. *$J(\cdot)$ is a well-defined, single-valued map from X onto X^*, which is a homeomorphism and is also monotone and of class $(S)_+$.*

Remark. It is not difficult to show that $J(\cdot)$ is uniformly continuous on a bounded subset of X.

Using the duality map $J(\cdot)$, we have the following criterion for maximal monotonicity (cf. Browder [5]).

Lemma 5. *A monotone operator $T : D \subseteq X \to 2^{X^*}\backslash\{\varnothing\}$ is maximal monotone if and only if for every $\lambda > 0$ (equivalently for some $\lambda > 0$) $R(T + \lambda J) = X^*$.*

The following approximate selection theorem due to Cellina [13] will be important in our extension of Browder's degree theory to a multivalued context.

Lemma 6 [13]. *If Y, V are Banach spaces, $B \subseteq Y$, and $G : B \to 2^V\backslash\{\varnothing\}$ is an u.s.c. multifunction with closed and convex values, then given $\varepsilon > 0$, there exists a continuous map $g_\varepsilon : B \to V$ such that*

$$g_\varepsilon(y) \in G((y + \mathring{B}_\varepsilon) \cap B) + \mathring{\widehat{B}}_\varepsilon$$

for all $y \in B$ and $g_\varepsilon(B) \subseteq \overline{\operatorname{conv}}\, G(B)$, with $\mathring{B}_\varepsilon = \{y \in Y : \|y\|_Y < \varepsilon\}$ and $\mathring{\widehat{B}}_\varepsilon = \{v \in V : \|v\|_V < \varepsilon\}$.

Remark. In particular, if $G(\cdot)$ is compact, then so is the approximate selector $g_\varepsilon(\cdot)$.

3. The degree for the mappings of the form $T + f + G$

Let X be a reflexive Banach space, equivalently renormed so that both X and X^* are locally, uniformly convex and let $J(\cdot)$ be the duality map corresponding to this locally uniformly convex norm. Assume that U is a bounded open set in X, $T : D \subseteq X \to 2^{X^*}\backslash\{\varnothing\}$ is maximal monotone, $f : \overline{U} \to X^*$ is a map of class $(S)_+$, and $G : \overline{U} \to 2^{X^*}\backslash\{\varnothing\}$ is a multifunction of class (P). In this section we will define a degree function $d(T + f + G, U, y_0)$ for all $y_0 \in X^*\backslash(T + f + G)(\partial U)$ and prove that such a degree function is unique among all possible degree functions satisfying the three characteristic properties of normalization, additivity on domain, and homotopy invariance, to be defined precisely in the present context in the sequel.

Recall that homotopy invariance is with respect to a certain class of permissible homotopies. We will now introduce those permissible homotopies for the maps T, f, and G. The permissible homotopies for T and f (see Definitions 6 and 7) are due to Browder [11, 12], while the permissible homotopies for G (see Definition 8) are a natural extension to multifunctions of the permissible homotopies for compact maps used in the Leray-Schauder degree theory.

Definition 7 [11]. A family of maximal monotone maps $\{T_t : t \in [0, 1]\}$ is said to be a "pseudomonotone homotopy" of maximal monotone maps if it satisfies the mutually equivalent conditions:

(i) Suppose that $t_n \to t$ in $[0, 1]$, $[x_n, x_n^*] \in \operatorname{Gr} T_{t_n}$ with $x_n \xrightarrow{w} x$ in X, $x_n^* \xrightarrow{w} x^*$ in X^*, and $\overline{\lim}\langle x_n^*, x_n\rangle \leq \langle x^*, x\rangle$. Then $[x, x^*] \in \operatorname{Gr} T_t$ and $\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle$.
(ii) $\varphi(t, x^*) = (T_t + J)^{-1}(x^*)$ is continuous from $[0, 1] \times X^*$ into X, with both X^* and X furnished with their norm topologies.
(iii) For each $x^* \in X^*$, $t \to \varphi(t, x^*) = (T_t + J)^{-1}(x^*)$ is continuous from $[0, 1]$ into X endowed with the norm topology.
(iv) Given $[x, x^*] \in \operatorname{Gr} T_t$ and $t_n \to t$ in $[0, 1]$, then there exists a sequence $[x_n, x_n^*] \in \operatorname{Gr} T_{t_n}$ such that $x_n \to x$ in X and $x_n^* \to x^*$ in X^* (i.e. $\operatorname{Gr} T_t \subseteq \underline{\lim} \operatorname{Gr} T_t$, which is of course equivalent to saying that $t \to \operatorname{Gr} T_t$ is l.s.c. from $[0, 1]$ into $2^{X \times X^*} \backslash \{\varnothing\}$).

The admissible homotopies for f are given in the next definition.

Definition 8 [12]. Let $\{f_t : t \in [0, 1]\}$ be a parameter family of maps from $\overline{U}$ into X^*. Then $\{f_t\}$ is said to be a "homotopy of class $(S)_+$", if for any $\{x_n\}_{n \geq 1} \subseteq \overline{U}$ for which we have $x_n \xrightarrow{w} x$ in X and for any $\{t_n\}_{n \geq 1} \subseteq [0, 1]$ such that $t_n \to t$ for which

$$\overline{\lim}\langle f_{t_n}(x_n), x_n - x\rangle \leq 0$$

we have that $x_n \to x$ in X and $f_{t_n}(x_n) \xrightarrow{w} f_t(x)$ in X^*.

Finally we introduce the family of admissible homotopies for the multivalued perturbation $G(\cdot)$.

Definition 9. A one-parameter family of multifunctions $G_t : \overline{U} \to 2^{X^*} \backslash \{\varnothing\}$, $t \in [0, 1]$, is said to be a "homotopy class (P)" if $(t, x) \to G_t(x)$ is u.s.c. from $[0, 1] \times \overline{U}$ into $2^{X^*} \backslash \{\varnothing\}$, for every $[t, x] \in T \times \overline{U}$ $G_t(x)$ is a closed and convex subset of X^*, and $\overline{\{\bigcup G_t(x) : t \in [0, 1], x \in \overline{U}\}}$ is compact in X^*.

The next proposition paves the way for the eventual definition of the degree function on maps of the form $T + f + G$ by producing a crucial approximation to it on which Browder's degree function can be defined. From Lemma 6, we know that if $G : \overline{U} \to 2^{X^*} \backslash \{\varnothing\}$ is a multifunction of class (P) and $\varepsilon > 0$, then we can find $g_\varepsilon : \overline{U} \to X^*$ a continuous function such that $g_\varepsilon(\overline{U}) \subseteq \overline{\operatorname{conv}}\, G(\overline{U})$ and for all $x \in \overline{U}$ $g_\varepsilon(x) \in G((x + \mathring{B}_\varepsilon) \cap \overline{U}) + \mathring{B}_\varepsilon^*$ where $\mathring{B}_\varepsilon = \{x \in X : \|x\| < \varepsilon\}$ and $\mathring{B}_\varepsilon^* = \{x \in X^* : \|x^*\|_* < \varepsilon\}$. In what follows $g_\varepsilon(\cdot)$ will denote this approximate selector of $G(\cdot)$.

Proposition 10. *Let U be a bounded open set in X, $T : D \subseteq X \to 2^{X^*} \backslash \{\varnothing\}$ a maximal monotone map with $0 \in T(0)$, $f : \overline{U} \to X^*$ a bounded map of class*

$(S)_+$, $G : \overline{U} \to 2^{X^*}\setminus\{\varnothing\}$ a multifunction of class (P). *Let $y_0^* \in X^*$ such that $y_0^* \notin (T+f+G)(\partial U)$. Then*

(i) *$y_0^* \notin (T+f+g_\varepsilon)(\partial U)$ for all $\varepsilon > 0$ small;*

(ii) *$f+g_\varepsilon$ is a mapping of class $(S)_+$ and so by Theorem* A, *the Browder degree $d(T+f+g_\varepsilon, U, y_0^*)$ is defined for all $\varepsilon > 0$ small;*

(iii) *there is $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$ and all approximations $g_\varepsilon(\cdot)$, $d(T+f+g_\varepsilon, U, y_0^*)$ has the same value (that is, $\{d(T+f+g_\varepsilon, U, y_0^*) : \varepsilon > 0$ small$\}$ stabilizes).*

Proof. From the remark following Lemma 6, we know that $g_\varepsilon(\cdot)$ is compact and so $f+g_\varepsilon$ is of class $(S)_+$. Now suppose that (i) were not true. Then we could find a sequence $\varepsilon_n \downarrow 0$ and $\{u_n\}_{n\geq 1} \subseteq \partial U$ with $u_n \xrightarrow{w} u$ in X such that

$$y_0^* \in (T+f+g_{\varepsilon_n})(u_n), \quad n \geq 1.$$

Let $v_n^* = y_0^* - (f(u_n) + g_{\varepsilon_n}(u_n))$. Then $v_n^* \in T(u_n)$. By passing to a subsequence if necessary, we may assume that $g_{\varepsilon_n}(u_n) \to g^*$, $f(u_n) \xrightarrow{w} f^*$, and $v_n^* \xrightarrow{w} v^*$ in X^*. Thus $v^* = y_0^* - (f^* + g^*)$. We have

$$\lim\langle v_n^* + f(u_n), u_n - u\rangle = \lim\langle y_0^* - g_{\varepsilon_n}(u_n), u_n - u\rangle = 0.$$

Since by hypothesis f is of class $(S)_+$ (thus demicontinuous), it is pseudomonotone and so $\underline{\lim}\langle f(u_n), u_n - u\rangle \geq 0$. Thus

$$\begin{aligned}&\overline{\lim}\langle v_n^*, u_n - u\rangle \leq 0\\ &\quad \Rightarrow v \in T(u) \text{ and } \langle v_n^*, u_n\rangle \to \langle v^*, u\rangle \quad (\text{since } T \text{ is maximal monotone}).\end{aligned}$$

Therefore $\langle f(u_n), u_n - u\rangle \to 0$ as $n \to \infty$ and so $u_n \to u$ in X and $f(u) = f^*$. Hence $u \in \partial U$. Since $g_{\varepsilon_n}(u_n) \in G((u_n + \mathring{B}_{\varepsilon_n}) \cap \overline{U}) + \mathring{B}_{\varepsilon_n}^*$, $u_n \to u$ in X, and G is of class (P), by standard arguments we can check that $g^* \in G(u)$. So finally we get that $y_0^* \in (T+f+G)(u)$ with $u \in \partial U$, a contradiction. Therefore we have established (i) and (ii).

To prove (iii), we proceed again by contradiction. So suppose that there exists $0 < \varepsilon_n \leq \delta_n \to 0$ such that

$$d(T+f+g_{\varepsilon_n}, U, y_0^*) \neq d(T+f+g_{\delta_n}, U, y_0^*).$$

Then from the homotopy invariance property of Browder's degree function (cf. Theorem A), we get $t_n \to t$ in $[0, 1]$ and $u_n \in \partial U$ such that

$$y_0^* = (T+f+t_n g_{\varepsilon_n} + (1-t_n) g_{\delta_n})(u_n).$$

Note that $(t_n g_{\varepsilon_n}(x) + (1-t_n) g_{\delta_n}(x)) \in G((x + \mathring{B}_{\delta_n}) \cap \overline{U}) + \mathring{B}_{\delta_n}^*$ for all $x \in \overline{U}$ and so arguing as in the proof of part (i), we get $y_0^* \in (T+f+G)(\partial U)$, a contradiction. So the proof is complete. Q.E.D.

In the light of this proposition, the following definition makes sense:

Definition 11. We define $d(T+f+G, U, y_0^*)$ to be the common value for $\varepsilon > 0$ sufficiently small of $d(T+f+g_\varepsilon, U, y_0^*)$ (this last degree being the Browder degree).

The next theorem shows that the degree function just defined has the three characteristic properties of normalization (with normalizing map the duality map J), of additivity on domain, and of homotopy invariance (with admissible homotopies being given by Definitions 7, 8, and 9).

Theorem 12. *The degree function defined by Definition* 10 *has the following properties*:

(i) *Normalization*: $d(J, U, y_0^*) = 1$ *for all* $y_0^* \in J(U)$.

(ii) *Additivity on domain*: *If* U_1, U_2 *are disjoint open subsets of* U *and* $y_0^* \notin (T+f+G)(\overline{U}\backslash(U_1 \cup U_2))$, *then*

$$d(T+f+G, U, y_0^*) = d(T+f+G, U_1, y_0^*) + d(T+f+G, U_2, y_0^*).$$

(iii) *Homotopy invariance*: *Let* $\{T_t\}_{t\in[0,1]}$ *be a pseudomonotone homotopy of maximal monotone maps from* X *into* 2^{X^*} *with* $0 \in T_t(0)$ *for all* $t \in [0, 1]$, $\{f_t\}_{t\in[0,1]}$ *is a homotopy of class* $(S)_+$ *of maps from* $\overline{U}$ *into a bounded subset of* X^*, *and* $\{G_t\}_{t\in[0,1]}$ *is a homotopy of class* (P) *of multifunctions from* $\overline{U}$ *into the nonempty, closed, and convex subsets of* X^*. *Let* $y^* : [0, 1] \to X^*$ *be a continuous map such that* $y^*(t) \notin (T_t + f_t + G_t)(\partial U)$ *for all* $t \in [0, 1]$. *Then* $d(T_t + f_t + G_t, U, y(t))$ *is independent of* $t \in [0, 1]$.

Proof. (i) This property follows immediately from Theorem A.

(ii) This property too follows from Theorem A, since by Definition 10, $d(T+f+G, U, y_0^*) = d(T+f+g_\varepsilon, U, y_0^*)$ for all $\varepsilon > 0$ sufficiently small and $f + g_\varepsilon$ is of class $(S)_+$ since it is a compact perturbation of a $(S)_+$ map.

(iii) Let $G(t, x) = G_t(x)$. Recalling (cf. Definition 8) that $(t, x) \to G(t, x)$ is u.s.c., we can apply Lemma 6 with $B = [0, 1] \times \overline{U}$ and, for any $\varepsilon > 0$, get a continuous function $g_\varepsilon(t, x)$ from B into $\overline{\text{conv}}\, G([0, 1], \overline{U})$ such that

$$g_\varepsilon(t, x) \in G(([t-\varepsilon, t+\varepsilon], x + \mathring{B}_\varepsilon) \cap B) + \mathring{B}_\varepsilon^* \quad \text{for all } (t, x) \in B.$$

We claim that for $\varepsilon > 0$ small enough, $y^*(t) \notin (T_t + f_t + g_{t,\varepsilon})(\partial U)$ for all $t \in [0, 1]$, with $g_{t,\varepsilon}(x) = g_\varepsilon(t, x)$. Assume the contrary. We then have $t_n \to t$ in $[0, 1]$, $\varepsilon_n \downarrow 0$, $\{u_n\}_{n\geq 1} \subseteq \partial U$ with $u_n \xrightarrow{w} u$ in X, and $y^*(t_n) \in (T_{t_n} + f_{t_n} + g_{t_n,\varepsilon_n})(u_n)$. Let $v_n^* = y^*(t_n) - (f_{t_n}(u_n) + g_{t_n,\varepsilon_n}(u_n))$. So $v_n^* \in T_{t_n}(u_n)$ for all $n \geq 1$. By passing to a subsequence if necessary, we may assume that $y^*(t_n) \to y^*(t)$, $v_n^* \xrightarrow{w} v^*$, $f_{t_n}(u_n) \xrightarrow{w} f^*$, and $g_{t_n,\varepsilon_n}(u_n) \to g^*$. Hence

$$\lim\langle v_n^* + f_{t_n}(u_n), u_n - u\rangle = 0.$$

Also since $\{f_t\}$ is a homotopy of class $(S)_+$, we have

$$\varliminf\langle f_{t_n}(u_n), u_n - u\rangle \geq 0.$$

So we get

$$\varlimsup\langle v_n^*, u_n - u\rangle \leq 0$$

and this by Definition 7 implies that $v^* \in T_t(u)$ and $\langle v_n^*, u_n\rangle \to \langle v^*, u\rangle$. Thus

$$\lim\langle f_{t_n}(u_n), u_n - u\rangle = 0$$

and so we have $u_n \to u$ in X and $f_{t_n}(u_n) \xrightarrow{w} f_t(u)$ in X^* (cf. Definition 8). Also it is easy to check that $g^* \in G_t(u)$. All these facts combined tell us that

$$y^*(t) \in (T_t + f_t + G_t)(u)$$

with $u \in \partial U$, which is a contradiction. So indeed for $\varepsilon > 0$ small enough, we have that $y^*(t) \notin (T_t + f_t + g_{t,\varepsilon})(\partial U)$ for all $t \in [0, 1]$.

It is routine to verify that $\{f_t + g_{t,\varepsilon}\}_{t\in[0,1]}$ is a homotopy of class $(S)_+$. So by Theorem A, we have that $d(T_t + f_t + g_{t,\varepsilon}, U, y^*(t))$ is independent of $t \in [0, 1]$ for $\varepsilon > 0$ small enough. It only remains to show that for $\varepsilon > 0$ sufficiently small and for each $t \in [0, 1]$ fixed, we have $d(T_t + f_t + g_{t,\varepsilon}, U, y^*(t)) = d(T_t + f_t + G_t, U, y^*(t))$.

Fix $t \in [0, 1]$. Then by hypothesis $G_t : \overline{U} \to 2^{X^*}\setminus\{\varnothing\}$ is a multifunction of class (P). Apply Lemma 5 to $G_t(\cdot)$ with $B = \overline{U}$, to get $g_\varepsilon : \overline{U} \to X^*$ a compact map which satisfies

$$g_\varepsilon(x) \in G_t((x + \mathring{B}_\varepsilon) \cap \overline{U}) + \mathring{B}^*_\varepsilon$$

for all $x \in \overline{U}$. Consider the affine homotopy $sg_\varepsilon + (1 - s)g_{t,\varepsilon}$ with variable $s \in [0, 1]$. The same arguments used before show that

$$y^*(t) \notin (T_t + f_t + sg_\varepsilon + (1 - s)g_{t,\varepsilon})(\partial U)$$

for $s \in [0, 1]$ and $\varepsilon > 0$ small enough. Then Theorem A tells us that

$$d(T_t + f_t + g_{t,\varepsilon}, U, y^*(t)) = d(T_t + f_t + g_\varepsilon, U, y^*(t))$$

for $\varepsilon > 0$ sufficiently small. But for $\varepsilon > 0$ sufficiently small the last degree equals $d(T_t + f_t + G_t, U, y^*(t))$ (cf. Definition 11). Consequently for every fixed $t \in [0, 1]$

$$d(T_t + f_t + G_t, U, y^*(t)) = d(T_t + f_t + g_{t,\varepsilon}, U, y^*(t))$$

for small $\varepsilon > 0$. Since $t \in [0, 1]$ was arbitrary, we have proved the homotopy invariance property of the degree function and so the proof of Theorem 12 is complete. Q.E.D.

Next we establish the uniqueness of the degree defined above with respect to the three properties of Theorem 12.

Theorem 13. *There exists exactly one degree function on the class of maps $T + f + G$, with T maximal monotone, f bounded and of class $(S)_+$, and G a multifunction of class* (P), *which satisfies the normalization and additivity properties of Theorem* 12 *and is also invariant under all affine homotopies of the form $(1-t)(T+f+G)+tf_1$ with $t \in [0, 1]$, T maximal monotone, f_1, f bounded and of class $(S)_+$, and G a multifunction of class* (P).

Proof. Let d_1 be such a degree function. By setting $G \equiv 0$, from Theorem A, we have that d_1 coincides with Browder's degree function, which is uniquely defined on maps of the form $T + f$. Using the above affine homotopy we will show that this unique identification carries on to the broader class $(T + f + G)$. Suppose $y_0^* \notin (T + f + G)(\partial U)$. Consider the affine homotopy

$$(1 - t)(T + f + G) + t(T_\varepsilon + f + g_\varepsilon)$$

with $T_\varepsilon = (T^{-1} + \varepsilon J^{-1})^{-1}$ and $g_\varepsilon(\cdot)$ is as always the compact selector of $G(\cdot)$ guaranteed by Lemma 6 such that $g_\varepsilon(x) \in G((x + \mathring{B}_\varepsilon) \cap \overline{U}) + \mathring{B}^*_\varepsilon$, for all $x \in \overline{U}$. Using Definition 7 we can easily check that $(T_\varepsilon + f)$ is of class (S_+); hence $(T_\varepsilon + f + g_\varepsilon)$ is of class $(S)_+$ (recall the class $(S)_+$ is closed under compact perturbations). Since we have

$$d(T_\varepsilon + f + g_\varepsilon, U, y_0^*) = d_1(T_\varepsilon + f + g_\varepsilon, U, y_0^*)$$

then $d(T+f+G, U, y_0^*) \neq d_1(T+f+G, U, y_0^*)$ would imply that there exist $\varepsilon_n \downarrow 0$, $\{u_n\}_{n\geq 1} \subseteq \partial U$ with $u_n \xrightarrow{w} u$, and $t_n \to t$ in $[0, 1]$ such that

$$y_0^* \in (1-t_n)T(u_n) + t_n T_{\varepsilon_n}(u_n) + (1-t_n)G(u_n) + t_n g_{\varepsilon_n}(u_n) + f(u_n).$$

Assume that $v_n^* \in T(u_n)$ and $g_n^* \in G(u_n)$ are such that

$$y_0^* = (1-t_n)v_n^* + t_n T_{\varepsilon_n}(u_n) + (1-t_n)g_n^* + t_n g_{\varepsilon_n}(u_n) + f(u_n),$$

by passing to a subsequence if necessary, we may assume that $f(u_n) \xrightarrow{w} f^*$ and $(1-t_n)g_n^* + t_n g_{\varepsilon_n}(u_n) \to h^*$ in X^*. In what follows, we use the arguments of Browder [11] (see the proof of Theorem 12). Let $w_n^* = T_{\varepsilon_n}(u_n)$, then $w_n^* \in T(u_n - \varepsilon_n J^{-1}(w_n^*))$. This and $v_n^* \in T(u_n)$ imply that $\varepsilon_n \|w_n^*\|_*^2 \leq \langle w_n^*, u_n\rangle$ and $0 \leq \langle v_n^*, u_n\rangle$. Thus we obtain

$$t_n\varepsilon_n\|w_n^*\|^2 \leq \langle y_0^* - (1-t_n)g_n^* - t_n g_{\varepsilon_n}(u_n) - f(u_n), u_n\rangle \leq M, \qquad M > 0.$$

Hence $\{t_n\varepsilon_n\|w_n^*\|^2\}_{n\geq 1}$ is bounded and so $t_n\varepsilon_n\|w_n^*\| \to 0$ as $n \to \infty$. Let $[x, x^*] \in \operatorname{Gr} T$. Then

$$\langle w_n^* - x^*, u_n - \varepsilon_n J^{-1}(w_n^*) - x\rangle \geq 0, \quad \langle v_n^* - x^*, u_n - x\rangle \geq 0.$$

Thus $\langle w_n^* - x^*, u_n - x\rangle \geq \langle w_n^* - x^*, \varepsilon_n J^{-1}(w_n^*)\rangle \geq -\varepsilon_n\|w_n^*\|_*\|x^*\|_*$. Consequently, we have

$$\langle y_0^* - (1-t_n)g_n^* - t_n g_{\varepsilon_n}(u_n) - f(u_n) - x^*, u_n - x\rangle \geq -t_n\varepsilon_n\|w_n^*\|_*\|x^*\|_* \to 0.$$

Hence if we let $z_n^* = y_0^* - [(1-t_n)g_n^* + t_n g_{\varepsilon_n}(u_n) + f(u_n)]$, then $z_n^* \xrightarrow{w} z^*$ with $z^* = y_0^* - (h^* + f^*)$ and

$$\underline{\lim}\langle z_n^* - x^*, u_n - x\rangle \geq 0$$

and the latter means that $\underline{\lim}\langle z_n^*, u_n\rangle \geq \langle x^*, u - x\rangle + \langle z^*, x\rangle$.

On the other hand, since $(1-t_n)g_n^* + t_n g_{\varepsilon_n}(u_n) \to h^*$ in X^*, we have

$$\lim\langle z_n^* + f(u_n), u_n - u\rangle = 0.$$

Also $\underline{\lim}\langle f(u_n), u_n - u\rangle \geq 0$, since $f(\cdot)$ is of class $(S)_+$. Therefore we get

$$\overline{\lim}\langle z_n^*, u_n - u\rangle \leq 0 \Rightarrow \overline{\lim}\langle z_n^*, u_n\rangle \leq \langle z^*, u\rangle.$$

Thus $\langle z^*, u\rangle \geq \overline{\lim}\langle z_n^*, u_n\rangle \geq \underline{\lim}\langle z_n^*, u_n\rangle \geq \langle x^*, u-x\rangle + \langle z^*, x\rangle$. It follows that $0 \leq \langle z^* - x^*, u - x\rangle$ for all $[x, x^*] \in \operatorname{Gr} T$. Because of the maximal monotonicity of T, $z^* \in T(u)$. Then, by replacing $[x, x^*]$ by $[u, z^*]$, we get $\langle z_n^*, u_n\rangle \to \langle z^*, u\rangle$. Hence

$$\lim\langle f(u_n), u_n - u\rangle = 0.$$

So we conclude that $u_n \to u$ in X; hence $u \in \partial U$ and $f(u_n) \xrightarrow{w} f(u) = f^*$ in X^* (since $f(\cdot)$ is demicontinuous being of class $(S)_+$; cf. Definition 2). Also it is straightforward to show that $h^* \in G(u)$. Thus $y_0^* \in (T+f+G)(u)$, with $u \in \partial U$, a contradiction. Therefore the two degrees coincide and so we have established the uniqueness of the degree function on maps of the form $(T+f+G)$. Q.E.D.

Remarks. (1) It is clear that the degree function of Definition 11 can be extended to the broader class of maps of the form $T+f+G$, with f being pseudomonotone and bounded by defining

$$d(T+f+G, U, y_0^*) = \lim_{\varepsilon\downarrow 0} d(T+f+\varepsilon J+G, U, y_0^*).$$

Note that for any $\varepsilon > 0$, $f+\varepsilon J$ is of class $(S)_+$. This extended degree function has properties similar to the degree function for pseudomonotone maps defined by Browder [11].

(2) The condition $0 \in T(0)$ can always be satisfied by appropriately translating the domain and the operator. More precisely if $[x_0, x_0^*] \in \operatorname{Gr} T$, define $T_1 : (D - x_0) \subseteq X \to 2^{X^*}\backslash\{\varnothing\}$ by $T_1(x) = T(x + x_0) - x_0^*$. Clearly $T_1(\cdot)$ is still maximal monotone (if $T(\cdot)$ is) and $0 \in T(0)$. For the permissible maximal monotone homotopies the condition $0 \in T_t(0)$ can be replaced by the requirement that $[x_0, x_0^*] \in \operatorname{Gr} T_t$ for every $t \in [0, 1]$.

4. Degree for maps of the form $f + N_G$

As we already pointed out in the introduction, the condition that $G : \overline{U} \to 2^{X^*}\backslash\{\varnothing\}$ is compact translates into some growth condition on G when applied to partial differential inclusions. In Theorem B, this restriction was replaced by a sign condition. In this section, we pursue this idea and achieve a two-fold extension of Theorem B. On the one hand, we allow a multivalued function $G(x, r)$ in place of $g(x, r)$ and on the other hand, we relax the sign condition.

Let $Z \subseteq \mathbb{R}^n$ be an open set and for $m \geq 1$, $1 < p < \infty$, let $X = W_0^{m,p}(Z)$. Then its dual is $X^* = W^{-m,q}(Z)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $G : Z \times \mathbb{R} \to 2^{\mathbb{R}}\backslash\{\varnothing\}$ be a multifunction with compact, convex values such that $(x, r) \to G(x, r)$ is measurable and $r \to G(x, r)$ is u.s.c. It is well-known that under these assumptions we can write $G(x, r) = [\varphi(x, r), \psi(x, r)] = \{h \in \mathbb{R} : \varphi(x, r) \leq h \leq \psi(x, r)\}$ and $(x, r) \to \varphi(x, r), \psi(x, r)$ are both measurable while $r \to -\varphi(x, r), \psi(x, r)$ are both u.s.c. We want to impose sign conditions on $G(x, r)$ and so we make the following definition:

Definition 14. A multifunction $G(x, r)$ is said to satisfy the "sign condition" if the following properties hold:

(i) $G(x, r) = [\varphi(x, r), \psi(x, r)]$ is measurable in (x, r) and u.s.c. in r and for each $s > 0$, there exists $h_s(\cdot) \in L^1_{\mathrm{loc}}(Z)$ such that for $|r| \leq s$

$$|\varphi(x, r)|, |\psi(x, r)| \leq h_s(x) \quad \text{a.e. on } Z;$$

(ii) for all $x \in Z$, $\varphi(x, r)r \geq 0$ for $r \leq 0$ and $\psi(x, r)r \geq 0$ for $r \geq 0$.

If $G(x, r)$ is single valued, this definition coincides with the sign condition of Browder [12] (cf. Definition 5). As we already indicated earlier, we want to relax this condition. So we introduce

Definition 15. A multifunction $G(x, r)$ is said to satisfy the "generalized sign condition" if the following properties hold:

(i) $G(x, r) = [\varphi(x, r), \psi(x, r)]$ is measurable in (x, r), u.s.c. in r, and for each $s > 0$ there exists $h_s \in L^1(Z)$ such that for $|r| \leq s$

$$|\varphi(x, r)|, |\psi(x, r)| \leq h_s(x) \quad \text{a.e. on } Z;$$

(ii) there is an $r_0 > 0$ such that for all $x \in Z$

$$\varphi(x, r)r \geq 0 \text{ if } r \leq -r_0 \quad \text{and} \quad \psi(x, r)r \geq 0 \text{ if } r \geq r_0.$$

Let $G : Z \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\varnothing\}$ be a multifunction satisfying Definition 14 or 15. We formally define the Nemitsky operator N_G from $D \subseteq X = W_0^{m,p}(Z)$ into $2^{X^*} \backslash \{\varnothing\}$ by

$$N_G(u) = \{v \in W^{-m,q}(Z) \cap L^1_{\mathrm{loc}}(Z) : v(x) \in G(x, u(x)) \text{ a.e. on } Z\},$$

with $D = \{u \in W_0^{m,p}(Z) : N_G(u) \neq \varnothing\}$.

A multifunction $G(x, r)$ which is only measurable in x and u.s.c. in r is not in general jointly measurable (cf. [18]). That is why we need to assume joint measurability of $G(x, r)$. Note that if $G(x, r) = g(x, r)$ is single-valued, then this joint measurability is automatically satisfied by the Carathéodory conditions; i.e. $g(x, r)$ is measurable in x, continuous in r. This is the case in Browder [12]. Also note that our joint measurability hypothesis implies that for every $u : Z \to \mathbb{R}$ measurable, $x \to G(x, u(x))$ is measurable and so it has a measurable selector.

Let U be a bounded open set in $X = W_0^{m,p}(Z)$, $f : \overline{U} \to X^* = W^{-m,q}(Z)$ a bounded map of class $(S)_+$, and $y_0^* \in X^* \backslash (f + N_G)(\partial U)$. In this section we will define a degree for the triples $(f + N_G, U, y_0^*)$ and prove that it has the three characteristic properties of normalization, additivity on domain, and homotopy invariance. Since the case when $G(x, r)$ satisfies the generalized sign condition is more complicated and the other case can be treated in a similar fashion, we only present a detailed analysis of the former.

The following proposition which will be needed in the sequel is due to Brezis and Browder [2].

Proposition 16. *Let u be an element of $W_0^{m,p}(Z)$, T an element of $W^{-m,q}(Z) \cap L^1_{\mathrm{loc}}(Z)$ such that $T(x)u(x) \geq h(x)$ for some h summable function on Z. Let $\langle T, u \rangle$ denote the distribution action of T on u (i.e. the duality brackets for $[T, u] \in X^* \times X$). Then $T(\cdot)u(\cdot)$ is summable on Z and*

$$\langle T, u \rangle = \int_Z T(x)u(x)\,dx.$$

A critical step in defining $d(f + N_G, U, y_0^*)$ is to approximate $G(x, r)$ by single-valued, Carathéodory functions $g_\varepsilon(x, r)$ which satisfy the corresponding sign conditions. This is done in the next proposition.

Proposition 17. *If $G : Z \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\varnothing\}$ is a multifunction which satisfies the generalized sign condition and $\varepsilon > 0$, then there exists $g_\varepsilon : Z \times \mathbb{R} \to \mathbb{R}$, a Carathéodory function satisfying:*

(i) $g_\varepsilon(x, r) \in G(x, r + \mathring{B}'_\varepsilon) + \mathring{B}'_\varepsilon$ *for all* $(x, r) \in Z \times \mathbb{R}$ *and with* $\mathring{B}'_\varepsilon = (-\varepsilon, \varepsilon)$;

(ii) *for each* $s > 0$, *there exists* $h_s \in L^1(Z)$ *such that for* $|r| \leq s$

$$|g_\varepsilon(x, r)| \leq h_s(x)$$

and $h_s(\cdot)$ *can be chosen independent of* $\varepsilon > 0$;

(iii) *for all* $x \in Z$ *and all* $|r| \geq r_0 + 1$, $g_\varepsilon(x, r)r \geq 0$.

Proof. Let $\mu : Z \to \mathbb{R}$ be a continuous function such that $0 < \mu(x) \leq 1$ for all $x \in Z$ and $\int_Z \mu(x)\,dx < \infty$.

Step 1: Define $\varphi^*, \psi^* : Z \times \mathbb{R} \to \mathbb{R}$ by

$$\varphi^*(x, r) = \begin{cases} \varphi(x, r), & r \leq r_0, \\ \max[0, \varphi(x, r)], & r > r_0, \end{cases}$$

and

$$\psi^*(x, r) = \begin{cases} \min[0, \psi(x, r)], & r < -r_0, \\ \psi(x, r), & r \geq -r_0, \end{cases}$$

and $G^* : Z \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\varnothing\}$ by $G^*(x, r) = [\varphi^*(x, r), \psi^*(x, r)]$. It is clear that $G^*(x, r) \subseteq G(x, r)$ and that $G^*(x, r)$ is measurable in (x, r) and u.s.c. in r.

Fix $x \in Z$ and apply Lemma 6 on $G^*(x, \cdot)$ with $\delta > 0$ to get $\eta_\delta : \mathbb{R} \to \mathbb{R}$ a continuous map such that

$$\eta_\delta(r) \in G^*(x, r + \dot{B}'_\delta) + \dot{B}'_\delta$$

for all $r \in \mathbb{R}$ (recall $\dot{B}'_\delta = (-\delta, \delta)$). Take $\delta \leq \min[\frac{1}{2}, \varepsilon\mu(x)]$. We then have

$$\max\{y \in \mathbb{R} : y \in G^*(x, r + \dot{B}'_\delta)\} \leq 0 \quad \text{for } r < -(r_0 + \tfrac{1}{2})$$

and

$$\min\{y \in \mathbb{R} : y \in G^*(x, r + \dot{B}'_\delta)\} \geq 0 \quad \text{for } r > r_0 + \tfrac{1}{2}.$$

Define $\eta^*_\delta : \mathbb{R} \to \mathbb{R}$ by

$$\eta^*_\delta(r) = \begin{cases} \min[0, \eta_\delta(r)], & r \leq -(r_0 + 1), \\ \eta_\delta(r), & -(r_0 + \frac{1}{2}) \leq r \leq (r_0 + \frac{1}{2}), \\ \max[0, \eta_\delta(r)], & r \geq r_0 + 1, \end{cases}$$

and on the intervals $-(r_0 + 1) \leq r \leq -(r_0 + \frac{1}{2})$ and $(r_0 + \frac{1}{2}) \leq r \leq (r_0 + 1)$, use line segments to make continuous connections. It is easy to see that $\eta^*_\delta(\cdot)$ is continuous, $\eta^*_\delta(r)r \geq 0$ for $|r| \geq r_0 + 1$, and

$$\eta^*_\delta(r) \in G^*(x, r + \dot{B}'_\varepsilon) + \mu(x)\dot{B}'_\varepsilon.$$

Hence $\eta^*_\delta(r) \in G(x, r + \dot{B}'_\varepsilon) + \mu(x)\dot{B}'_\varepsilon$.

Step 2. Define a multifunction $\Gamma_\varepsilon : Z \to 2^{C(\mathbb{R}, \mathbb{R})}$ by

$$\Gamma_\varepsilon(x) = \{\eta \in C(\mathbb{R}, \mathbb{R}) : \eta(r) \in G(x, r + \dot{B}'_\varepsilon) + \mu(x)\dot{B}'_\varepsilon,\ \eta(r)r \geq 0 \text{ for } |r| \geq r_0 + 1\}.$$

From Step 1 above, we know that for every $x \in Z$, $\Gamma_\varepsilon(x) \neq \varnothing$.

Let $\widehat{G}(x, r) = G(x, r + B'_\varepsilon) + \mu(x)B'_\varepsilon$, $B'_\varepsilon = [-\varepsilon, \varepsilon]$. Since $r \to G(x, r)$ is u.s.c. so is $r \to \widehat{G}(x, r)$ and $\widehat{G}(x, r)$ is a bounded closed interval in $\mathbb{R}$. For any $v \in \mathbb{R}$, $\sigma(v, \widehat{G}(x, r)) = \sup[\sigma(v, G(x, r + u)) : u \in B'_\varepsilon] + \varepsilon\mu(x)$, where σ is the support function. Thus $x \to \sigma(v, \widehat{G}(x, r))$ is Lebesgue measurable and hence so is $x \to \widehat{G}(x, r)$.

Let $G(\Gamma_\varepsilon)$ be the graph of $\Gamma_\varepsilon(\cdot)$. We have

$$G(\Gamma_\varepsilon) = \{[x, \eta] \in Z \times C(\mathbb{R}, \mathbb{R}) : d(\eta(r), \widehat{G}(x, r)) = 0 \\ \text{for all } r \in \mathbb{R}, \text{ and } \eta(r)r \geq 0 \text{ for } |r| \geq r_0 + 1\}.$$

Assume that $\{r_n\}_{n\geq 1}$ is an enumeration of the rationals in $\mathbb{R}$ and $\{s_m\}_{m\geq 1}$ an enumeration of the rationals in $|r| \geq r_0 + 1$. Note that since $\widehat{G}(x, \cdot)$ is u.s.c., $r \to d(\eta(r), \widehat{G}(x, r))$ is l.s.c. for any $\eta \in C(\mathbb{R}, \mathbb{R})$. So we can write

$$G(\Gamma_\varepsilon) = \bigcap_{n\geq 1} \bigcap_{m\geq 1} \{[x, \eta] \in Z \times C(\mathbb{R}, \mathbb{R}) : d(\eta(r_n), \widehat{G}(x, r_n)) = 0,\ \eta(s_m)s_m \geq 0\}.$$

For each $n \geq 1$, $(x, \eta) \to d(\eta(r_n), \widehat{G}(x, r_n))$ is a Carathéodory function from $Z \times C(\mathbb{R}, \mathbb{R})$ into $\mathbb{R}_+$ (recall $C(\mathbb{R}, \mathbb{R})$ is a Fréchet space). So it is jointly measurable and therefore $\{(x, \eta) \in Z \times C(\mathbb{R}, \mathbb{R}) : d(\eta(r_n), \widehat{G}(x, r_n)) = 0\} \in \mathscr{L}(Z) \times B(C(\mathbb{R}, \mathbb{R}))$ for every $n \geq 1$ with $\mathscr{L}(Z)$ being the Lebesgue σ-field of Z and $B(C(\mathbb{R}, \mathbb{R}))$ the Borel σ-field of $C(\mathbb{R}, \mathbb{R})$. Consequently, $G(\Gamma_\varepsilon) \in \mathscr{L}(Z) \times B(C(\mathbb{R}, \mathbb{R}))$. Applying Aumann's selection theorem (cf. Wagner [21, Theorem 5.10]) we get $\gamma_\varepsilon : Z \to C(\mathbb{R}, \mathbb{R})$ a Lebesgue measurable map such that $\gamma_\varepsilon(x) \in \Gamma_\varepsilon(x)$ for all $x \in Z$.

Set $g_\varepsilon(x, r) = (\gamma_\varepsilon(x))(r)$. Then $g_\varepsilon : Z \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies conditions (i) and (iii) of the proposition. It is easy to see that condition (ii) is also satisfied and in fact $h_s(\cdot)$ can be chosen independent of $\varepsilon > 0$ small, since $\mu(\cdot) \in L^1(Z)$. Q.E.D.

Let $g_\varepsilon(x, r)$ be the Carathéodory approximate selector obtained in the above proposition. We need yet another approximation method, namely the truncation procedure on $g_\varepsilon(x, r)$.

Definition 18. Let $\{Z_k\}_{k\geq 1}$ be an increasing sequence of relatively compact, open subsets of Z such that $Z = \bigcup_{k=1}^\infty Z_k$. Let $\xi_k(\cdot)$ be the characteristic function of Z_k. Consider the truncation of g_ε at level k; i.e.

$$g_\varepsilon^k(x, r) = \begin{cases} g_\varepsilon(x, r) & \text{if } |g_\varepsilon(x, r)| \leq k, \\ k \operatorname{sign}(g_\varepsilon(x, r)) & \text{if } |g_\varepsilon(x, r)| \geq k. \end{cases}$$

We define the kth-approximant $N_\varepsilon^k = N_\varepsilon^k(g_\varepsilon)$ of the Nemitsky operator N_{g_ε}, as a map from X into X^*, by

$$N_\varepsilon^k(u)(x) = \xi_k g_\varepsilon^k(x, u(x)).$$

It is then clear that each N_ε^k is a compact map of X into $X^* \cap L^\infty(Z)$.

In what follows, $g_\varepsilon(x, r)$ will be the Carathéodory approximate selector of G guaranteed by Proposition 17 and $g_\varepsilon^k(x, r)$ the corresponding truncation and N_ε^k its Nemitsky operator. Also by $\lambda(\cdot)$ we denote the Lebesgue measure on the set Z.

Theorem 19. *If U is a bounded open set of $X = W_0^{m,p}(Z)$, f a bounded map of class $(S)_+$ from $\overline{U}$ into $X^* = W^{-m,q}(Z)$, $G : Z \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\varnothing\}$ is a multifunction satisfying the generalized sign condition, and $y_0^* \in X^*$ such that $y_0^* \notin (f + N_G)(\partial U)$, then*

(i) *$y_0^* \notin (f + N_\varepsilon^k)(\partial U)$ for $k \geq 1$ sufficiently large and ε sufficiently small. Since $(f + N_\varepsilon^k)$ is of class $(S)_+$, the Browder degree $d(f + N_\varepsilon^k, U, y_0^*)$ is well defined;*

(ii) *for $k \geq 1$ sufficiently large and $\varepsilon > 0$ sufficiently small, $d(f + N_\varepsilon^k, U, y_0^*)$ is independent of k, ε, and the selector $g_\varepsilon(\cdot)$ from Proposition 17. So we can define this ultimate common value to be the degree $d(f + N_G, U, y_0^*)$.*

Proof. (i) Suppose that the conclusion was false. Then we can find $\{u_k\}_{k\geq 1} \subseteq \partial U$ and $\varepsilon_k \downarrow 0$ such that

$$f(u_k) + N_{\varepsilon_k}^k(u_k) = y_0^*.$$

As before, by passing to a subsequence if necessary, we may assume that $u_k \xrightarrow{w} u$ in X, $f(u_k) \xrightarrow{w} f^*$, and $N_{\varepsilon_k}^k(u_k) \xrightarrow{w} v^*$ in X^*. For any bounded subdomain Z' of Z, by Sobolev's embedding theorem we know that $W_0^{m,p}(Z')$ embeds compactly in $L^1(Z')$. Hence we may assume that $u_k(x) \to u(x)$ a.e. on Z.

Contrary to the single-valued situation considered by Browder [12] (cf. Theorem B), where $N_k(u_k) \to N_g(u)$ in $L^1(Z')$, in the present multivalued context, $N_{\varepsilon_k}^k(u_k)(\cdot)$ is not pointwise convergent in general. It is only weakly convergent in $L^1(Z)$ as we will show next. Nevertheless, using this weak convergence of $\{N_{\varepsilon_k}^k(u_k)(\cdot)\}_{k\geq 1}$ in $L^1(Z)$ we will arrive at a contradiction, establishing part (i) of our theorem.

First we will show that

$$\sup_{k\geq 1} \int_Z |N_{\varepsilon_k}^k(u_k)(x)|\, dx < \infty .$$

To this end, let $Z(k) = \{x \in Z : |u_k(z)| \leq r_0 + 1\}$. Observe that for every $k \geq 1$ and every $x \in Z \backslash Z(k)$, we have $0 \leq N_{\varepsilon_k}^k(u_k)(x)u_k(x)$. Then we have

$$\begin{aligned}
M &\geq \int_Z N_{\varepsilon_k}^k(u_k)(x)u_k(x)\, dx \\
&= \int_{Z(k)} N_{\varepsilon_k}^k(u_k)(x)u_k(x)\, dx + \int_{Z\backslash Z(k)} N_{\varepsilon_k}^k(u_k)(x)u_k(x)\, dx \\
&\geq -(r_0+1)\int_{Z(k)} h_{r_0+1}(x)\, dx + (r_0+1)\int_{Z\backslash Z(k)} |N_{\varepsilon_k}^k(u_k)(x)|\, dx \\
&\Rightarrow \int_{Z\backslash Z(k)} |N_{\varepsilon_k}^k(u_k)(x)|\, dx \leq \frac{M}{r_0+1} + \int_Z h_{r_0+1}(x)\, dx \\
&\Rightarrow \int_Z |N_{\varepsilon_k}^k(u_k)(x)|\, dx \leq \frac{M}{r_0+1} + 2\int_Z h_{r_0+1}(x)\, dx .
\end{aligned}$$

Now let $\widehat{Z} \subseteq Z$ be measurable with $\lambda(\widehat{Z}) < \infty$ and $l > r_0 + 1$. Let $\widehat{Z}_-^k = \{x \in \widehat{Z} : |u_k(x)| \leq l\}$ and $\widehat{Z}_+^k = \{x \in \widehat{Z} : |u_k(z)| > l\}$. We have

$$\begin{aligned}
\int_{\widehat{Z}} |N_{\varepsilon_k}^k(u_k)(x)|\, dx &= \int_{\widehat{Z}_-^k} |N_{\varepsilon_k}^k(u_k)(x)|\, dx + \int_{\widehat{Z}_+^k} |N_{\varepsilon_k}^k(u_k)(x)|\, dx \\
&\leq \int_{\widehat{Z}} h_l(x)\, dx + \frac{1}{l}\left(M + (r_0+1)\int_Z h_{r_0+1}(x)\, dx\right) .
\end{aligned}$$

Since $l > r_0 + 1$ was arbitrary and $h_l(\cdot) \in L^1(Z)$, it is immediate from the above inequality that $\sup_{k\geq 1} \int_{Z'} |N_{\varepsilon_k}^k(u_k)(x)|\, dx \to 0$ as $\lambda(Z') \to 0$, and for every $\varepsilon > 0$ there is a $\widehat{Z} \subseteq Z$, $\lambda(\widehat{Z}) < \infty$ such that $\int_{Z\cap\widehat{Z}^c} |N_{\varepsilon_k}^k(u_k)(x)|\, dx < \varepsilon$. So finally invoking the Dunford-Pettis theorem (see Dunford-Schwartz [15, p. 347]), we get that $\{N_{\varepsilon_k}^k(u_k)(\cdot)\}_{k\geq 1}$ is relatively sequentially weakly compact in $L^1(Z)$. Hence we may assume that $N_{\varepsilon_k}^k(u_k) \xrightarrow{w} v_1^*$ in $L^1(Z)$.

Now we will show that $v_1^* u \in L^1(Z)$ and that the following inequality holds:

$$\int_Z v_1^*(x)u(x)\, dx \leq \underline{\lim} \int_Z N_{\varepsilon_k}^k(u_k)(x)u_k(x)\, dx .$$

Define $Z_- = \{x \in Z : |u(x)| \leq r_0 + 1\}$ and $Z_+ = \{x \in Z : |u(x)| > r_0 + 1\}$. Since $v_1^* \in L^1(Z)$, it is clear then that $v_1^* u \in L^1(Z_-)$. We claim that

$v_1^*(x)u(x) \geq 0$ a.e. on Z_+. In fact for any $Z_1 \subseteq Z_+$ with $\lambda(Z_1) < \infty$ and any $\delta > 0$ by Egorov's and Lusin's theorems we can find a closed set $Z_\delta \subseteq Z_1$ with $\lambda(Z_1 \setminus Z_\delta) < \delta$ such that $u_k(z) \to u(z)$ uniformly on Z_δ and $u(\cdot)$ is continuous on Z_δ. Hence $u_k \to u$ in $L^\infty(Z_\delta)$ and so since $N_{\varepsilon_k}^k(u_k) \xrightarrow{w} v_1^*$ in $L^1(Z_\delta)$ we get that for any $B \subseteq Z_\delta$ measurable

$$\int_B v_1^*(x)u(x)\,dx = \lim \int_B N_{\varepsilon_k}^k(u_k)(x)u_k(x)\,dx \geq 0$$
$$\Rightarrow 0 \leq v_1^*(x)u(x) \quad \text{a.e. on } Z_\delta .$$

Since $\chi_{Z_\delta}(x)v_1^*(x)u(x)$ converges to $\chi_{Z_1}(x)v_1^*(x)u(x)$ in λ-measure we get that $0 \leq v_1^*(x)u(x)$ a.e. on Z_1. Finally recall that $Z_1 \subseteq Z_+$ with $\lambda(Z_1) < \infty$ was arbitrary and Z_+ is σ-finite to conclude that $0 \leq v_1^*(x)u(x)$ a.e. on Z_+.

Next we prove that $\int_{Z_+} v_1^*(x)u(x)\,dx \leq \underline{\lim} \int_{Z_+} N_{\varepsilon_k}^k(u_k)(x)u_k(x)\,dx$. For any $\delta > 0$, there exists $Z_1 \subseteq Z_+$ such that $\lambda(Z_1) < \infty$ and

$$\sup_{k \geq 1} \int_{Z_+ \setminus Z_1} |N_{\varepsilon_k}^k(u_k)(x)|\,dx < \infty .$$

If $Z_\delta \subseteq Z_1$ is as above, we define

$$Z_-^k(\delta) = \{x \in Z_+ \cap Z_\delta^c : |u_k(x)| \leq r_0 + 1\}$$

and

$$Z_+^k(\delta) = \{x \in Z_+ \cap Z_\delta^c : |u_k(x)| > r_0 + 1\} .$$

Then $|N_{\varepsilon_k}^k(u_k)(x)u_k(x)| \leq (r_0+1)h_{r_0+1}(x)$ on $Z_-^k(\delta)$ and $N_{\varepsilon_k}^k(u_k)(x)u_k(x) \geq 0$ on $Z_+^k(\delta)$. Clearly $\lambda(Z_1 \cap Z_-^k(\delta)) \leq \lambda(Z_1 \cap Z_\delta^c) < \delta$. Since $h_{r_0+1}(\cdot) \in L^1(Z)$, we have

$$\sup_{k \geq 1} \left[(r_0 + 1) \int_{Z_1 \cap Z_-^k(\delta)} h_{r_0+1}(x)\,dx \right] = \rho(\delta) \to 0$$

as $\delta \downarrow 0$. Thus

$$\begin{aligned}
&\underline{\lim} \int_{Z_+} N_{\varepsilon_k}^k(u_k)(x)u_k(x)\,dx \\
&= \underline{\lim} \left[\int_{Z_\delta} N_{\varepsilon_k}^k(u_k)(x)u_k(x)\,dx + \int_{Z_-^k(\delta)} N_{\varepsilon_k}^k(u_k)(x)u_k(x)\,dx \right. \\
&\qquad\qquad \left. + \int_{Z_+^k(\delta)} N_{\varepsilon_k}^k(u_k)(x)u_k(x)\,dx \right] \\
&\geq \lim \int_{Z_\delta} N_{\varepsilon_k}^k(u_k)(x)u_k(x)\,dx - (r_0 + 1) \int_{Z_1 \cap Z_-^k(\delta)} h_{r_0+1}(x)\,dx \\
&\quad - \sup_{k \geq 1} \int_{Z_+ \setminus Z_1} |N_{\varepsilon_k}^k(u_k)(x)|\,dx \\
&\geq \int_{Z_\delta} v_1^*(x)u(x)\,dx - \rho(\delta) - \delta .
\end{aligned}$$

Since $\delta > 0$ was arbitrary, $v_1^*(x)u(x) \geq 0$ a.e. on Z_+ and $\rho(\delta) \to 0$ as $\delta \downarrow 0$, we get that $v_1^*u \in L^1(Z_+)$ and

$$\int_{Z_+} v_1^*(x)u(x)\,dx \leq \underline{\lim} \int_{Z_+} N_{\varepsilon_k}^k(u_k)(x)u_k(x)\,dx .$$

Now we will show that the same inequality holds also over Z_-. For any $\delta > 0$, find a measurable set $Z_1 \subseteq Z_-$ with $\lambda(Z_1) < \infty$ such that

$$\int_{Z_- \cap Z_1^c} |v_1^*(x)u(x)|\,dx < \delta$$

and

$$(r_0 + 1)\int_{Z_- \cap Z_1^c} h_{r_0+1}(x)\,dx < \delta.$$

Recall that $v_1^* u \in L^1(Z_-)$ and $h_{r_0+1}(\cdot) \in L^1(Z)$ and so the above choice is possible. Let $\widehat{Z}_\delta \subseteq Z_1$ with $\lambda(Z_1 \backslash \widehat{Z}_\delta) < \delta$ such that $u_k(x) \to u(x)$ uniformly on $\widehat{Z}_\delta$ and $u(\cdot)$ is continuous on $\widehat{Z}_\delta$. Define

$$\widehat{Z}_-^k(\delta) = \{x \in Z_- \cap \widehat{Z}_\delta^c : |u_k(x)| \le r_0 + 1\}$$

and

$$\widehat{Z}_+^k(\delta) = \{x \in Z_- \cap \widehat{Z}_\delta^c : |u_k(x)| > r_0 + 1\}.$$

Hence since $\int_{\widehat{Z}_\delta} v_1^*(x)u(x)\,dx = \lim \int_{\widehat{Z}_\delta} N_{\varepsilon_k}^k(u_k)(x)u_k(x)\,dx$, we have

$$\begin{aligned}\int_Z v_1^*(x)u(x)\,dx &\le \lim \int_{\widehat{Z}_\delta} N_{\varepsilon_k}^k(u_k)(x)u_k(x)\,dx + \int_{Z_- \cap \widehat{Z}_\delta^c} |v_1^*(x)u(x)|\,dx \\ &\le \underline{\lim} \int_{Z_-} N_{\varepsilon_k}^k(u_k)(x)u_k(x)\,dx + \rho(\delta) + \int_{Z_- \cap \widehat{Z}_\delta^c} |v_1^*(x)u(x)|\,dx\end{aligned}$$

where $\rho(\delta) = \sup_{k \ge 1}[(r_0 + 1)\int_{\widehat{Z}_-^k(\delta)} h_{r_0+1}(x)\,dx]$. Recalling our initial choice of $Z_1 \subseteq Z$, we see that $\rho(\delta) \to 0$ and $\int_{Z_- \cap \widehat{Z}_\delta^c} |v_1^*(x)u(x)|\,dx \to 0$ as $\delta \downarrow 0$. So we have

$$\int_{Z_-} v_1^*(x)u(x)\,dx \le \underline{\lim} \int_{Z_-} N_{\varepsilon_k}^k(u_k)(x)u_k(x)\,dx.$$

Therefore, we finally have that $v_1^* u \in L^1(Z)$ and

$$\int_Z v_1^*(x)u(x)\,dx \le \underline{\lim} \int_Z N_{\varepsilon_k}^k(u_k)(x)u_k(x)\,dx.$$

Recapitulating, we have that $N_{\varepsilon_k}^k(u_k) \xrightarrow{w} v_1^*$ in $L^1(Z)$ and $N_{\varepsilon_k}^k(u_k) \xrightarrow{w} v^*$ in $X^* = W^{-m,q}(Z)$. Since both modes of convergence imply weak convergence in the space of distributions $\mathscr{D}(Z)'$, we get that $v_1^* = v^* = y_0^* - f^*$ and so $v_1^* \in W^{-m,q}(Z)$.

Define $h : Z \to \mathbb{R}$ by

$$h(x) = \begin{cases} 0 & \text{if } |u(x)| > r_0 + 1, \\ v_1^*(x)u(x) & \text{if } |u(x)| \le r_0 + 1. \end{cases}$$

Then $h \in L^1(Z)$ and $v_1^*(x)u(x) \ge h(x)$ a.e. on Z. So by Proposition 16, we have that

$$\langle v^*, u\rangle = \int_Z v_1(x)u(x)\,dx.$$

Hence, since $y_0^* = f(u_k) + N_{\varepsilon_k}^k(u_k)$, we have

$$\begin{aligned}\overline{\lim}\langle f(u_k), u_k - u\rangle &= \langle y_0^* - f^*, u\rangle - \underline{\lim} \int_Z N_{\varepsilon_k}^k(u_k)(x)u_k(x)\,dx \\ &\leq \langle y_0^* - f^*, u\rangle - \int_Z v_1^*(x)u(x)\,dx \\ &= \langle y_0^* - f^*, u\rangle - \langle v^*, u\rangle = 0.\end{aligned}$$

Since f is a map of class $(S)_+$, we get that $u_k \to u$ in $W_0^{m,p}(Z)$ and so $u \in \partial U$. Furthermore, $f(u_k) \xrightarrow{w} f(u)$ in X^* and so $f^* = f(u)$. Also it is straightforward to check that $v^* \in N_G(u)$. Thus we have shown that $y_0^* \in (f + N_G)(u)$ with $u \in \partial U$, a contradiction. This completes the proof of part (i) of the theorem.

(ii) Again we proceed by contradiction. Then we can find sequences $n_k \geq 1$, $\varepsilon_k > 0$, $\delta_k > 0$ such that $n_k \to \infty$, $\varepsilon_k \downarrow 0$, and $\delta_k \downarrow 0$ as $k \to \infty$ and furthermore

$$d(f + N_{\varepsilon_k}^k, U, y_0^*) \neq d(f + N_{\delta_k}^{n_k}, U, y_0^*).$$

By the homotopy invariance property of the degree function for affine homotopies of class $(S)_+$, we know that we can find $s_k = [0, 1]$ and $u_k \in \partial U$ such that

$$f(u_k) + (1 - s_k)N_{\varepsilon_k}^k(u_k) + s_k N_{\delta_k}^k(u_k) = y_0^*.$$

Without loss of generality, we may assume that the above equation holds for all $k \geq 1$. Let $v_k^* = (1 - s_k)N_{\varepsilon_k}^k(u_k) + s_k N_{\delta_k}^k(u_k)$. We may assume that $s_k \to s$ in $[0, 1]$, $u_k \xrightarrow{w} u$ in X and $f(u_k) \xrightarrow{w} f^*$, $v_k^* \xrightarrow{w} v^*$ in X^*. We have

$$\int_Z v_k^*(x)u_k(x)\,dx = \langle y_0^* - f(u_k), u_k\rangle \leq M.$$

Let $\hat{g}_k(x, r) = (1 - s_k)\xi_k(x)g_{\varepsilon_k}^k(x, r) + s_k\xi_k(x)g_{\delta_k}^{n_k}(x, r)$. Then $v_k^*(x) = \hat{g}_k(x, u_k(x))$. It is easy to see that $\hat{g}_k(x, r)$ satisfies (i)–(iii) of Proposition 16, with $g_\varepsilon(x, r)$ replaced by $\hat{g}_k(x, r)$ and ε replaced by $\max[\varepsilon_k, \delta_k]$.

With the same argument as in part (i), we can show that $v_k^* \xrightarrow{w} v_1^*$ in $L^1(Z)$, $v_1^*(x)u(x) \geq 0$ a.e. on the set $\{x \in Z : |u(x)| > r_0 + 1\}$, $v_1^* u \in L^1(Z)$, and

$$\int_Z v_1^*(x)u(x)\,dx \leq \underline{\lim} \int_Z v_k^*(x)u_k(x)\,dx$$

(recall that because of the compact embedding of $W_0^{m,p}(Z')$ into $L^1(Z')$ for any $Z' \subseteq Z$ bounded, we may assume that $u_k(x) \to u(x)$ a.e. on Z). As before we get $v^* = v_1^*$ and $\langle v^*, u\rangle = \int_Z v_1^*(x)u(x)\,dx$ and $v_1^* \in N_G(u)$. Finally since f is of class $(S)_+$ and $\overline{\lim}\langle f(u_k), u_k - u\rangle \leq 0$, we have $u_k \to u$ in X; hence $u \in \partial U$ and $f(u_k) \xrightarrow{w} f(u) = f^*$. Thus we get $y_0^* \in (f + N_G)(u)$ with $u \in \partial U$, which is a contradiction.

Therefore, we have proved part (ii) and the proof of the theorem is complete. Q.E.D.

For the degree function established with the previous theorem, we will prove the three characteristic properties. For this we need to introduce the permissible homotopies for the multifunction $G(x, r)$.

Definition 20. Let $\{G_t(x, r)\}_{t\in[0,1]}$ be a family of multifunctions from $Z \times \mathbb{R}$ into $2^{\mathbb{R}}\backslash\{\varnothing\}$. Such a family is said to be a "permissible homotopy of multifunctions" satisfying the generalized sign condition, if the following conditions are satisfied:

(i) $G_t(x, r) = [\varphi(x, r, t), \psi(x, r, t)]$ is measurable in (x, r, t) and u.s.c. in (r, t); for each $s > 0$, there exists $h_s \in L^1(Z)$ such that for $|r| \leq s$ and all $t \in [0, 1]$

$$\max[|\varphi(x, r, t)|, |\psi(x, r, t)|] \leq h_s(x).$$

(ii) There is an $r_0 > 0$ such that for all $(x, t) \in Z \times [0, 1]$ we have

$$\varphi(x, r, t)r \geq 0 \text{ for } r \leq -r_0, \qquad \psi(x, r, t)r \geq 0 \text{ for } r \geq r_0.$$

Having defined the permissible homotopies for the multifunction $G(x, r)$ we can now introduce the permissible homotopies for the degree function defined by Theorem 19.

Definition 21. The class H of permissible homotopies of maps of the form $f + N_G$ consists of all homotopies $h_t = (f_t + N_{G_t})$, $t \in [0, 1]$, with $\{f_t\}_{t\in[0,1]}$ a bounded homotopy of class $(S)_+$ and $\{G_t\}_{t\in[0,1]}$ a permissible homotopy of multifunctions satisfying the generalized sign condition.

Theorem 22. *The degree function defined in Theorem* 18 *has the following properties*:

(i) *Normalization*: $d(J, U, y_0^*) = 1$ *if* $y_0^* \in J(U)$.

(ii) *Additivity on domain*: *If* U_1, U_2 *are disjoint open subsets of* U *such that* $y_0^* \notin (f + N_G)(\overline{U}\backslash(U_1 \cup U_2))$, *then*

$$d(f + N_G, U, y_0^*) = d(f + N_G, U_1, y_0^*) + d(f + N_G, U_2, y_0^*).$$

(iii) *Homotopy invariance*: *Let* $\{h_t = f_t + N_{G_t}\}_{t\in[0,1]}$ *be a homotopy in the class* H *and let* $y^* : [0, 1] \to X^*$ *a continuous map such that* $y_t^* \notin (f_t + N_{G_t})(\partial U)$ *for all* $t \in [0, 1]$. *Then* $d(f_t + N_{G_t}, U, y_t^*)$ *is independent of* $t \in [0, 1]$.

Proof. Properties (i) and (ii) are obvious. To establish property (iii), first we obtain a single-valued approximate selector $g_\varepsilon(x, r, t)$ of $G_t(x, r)$ which is measurable in x and continuous in (r, t) (cf. Proposition 17) and satisfies all conditions of Proposition 17 uniformly in $t \in [0, 1]$, and then repeat the arguments employed in the proof of Theorem 19, using the fact that Browder's degree function on maps of class $(S)_+$ is homotopy invariant. Q.E.D.

Remark. The degree function defined by Theorem 19 on triples $(f+N_G, U, y_0^*)$ is not unique in general, since not every approximate continuous selector $g_\varepsilon(x, r)$ of $G(x, r)$ necessarily satisfies the same sign condition as G.

A careful reading of the proof of Theorem 19 shows that in the definition of the generalized sign condition, we had to assume that the control function $h_s(\cdot) \in L^1(Z)$. If $G(x, r)$ satisfies the sign condition of Definition 14, then we only need to assume that $h_s(\cdot) \in L^1_{\mathrm{loc}}(Z)$ (see also Browder [12]).

Definition 23. The class H_1 of permissible homotopies of maps of the form $f + N_G$ with $G(x, r)$ satisfying the sign condition (cf. Definition 14) consists of all homotopies $\{f_t + N_{G_t}\}_{t\in[0,1]}$ with $\{f_t\}_{t\in[0,1]}$ a bounded homotopy of class $(S)_+$ and $\{G_t\}_{t\in[0,1]}$ a family of multifunctions such that $G_t(x, r) = [\varphi(x, r, t), \psi(x, r, t)]$ is measurable in (x, r, t), u.s.c. in (r, t), and

(i) for every $s > 0$, there is $h_s \in L^1_{\mathrm{loc}}(Z)$ such that for all $x \in Z$, $|r| \le s$, and $t \in [0, 1]$
$$\max[|\varphi(x, r, t)|, |\psi(x, r, t)|] \le h_s(x),$$
(ii) $\varphi(x, r, t)r \ge 0$ for all $r \le 0$, $\psi(x, r, t)r \ge 0$ for all $r \ge 0$.

Theorem 24. *The same approach as in Theorem* 19 *will define a degree function* $d(f + N_G, U, y_0^*)$ *with* f *of class* $(S)_+$, G *a multifunction which satisfies the sign condition (cf. Definition* 14*). In addition, this degree function has the three characteristic properties of normalization, additivity on domain, and invariance under homotopies of class* H_1.

Proof. As in Proposition 17, we can obtain a Carathéodory approximate selector $g_\varepsilon(x, r)$ of $G(x, r)$ satisfying the sign condition; i.e. $g_\varepsilon(x, r)r \ge 0$ for all $r \in \mathbb{R}$. Then we use this selector in the process of constructing the degree function. Q.E.D.

To have uniqueness of the degree function, we need to restrict the class of multifunctions $G(x, r)$.

Definition 25. $G : Z \times \mathbb{R} \to 2^{\mathbb{R}}\backslash\{\varnothing\}$ is a multifunction satisfying the "strict sign condition" if the following hold:

(i) $G(x, r) = [\varphi(x, r), \psi(x, r)]$ and is measurable in (x, r) and u.s.c. in r;
(ii) $\psi(x, r)r \ge 0$ for $r \le 0$ and $\varphi(x, r)r \ge 0$ for $r \ge 0$;
(iii) for any $s > 0$, there exists $h_s \in L^1_{\mathrm{loc}}(Z)$ such that for $|r| \le s$
$$\max[|\varphi(x, r)|, |\psi(x, r)|] \le h_s(x).$$

As in Definition 23, we can define the class H_2 of all permissible homotopies of maps of the form $f + N_G$ with $G(x, r)$ satisfying the strict sign condition.

Theorem 26. *Restricted to multifunctions* $G(x, r)$ *satisfying the strict sign condition and to the corresponding homotopy class* H_2, *the degree function defined by Theorem* 24 *is the only one having the three characteristic properties of normalization, additivity on domain, and homotopy invariance.*

Proof. Suppose that there were another degree function d_1 different from d obtained in Theorem 24; i.e. $d_1(f + N_G, U, y_0^*) \ne d(f + N_G, U, y_0^*)$ on an admissible triple $(f + N_G, U, y_0^*)$. Take a qualified Carathéodory approximate selector $g_\varepsilon(x, r)$ of $G(x, r)$. Then from the definition of d (cf. Theorem 19), we know that
$$d(f + N_G, U, y_0^*) = d(f + N_\varepsilon^k, U, y_0^*)$$
for $k \ge 1$ large enough and $\varepsilon > 0$ small enough. Recalling that Browder's degree function is unique on maps of class $(S)_+$ (cf. Browder [11, Proposition 14]), we have $d(f + N_\varepsilon^k, U, y_0^*) = d_1(f + N_\varepsilon^k, U, y_0^*)$. Consequently, $d_1(f + N_\varepsilon^k, U, y_0^*) \ne d_1(f + N_G, U, y_0^*)$. Because d_1 is homotopy invariant, we can find $s_k \to s$ in $[0, 1]$, $\varepsilon_k \to 0$, $u_k \in \partial U$, and $u_k \xrightarrow{w} u$ in X, such that
$$y_0^* \in (f + (1 - s_k)N_{\varepsilon_k}^k + s_k N_G)(u_k).$$

So there exists $g_k^* \in N_G(u_k)$ such that if $v_k^* = (1-s_k)N_{\varepsilon_k}^k(u_k) + s_k g_k^*$, we have $y_0^* = f(u_k) + v_k^*$. Since $G(x, r)$ satisfies the sign condition and $g_k^*(x, r) \in G(x, u_k(x))$ a.e. on Z, we have $v_k^*(x)u_k(x) \geq 0$ a.e. on Z and since $g_k^* \in L^1_{\text{loc}}(Z)$ (cf. Definition 25(iii)), we have $v_k^* \in L^1_{\text{loc}}(Z)$. Apply Proposition 16 to get that

$$\int_Z v_k^*(x)u_k(x)\,dx = \langle v_k^*, u_k\rangle = \langle y_0^* - f(u_k), u_k\rangle \leq M.$$

By slight modifications of the arguments in the proof of Theorem 19, we can prove that there exists $v_0^* \in L^1_{\text{loc}}(Z)$ such that $v_k^* \xrightarrow{w} v_0^*$ in $L^1(Z')$ for any $Z' \subseteq Z$ bounded (since $\{v_k^*\}_{k\geq 1} \subseteq L^1(Z')$ is uniformly integrable). Also as always, we assume that $u_k(x) \to u(x)$ a.e. on Z, and $f(u_k) \xrightarrow{w} f^*$ in X^*. Then we can proceed and show that

$$0 \geq v_0^*(x)u(x) \quad \text{a.e. on } Z$$

and

$$\int_Z v_0^*(x)u(x)\,dx \leq \underline{\lim} \int_Z v_k^*(x)u_k(x)\,dx.$$

Only the second inequality requires some work. Let $Z' \subseteq Z$ with $\lambda(Z') < \infty$. Given $\delta > 0$ find $Z_\delta \subseteq Z'$ closed with $\lambda(Z' \backslash Z_\delta) < \delta$ such that on Z_δ, $u_k(x) \to u(x)$ uniformly and $u(\cdot)$ is continuous. Thus we get

$$\int_{Z_\delta} v_0^*(x)u(x)\,dx = \lim \int_{Z_\delta} v_k^*(x)u_k(x)\,dx \leq \underline{\lim} \int_Z v_k^*(x)u_k(x)\,dx \leq M.$$

Since $\delta > 0$ and $Z' \subseteq Z$ were arbitrary, using the monotonicity of the measure $m(Z) = \int_A v_0^*(x)u(x)\,dx$, $A \subseteq Z$ measurable, and the σ-finiteness of Z we finally get that

$$\int_Z v_0^*(x)u(x)\,dx \leq \underline{\lim} \int_Z v_k^*(x)u_k(x)\,dx.$$

Using this inequality and the fact that $v_0^* = y_0^* - f^*$ (cf. Proposition 16) we then can show that

$$\overline{\lim}\langle f(u_k), u_k - u\rangle \leq 0.$$

Hence, since f is of class $(S)_+$, $u_k \to u$ in X and $f(u_k) \xrightarrow{w} f(u) = f^*$ in X^*. Thus $x \in \partial U$. Also by standard arguments, we can show that $v_0^*(x) \in G(x, u(x))$ a.e. on Z and so $v_0^* \in N_G(u)$. So finally we have $y_0^* \in (f + N_G)(u)$ with $u \in \partial U$, a contradiction. Therefore $d = d_1$. Q.E.D.

Remark. The degree function defined in this section can be extended further in the weak sense (see Browder [11, Definition 6]), to the larger class $f + N_G$ where $G(x, r)$ is a multifunction as before, but f is a bounded pseudomonotone map instead of a map of class $(S)_+$ (since we have included the demicontinuity condition in the definition of class $(S)_+$—cf. Definition 2—we see that this new class is indeed broader). Then the degree function is defined by

$$d(f + N_G, U, y_0^*) = \lim_{\varepsilon \downarrow 0}(f + \varepsilon J + N_G, U, y_0^*),$$

where the degrees in the limit are defined as in this section since $f+\varepsilon J$ is of class $(S)_+$.

5. Examples

In this section we present some examples of multivalued elliptic and parabolic partial differential equations, where our abstract results apply.

(I) Let Z be a bounded domain in $\mathbb{R}^N$ with smooth boundary Γ. We consider the following nonlinear multivalued partial differential equation in divergence form:

$$(*)_1 \qquad \begin{cases} \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \eta(u(x))) + G(x, \xi(u(x))) \ni h(x) \quad \text{on } Z, \\ D^\beta x|_\Gamma = 0, \qquad |\beta| \le m-1, \end{cases}$$

where $\eta(u) = \{D^\alpha u : |\alpha| \le m\}$, $\xi(u) = \{D^\alpha u : |\alpha| \le m-1\}$ and the function $A_\alpha(x, \eta)$ maps $Z \times \mathbb{R}^{N_m}$ into $\mathbb{R}$ (with $N_m = \frac{(N+m)!}{N!\,m!}$). In what follows, we split the η-variable into two parts $\eta = (\xi, \zeta)$, with $\xi = (\xi_\alpha : |\alpha| \le m-1)$ and $\zeta = (\zeta_\alpha : |\alpha| = m)$ and impose separate conditions on them.

Our hypotheses on the data of $(*)_1$ are the following:

$H(A)$: $A_\alpha : Z \times \mathbb{R}^{N_m} \to \mathbb{R}$ is a function such that

(1) $x \to A_\alpha(x, \eta)$ is measurable, $\eta \to A_\alpha(x, \eta)$ is continuous, and

$$|A_\alpha(x, \eta)| \le a_1(x) + c_1\|\eta\|^{p-1} \quad \text{a.e. on } Z$$

with $a_1(\cdot) \in L^1(Z)$, $p > 2$,

(2) $\sum_{|\alpha|=m}[A_\alpha(x, \xi, \zeta) - A_\alpha(x, \xi, \zeta')](\zeta_\alpha - \zeta'_\alpha) > 0$ for all $\zeta \ne \zeta'$, $x \in Z$, and $\xi \in \mathbb{R}^{N_{m-1}}$ (i.e. we have monotonicity only on the principal part term; this condition is known as the Leray-Lions condition);

(3) there exist constant $c > 0$ and $\beta(\cdot) \in L^1(Z)_+$ such that

$$\sum_{|\alpha|\le m} A_\alpha(x, \eta)\eta_\alpha \ge c\|\eta\|^p - \beta(x) \quad \text{a.e. on } Z,\ \eta \in \mathbb{R}^{N_m}.$$

$H(G)$: $G : Z \times \mathbb{R}^{N_{m-1}} \to 2^{\mathbb{R}} \setminus \{\varnothing\}$ is a multifunction such that

(1) $(x, \xi) \to G(x, \xi) = [\varphi(x, \xi), \psi(x, \xi)]$ is measurable and $\xi \to G(x, \xi)$ is u.s.c., and

(2) $|G(x, \xi)| = \max[|\varphi(x, \xi)|, |\psi(x, \xi)|] \le a_2(x) + c_2\|\xi\|$ a.e. on Z and $a_2(\cdot) \in L^q(Z)$.

In this case $X = W_0^{m,p}(Z)$, which is separable, reflexive, and uniformly convex. Let $\widehat{A} : W_0^{m,p}(Z) \to X^* = W^{-m,q}(Z)$ be defined by

$$\langle \widehat{A}(u), v\rangle = \int_Z \sum_{|\alpha|\le m} A_\alpha(x, \eta(u(x)))D^\alpha v(x)\,dx$$

with $\langle \cdot, \cdot\rangle$ being the duality brackets for the pair $(W_0^{m,p}(Z), W^{-m,q}(Z))$.

Proposition 27. *If hypothesis $H(A)$ holds, then $\widehat{A}(\cdot)$ is of class $(S)_+$.*

Proof. Demicontinuity (in fact continuity) of $\widehat{A}(\cdot)$ follows from hypothesis $H(A)(1)$ and Krasnosel'skii's theorem on the continuity of the Nemitsky operator.

Now let $u_n \xrightarrow{w} u$ in X and assume that

$$\begin{aligned}\overline{\lim}\langle \widehat{A}(u_n), u_n - u\rangle &= \overline{\lim}\langle \widehat{A}(u_n) - \widehat{A}(u), u_n - u\rangle \\ &= \overline{\lim} \int_Z \sum_{|\alpha|\le m} (A_\alpha(x, \xi(u_n(x)), \zeta(u_n(x))) \\ &\qquad - A_\alpha(x, \xi(u(x)), \zeta(u(x)))D^\alpha(u_n(x) - u(x))\, dx \le 0.\end{aligned}$$

Since $W_0^{m,p}(Z)$ embeds compactly into $W_0^{m-1,p}(Z)$, by passing to a subsequence if necessary, we may assume that $\xi(u_n(\cdot)) \to \xi(u(\cdot))$ in $L^p(Z)^{N_{m-1}}$. So the above inequality becomes

$$\overline{\lim} \int_Z \sum_{|\alpha|=m} (A_\alpha(x, \xi(u(x)), \zeta(u_n(x))) - A_\alpha(x, \xi(u(x)), \zeta(u(x)))) \cdot D^\alpha(u_n(x) - u(x))\, dx \le 0.$$

But note that hypothesis $H(A)(2)$ implies that $\zeta \to \sum_{|\alpha|=m} A_\alpha(x, \xi, \zeta)$ is strictly increasing, while hypothesis $H(A)(3)$ implies that

$$\sum_{|\alpha|=m} A_\alpha(x, \xi, \zeta)\zeta_\alpha \ge c\|\zeta\|^p - \gamma(x) \quad \text{a.e. on } Z$$

with $\gamma(\cdot) \in L^1(Z)$. So we can apply a result of Browder [6] and get that the principal part defines an operator of class $(S)_+$ and so $D^\alpha u_n \to D^\alpha u$ in $L^p(Z)$ for $|\alpha| = m$. Thus $\widehat{A}(\cdot)$ is of class $(S)_+$. Q.E.D.

Next let $N_G : X \to 2^{X^*}\backslash\{\varnothing\}$ be defined by

$$N_G(u) = \{g \in X^* : g(x) \in G(x, \xi(u(x))) \text{ a.e.}\}.$$

Observe that because of hypothesis $H(G)(2)$ we have

$$|N_G(u)| = \sup\{\|g\|_q, g \in N_G(u)\} \le \hat{a}_2 + \hat{c}_2\|u\|$$

(here $\|\cdot\|$ denotes the norm in $W_0^{m,p}(Z)$ and $\|\cdot\|_q$ the norm in $L^q(Z)$). Since $L^q(Z)$ embeds into $W^{-m,q}(Z)$ compactly, we have that $N_G(\cdot)$ is compact with closed and convex values, and it is easy to see that it is u.s.c. (cf. hypothesis $H(G)(1)$).

Therefore we can state the following existence theorem for $(*)_1$.

Theorem 28. *If hypotheses $H(A)$ and $H(G)$ hold and $h \in W^{-m,q}(Z)$, then problem $(*)_1$ admits a generalized solution $u \in W_0^{m,p}(Z)$.*

Proof. To use degree-theoretic techniques, we need to establish an a priori bound for the solutions of $tJ(u) + (1-t)\widehat{A}(u) + (1-t)N_G(u) = h$, $t \in [0, 1]$ and $J : X \to X^*$ the duality map. So suppose $u \in W_0^{m,p}(Z)$ is such a solution. Then there exists $g \in N_G(u)$ such that

$$\begin{aligned}&tJ(u) + (1-t)\widehat{A}(u) + (1-t)g = h \\ &\quad\Rightarrow t\langle J(u), u\rangle + (1-t)\langle \widehat{A}(u), u\rangle + (1-t)\langle g, u\rangle = \langle h, u\rangle.\end{aligned}$$

Because of hypothesis $H(A)(3)$, we have for some $\hat{c}, \hat{\beta} > 0$

$$t\|u\|^2 + (1-t)\hat{c}\|u\|^p - (1-t)\hat{\beta} \le [\|h\|_* + (1-t)\|g\|_*]\|u\|, \qquad t \in [0, 1]$$

(with $\|\cdot\|_*$ denoting the norm of $W^{-m,q}(Z)$). So

$$t\|u\|^2 + (1-t)\hat{c}\|u\|^p \leq [\|h\|_* + \hat{a}_2 + (1-t)\hat{c}_3\|u\|]\|u\| + \hat{\beta}.$$

Using Young's inequality (recall $p > 2$), we deduce that there exists $M_1 > 0$ such that $\|u\| \leq M_1$ for all such solutions $u(\cdot)$. Take $M_2 > M_1$ and define

$$U = \{u \in W_0^{m,p}(Z) \colon \|u\| < M_2\}.$$

We choose M_2 so that $h \in J(U)$. Then clearly

$$h \notin (tJ + (1-t)\widehat{A} + (1-t)N_G)(\partial U)$$

and so $d(tJ + (1-t)\widehat{A} + (1-t)N_G, U, h)$ is well defined for all $t \in [0, 1]$ (cf. §3). Thus by the homotopy invariance we have

$$d(tJ + (1-t)\widehat{A} + (1-t)N_G, U, h) = d(J, U, h) = 1.$$

So $(*)_1$ admits a generalized solution. Q.E.D.

If $G(x, \cdot)$ depends only on u and not on any of its derivatives, then to have the compactness of the multivalued Nemitsky operator $N_G(\cdot)$, we can allow a more general growth condition than the sublinear one. Indeed we have the following proposition:

Proposition 29. *If hypothesis $H(G)(1)$ holds and $|G(x, r)| \leq a_2(x) + c_2|r|^\theta$ a.e. with*

$$\theta < \frac{N_p - N + mp}{N - mp} \text{ if } N \geq mp \quad \text{and} \quad \theta < \infty \text{ if } N \leq mp$$

then $N_G : X \to 2^{X^*}\backslash\{\varnothing\}$ is compact.

Proof. By Sobolev's embedding theorem, we know that $W_0^{m,p}(Z)$ embeds into $L^r(Z)$ continuously and densely provided that $\frac{1}{r} \geq \frac{1}{p} - \frac{m}{N}$. Furthermore, the embedding is compact provided the inequality is strict. We have $r \leq p^* = \frac{Np}{N-mp}$.

Let $r' \geq 1$ be the conjugate exponent of r (i.e. $\frac{1}{r} + \frac{1}{r'} = 1$). If $\theta = \frac{p^*}{r'}$, then $N_G(u) \subseteq W^{-m,q}(Z)$. So to have the supremum of all possible exponents, we need to maximize $\frac{1}{r'}$, hence minimize $\frac{1}{r}$. But this last infimum is $\frac{1}{p^*}$. So the supremum of θ is

$$p^*\left(1 - \frac{1}{p^*}\right) = p^* - 1 = \frac{Np - N + mp}{N - mp}.$$

Therefore if $\theta < \frac{Np-N+mp}{N-mp}$, we have the compactness of N_G. Q.E.D.

Remarks. (1) This critical exponent is consistent with the one established by Pohozaev [19] and DeFigueiredo-Lions-Nussbaum [14]. They considered the Laplace equation $-\Delta u = u^\theta$ on a ball in $\mathbb{R}^N$, $N \geq 3$, with Dirichlet boundary conditions. Pohozaev [19] showed that for $\theta = \frac{N+2}{N-2}$ there,is no positive solution to this problem. Later DeFigueiredo-Lions-Nussbaum [14] proved that for $\theta < \frac{N+2}{N-2}$, we have a priori estimates in the L^∞-norm for the positive solutions. Note that the critical exponent provided by Proposition 29 reduces exactly to $\frac{N+2}{N-2}$ if $m = 1$, $p = 2$, the situation in Pohozaev [19] and DeFigueiredo-Lions-Nussbaum [14].

(2) Since $N_G(\cdot)$ is compact, in principle we can apply degree-theoretic techniques using $N_G(\cdot)$. But in this general case, the derivation of a priori estimates is extremely difficult.

Of course, we can drop the sublinear growth condition $H(G)(2)$ and replace it by a strict sign condition assuming that $G(z, \cdot)$ depends only on u and not on any of its derivatives. So our hypothesis on $G(x, r)$ is now the following:

$H(G)_1$: $G: Z \times \mathbb{R} \to 2^{\mathbb{R}}\backslash\{\varnothing\}$ is a multifunction such that

(1) $G(x, r) = [\varphi(x, r), \psi(x, r)]$ is measurable in (x, r) and u.s.c. in r;

(2) $\psi(x, r)r \geq 0$ if $r \leq 0$, $\varphi(x, r)r \geq 0$ if $r \geq 0$, and for each $s > 0$ there is $h_s(\cdot) \in L^1_{\mathrm{loc}}(Z)$ such that $|G(x, r)| = \max[|\varphi(x, r)|, |\psi(x, r)|] \leq h_s(x)$ a.e. on Z for $|r| \leq s$.

Using similar arguments as in the proof of Theorem 28, we can have

Theorem 30. *If hypotheses $H(A)$ and $H(G)_1$ hold and $h \in W^{-m,q}(Z)$, then problem $(*)_1$ has a generalized solution $u \in W_0^{m,p}(Z)$.*

(II) Now we consider a multivalued parabolic partial differential equation. So let $S = [0, b]$ and $Z \subseteq \mathbb{R}^n$ a bounded domain with smooth boundary Γ. By Q we will denote the cylinder $S \times Z$. We consider the following initial-boundary value nonlinear parabolic problem:

$(*)_2$
$$\begin{cases} \dfrac{\partial x}{\partial t} + \displaystyle\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D_\alpha A_\alpha(t, x, \eta(u(t, x))) + G(t, x, u(t, x)) \ni h(t, x) \text{ on } Q, \\ u|_{T\times\Gamma} = 0, \qquad u(0, x) = u_0(x) \quad \text{a.e. on } Z \end{cases}$$

where as before $\eta(u) = \{D^\alpha u : |\alpha| \leq m\}$, $\xi(u) = \{D^\alpha u : |\alpha| \leq m - 1\}$, and the function A_α maps $S \times Z \times \mathbb{R}^{N_m}$ into $\mathbb{R}$. Again we split η into two parts, $\eta = (\xi, \zeta)$ with $\xi = (\xi_\alpha : |\alpha| \leq m - 1)$ and $\zeta = (\zeta_\alpha : |\alpha| = m)$.

We will need the following hypotheses on the data:

$H(A)_1$: $A_\alpha : S \times Z \times \mathbb{R}^{N_m} \to \mathbb{R}$ is a function such that

(1) $(t, x) \to A_\alpha(t, x, \eta)$ is measurable, $\eta \to A_\alpha(t, x, \eta)$ is continuous, and
$$|A_\alpha(t, x, \eta)| \leq a_1(t, x) + c_1\|\eta\|^{p-1} \quad \text{a.e. on } Q$$
with $a_1 \in L^q(Q)$ and $c_1 > 0$ ($p \geq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$);

(2) $\sum_{|\alpha|=m}[A_\alpha(t, x, \xi, \zeta) - A_\alpha(t, x, \xi, \zeta')](\zeta_a - \zeta'_a) > 0$ for all $\zeta \neq \zeta'$ and all $(t, x, \xi) \in S \times Z \times \mathbb{R}^{N_{m-1}}$;

(3) there exist $c > 0$ and $\beta \in L^1(Q)_+$ such that
$$\sum_{|\alpha|\leq m} A_\alpha(t, x, \eta)\eta_\alpha \geq c\|\eta\|^p - \beta(t, x) \quad \text{a.e. on } Q$$
for all $\xi \in \mathbb{R}^{N_m}$.

$H(G)_2$: $G: S \times Z \times \mathbb{R} \to 2^{\mathbb{R}}\backslash\{\varnothing\}$ is a multifunction such that

(1) $G(t, x, r) = [\varphi(t, x, r), \psi(t, x, r)]$ is measurable in (t, x, r) and u.s.c. in r;

(2) $\psi(t, x, r)r \geq 0$ for $r \leq 0$ and $\varphi(t, x, r)r \geq 0$ for $r \geq 0$;

(3) $|G(t, x, r)| = \max[|\varphi(t, x, r)|, |\psi(t, x, r)|] \leq a_2(t, x) + c_2|r|^{p/q}$ a.e., $a_2(\cdot, \cdot) \in L^q(Q)$, $c_2 > 0$.

Theorem 31. *If hypotheses $H(A)_1$ and $H(G)_2$ hold, $u_0 \in L^2(Z)$ and $h \in L^q(S, W^{-m,q}(Z))$, then problem $(*)_2$ admits a generalized solution in $W_0^{m,p}(Z)$.*

Proof. Let $Y = W_0^{m,p}(Z)$ and $Y^* = W^{-m,q}(Z)$. Define $T : D \subseteq L^p(T, Y) \to L^q(T, Y^*)$ by $T(u) = \frac{du}{dt}$, with $D = \{u \in L^p(S, Y) : \dot{u} \in L^q(S, Y^*)$ and $u(0) = u_0(\cdot)\}$ (recall that the space $W_{pq}(S)$ of functions $u \in L^p(S, Y)$ such that $\dot{u} \in L^q(S, Y^*)$ embeds continuously into $C(S, H)$, $H = L^2(Z)$ and so the condition $u(0) = u_0(\cdot)$ makes sense).

We claim that $T(\cdot)$ is maximal monotone from $L^p(T, Y)$ into $L^q(T, Y^*)$. Clearly $T(\cdot)$ is monotone. According to Lemma 5 in order to establish the maximality of $T(\cdot)$, we need to show that $R(T + \widehat{J}_1) = L^q(S, Y^*)$ with $\widehat{J}_1$ being defined by $\widehat{J}_1(u)(t) = J(u(t))\|u(t)\|^{p-2}$, $u \in L^p(S, Y)$, $t \in [0, b]$, and J is the duality map from Y into Y^*. Let $g \in L^q(S, Y^*)$ and consider the Cauchy problem

$$\dot{u}(t) + J(u(t))\|u(t)\|^{p-2} = g(t) \quad \text{a.e.}, \qquad u(0) = u_0.$$

Since $J(\cdot)$ is maximal monotone, demicontinuous, by a well-known existence result for evolution equations, we deduce that the above problem has a unique solution $u \in W_{pq}(S)$. So $R(T + \widehat{J}_1) = L^q(S, Y^*)$ and this proves the maximality of T. Hence $\operatorname{Gr} T$ is a closed subspace of $L^p(S, Y) \times L^q(S, Y^*)$. Therefore D equipped with the graph norm $\|u\|_D = \|u\|_{L^p(S,Y)} + \|Tu\|_{L^q(S,Y^*)}$ is a separable reflexive Banach space. By Troyanski's theorem we may assume that both D and D^* are locally uniformly convex spaces.

Let $A : S \times Y \to Y^*$ be defined by

$$\langle A(t, u), u\rangle = \int_Z \sum_{|\alpha| \le m} A_\alpha(t, x, \eta(u(x)))D^\alpha u(x)\, dx$$

and let $\widehat{A} : D \to D^*$ be defined by $\widehat{A}(u)(t) = A(t, u(t))$.

Using Krasnosel'skii's theorem, we see that $\widehat{A}(\cdot)$ is continuous. Also we will show that $\widehat{A}(\cdot)$ is of class $(S)_+$. To this end, let $u_n \xrightarrow{w} u$ in D and assume that

$$\overline{\lim}((\widehat{A}(u_n), u_n - u)) \le 0$$

with $((\cdot, \cdot))$ denoting the duality brackets for the pair (D, D^*). From Theorem 5.1, p. 58 of Lions [17] we know that D embeds compactly in $L^p(Q)$. So for all $|\alpha| \le m - 1$, we have $D^\alpha u_n \to D^\alpha u$ in $L^p(Q)$ and $D^\alpha u_n(t, x) \to D^\alpha u(t, x)$ a.e. Using hypothesis $H(A)_1$ and the result of Browder [6] as in the proof of Proposition 27, we can get that $D^\alpha u_n \to D^\alpha u$ in $L^p(Q)$ for $|\alpha| = m$ and so conclude that indeed $\widehat{A}$ is of class $(S)_+$ and of course bounded.

Next let $\widehat{G} : D \to 2^{D^*} \backslash \{\varnothing\}$ be defined by

$$\widehat{G}(u) = \{g \in L^q(Q) : g(t, z) \in G(t, z, u(t, z))\}.$$

First note that $\widehat{G}(\cdot)$ is bounded into $L^q(Q)$ (cf. hypothesis $H(G)_2(3)$) and since $L^q(Q)$ embeds compactly into D^*, we deduce that $\widehat{G}(\cdot)$ is a compact multifunction. Also it is easy to see that $\widehat{G}(\cdot)$ is u.s.c. (hence $\widehat{G}(\cdot)$ is of class (P); cf. §3).

Now consider the map $sJ+(1-s)(T+\widehat{A}+\widehat{G})u$, $s\in[0,1]$ and the operator inclusion $sJu+(1-s)(T+\widehat{A}+G)u\ni h$. Let $u\in D$ be a solution of this inclusion. Hence there is a $\hat{g}\in\widehat{G}(u)$ such that

$$s((\widehat{J}u,u))+(1-s)[((Tu,u))+((\widehat{A}u,u))+((\hat{g},u))]=((h,u)).$$

Note that $((\hat{g},u))\geq 0$ (cf. hypothesis $H(G)_2$), $((\widehat{A}u,u))\geq c\|u\|^p_{L^p(S,Y)}-\hat{\beta}$ (cf. hypothesis $H(A)_1$), and $((Tu,u))=((\dot{u},u))=\frac{1}{2}\|u(b)\|^2_{L^2(Z)}-\frac{1}{2}\|u_0\|^2_{L^2(Z)}$. So we get

$$s\|u\|^2_D+(1-s)c\|u\|^p_{L^p(S,Y)}\leq\|h\|_{L^q(S,Y)}\|u\|_{L^p(S,Y)}+\hat{\beta}+\frac{1}{2}\|u_0\|^2_{L^2(Z)}.$$

From this, using Young's inequality, we deduce that there is $M_1>0$ such that $\|u\|_{L^p(S,Y)}\leq M_1$. Thus there is $M_2>0$ such that $s\|u\|^2_D\leq M_2$. Our claim is that $\|u\|_D$ is uniformly bounded as u varies over all solutions of the original operator inclusion. Indeed if this is not the case, we can find solutions u_n of the operator inclusion with $s_n\in[0,1]$ such that $\|u_n\|_D\to\infty$; so $s_n\|u_n\|_D\to 0$, i.e. $s_n\to 0$. But note that for some $M_3>0$, $(1-s_n)\|Tu_n\|_{L^q(S,Y^*)}\leq s_n\|u_n\|_D+M_3\Rightarrow\|Tu_n\|_{L^q(S,Y^*)}$ is bounded; hence $\|u_n\|_D$ is bounded, a contradiction. Therefore there exists $M_4>0$ so that $\|u\|_D\leq M_4$ for every solution of the operator inclusion. Choose $M_5>M_4$ such that $h\in J(U)$ with $U=\{u\in D:\|u\|_D<M_5\}$. Then by homotopy invariance, we have

$$d(T+\widehat{A}+\widehat{G},U,h)=d(J,U,h)=1$$

$\Rightarrow(*)_2$ admits a generalized solution $u(\cdot)\in W_{pq}(S)$. Q.E.D.

Acknowledgement

The authors express their sincere thanks to the referee for the comments.

References

1. H. Amann and S. Weiss, *On the uniqueness of the topological degree*, Math. Z. **130** (1973), 39–54.
2. H. Brezis and F. E. Browder, *Some properties of higher order Sobolev spaces*, J. Math. Pures Appl. **61** (1982), 245–259.
3. F. E. Browder, *Nonlinear eigenvalue problems and Galerkin approximations*, Bull. Amer. Math. Soc. **74** (1965), 251–258.
4. ——, *Existence theorems for nonlinear partial differential equations*, Proc. Sympos. Pure Math., vol. 16, Amer. Math. Soc., Providence, RI, 1970, pp. 1–62.
5. ——, *Nonlinear operators and nonlinear equations in Banach spaces*, Proc. Sympos. Pure Math., vol. 18, part 2, Amer. Math. Soc., Providence, RI, 1975.
6. ——, *Nonlinear functional analysis and nonlinear integral equations of Hammerstein and Urysohn type*, Contributions to Nonlinear Analysis (E. Zarantonello, ed.), Academic Press, New York, 1971, pp. 425–500.
7. ——, *Degree of mapping for nonlinear operators of monotone type*, Proc. Nat. Acad. Sci. U.S.A. **80** (1983), 1771–1773.
8. ——, *Degree of mapping for nonlinear operators of monotone type: Densely defined mappings*, Proc. Nat. Acad. Sci. U.S.A. **80** (1983), 2405–2407.
9. ——, *Degree of mapping for nonlinear operators of monotone type: Strongly nonlinear mappings*, Proc. Nat. Acad. Sci. U.S.A. **80** (1983), 2408–2409.

10. ———, *The degree mapping and its generalizations*, Contemp. Math., vol. 2, Amer. Math. Soc., Providence, RI, 1983, pp. 15–39.

11. ———, *Fixed point theory and nonlinear problems*, Bull. Amer. Math. Soc. (N.S.) **9** (1983), 1–39.

12. ———, *Degree theory for nonlinear mappings*, Proc. Sympos. Pure Math., vol. 45, part I, Amer. Math. Soc., Providence, RI, 1986, pp. 203–206.

13. A. Cellina, *Approximations of set-valued functions and fixed point theorems*, Ann. Mat. Pura Appl. **82** (1969), 17–24.

14. D. DeFigueiredo, P.-L. Lions, and R. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations*, J. Math. Pures Appl. **61** (1982), 42–63.

15. N. Dunford and J. Schwartz, *Linear operators*. I, Wiley, New York, 1958.

16. L. Fuhrer, *Ein elementarer analytischer beweis zur eindeutigkeit des abbildungsgrades im* $\mathbb{R}^n$, Math. Nachr. **54** (1972), 259–267.

17. J.-L. Lions, *Quelques méthodes de resolution des problèmes aux limites non-lineaires*, Dunod, Paris, 1969.

18. N. S. Papageorgiou, *On measurable multifunctions with applications to random multivalued equations*, Math. Japon. **32** (1987), 437–464.

19. S. Pohozaev, *On eigenfunctions of the equation* $-\Delta u = \lambda f(u)$, Soviet Math. Dokl. **165** (1965), 36–39.

20. S. Troyanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math. **37** (1971), 173–180.

21. D. Wagner, *Survey of measurable selection theorems*, SIAM J. Control. Optim. **15** (1977), 859–903.

22. E. Zeidler, *Nonlinear functional analysis and its applications*. II, Springer, New York, 1990.

Department of Mathematics, Southwest Missouri State University, Springfield, Missouri 65804

E-mail address: `shh209f@vma.smsu.edu`

Department of Applied Mathematics, Florida Institute of Technology, Melbourne, Florida 32901-6988