BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 29, Number 3, September 2001

ON PARABOLIC SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS

BY

K. J. CHANG (張凱君) AND L. Y. TSAI (蔡隆義)

Abstract. In this paper, the existence of the solutions for nonlinear integro-differential systems is discussed. The comparison result is obtained by assuming the existence of weak upper and weak lower solutions for the given problem. Then, we use monotone method to construct two sequences which converge monotonically to the solution.

0. Introduction. The existence of solutions for a nonlinear integro-differential equation of the form

$$\frac{\partial u}{\partial t} - \Delta u = \Phi(x, t, u, F(u)),$$

is investigated recently by many authors, where $F(u)$ is a nonlinear integral operator. Such equations are used for the description of the physical processes in a nuclear reactor, of those of population, and of other processes. Several specific models in various fields in applied science can be found in [6]. Some classical existence results of parabolic integro-differential equations can be found in [7], [8], [9] and [10]. Carl gave the existence of weak solutions for nonlinear parabolic systems in [2] and [3].

Received by the editors August 17, 2000
AMS Subject Classification: 35K60.
Key words and phrases: Weak upper (lower) solutions.

In this paper, we shall consider the existence of weak solutions for non-linear integro-differential equations of the form

$$\frac{\partial u_k}{\partial t} - L_k u_k = g_k(x, t, u, F(u)),$$

here L_k is an uniformly elliptic operator. Constructive monotone-scheme is used. It is organized as follows. In section 1, some preliminaries and notations are stated. In section 2, we give a comparison theorem for weak upper and weak lower solution. By using such a comparison result and monotone method, we establish an existence and uniqueness result for a single parabolic integro-differential equation. In section 3, we shall follow the same framework of section 2 to discuss the existence and uniqueness of weak solution for a system of parabolic integro-differential equations. These results extend the result of [3] to the more general systems.

1. Preliminaries. Let Ω be a bounded domain in R^N with a regular boundary $\partial\Omega$. Let $Q_T = (0,T) \times \Omega$ and $\Gamma_T = (0,T) \times \partial\Omega$ where $T > 0$. Let $V = L^2(0,T; W^{1,2}(\Omega))$ and $V^* = L^2(0,T; (W^{1,2}(\Omega))^*)$, where $W^{1,2}(\Omega) = \{h | h \in L^2(\Omega), \frac{\partial h}{\partial x_i} \in L^2(\Omega)\}$ is the Sobolev space with its dual $(W^{1,2}(\Omega))^*$. We consider the solutions in the space $W = \{h | h \in V, \frac{\partial h}{\partial t} \in V^*\}$ (see [5]), where $\frac{\partial}{\partial t}$ denotes the distributional derivative in V^*. The spaces V, V^* and W are Banach spaces equipped with the norms $\|h\|_V^2 = \int_0^T \|h(\cdot,t)\|^2_{W^{1,2}(\Omega)} dt$, $\|h\|^2_{V^*} = \int_0^T \|h(\cdot,t)\|^2_{(W^{1,2}(\Omega))^*} dt$, and $\|h\|^2_W = \|h\|^2_V + \|\frac{\partial h}{\partial t}\|^2_{V^*}$ respectively. Denote W_0, V_0, and V_0^* the corresponding spaces if the Sovolev space $W^{1,2}(\Omega)$ in the definitions of W, V, and V^* is replaced by its subspace $W_0^{1,2}(\Omega)$, which is the space of all functions of $W^{1,2}(\Omega)$ with zero traces on $\partial\Omega$ (**[1]**).

Let B be a Banach space, we denote $B^n = B \times \cdots \times B$ by the n-dimensional Cartesian product of B, which is again a Banach space equipped with the norm $\|(u_1,\dots,u_n)\|_{B^n} = \sum_{i=1}^n \|u_i\|_B$. Hence W^n and W_0^n are understood. Denote L and L_k the differential operators of the divergence form

$$L = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial}{\partial x_j}\Big) - \sum_{i=1}^{N} b_i(x,t)\frac{\partial}{\partial x_i}$$

and

$$L_k = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}^k(x,t)\frac{\partial}{\partial x_j}\Big) - \sum_{i=1}^{N} b_i^k(x,t)\frac{\partial}{\partial x_i}$$

For a vector $u \in R^n$, we denote $[u]_k = (u_1,\dots,u_{k-1},u_{k+1},\dots,u_n)$. And for any $n\times n$ matrix $M = (M_{ij}), i,j = 1,\dots,n$, denote $[M]_k = (M_{k,1},\dots,M_{k,k-1},M_{k,k+1},\dots,M_{k,n}) \in R^{n-1}$ for each $k = 1,2,\dots,n$.

In order to prove the existence theorem for nonlinear problem, we need an existence and uniqueness result for linear coupled systems of the form

$$\begin{aligned} &\frac{\partial u_k}{\partial t} - L_k u_k + \sum_{\ell=1}^{n} P_{k\ell}u_\ell = \Phi_k \text{ in } Q_T \\ &u(x,0) = \varphi(x) \text{ in } \Omega \\ &u = 0 \text{ on } \Gamma_T \end{aligned} \tag{1.1}$$

where $\Phi_k \in V_0^*$ and $\varphi_k \in L^2(\Omega)$, $1 \le k \le n$.

Lemma 1.1. ([5]): *If each L_k is uniformly elliptic in Q_T and a_{ij}^k, b_i^k, and P_{kj} are real bounded measurable in Q_T, $1 \le i,j \le N, 1 \le k \le n$, then there exists a unique solution $u \in W^n$ of* (1.1) *such that $\|u\|_{W^n}^2 \le c(\|\varphi\|^2_{(L^2(\Omega))^n} + \|\Phi\|^2_{(V_0^*)^n})$. In particular, if $\Phi_k \in L^2(Q_T)$, then*

$$\|u\|_{W^n}^2 \le c(\|\varphi\|^2_{(L^2(\Omega))^n} + \|\Phi\|^2_{(L^2(Q_T))^n}).$$

since $L^2(Q_T)$ is continuously embedded in V_0^*.

2. Existence for a single equation. In this section, we shall consider the nonlinear initial-boundary-value problem of the form

$$\begin{aligned} &\frac{\partial u}{\partial t} - Lu = g(x,t,u,F(u)) \text{ in } Q_T \\ &u(x,0) = \varphi(x) \text{ in } \Omega \qquad (2.1) \\ &u(x,t) = 0 \text{ on } \Gamma_T \end{aligned}$$

where $u \in R^1$, and F is an integral operator. Assume that L is uniformly elliptic in Q_T, and that the coefficents a_{ij} and b_i are real bounded measurable in Q_T, and that $\varphi \in L^2(\Omega)$. Denote the order interval $[u,v]$ in $L^2(Q_T)$ by

$$[u,v]=\{z \in L^2(Q_T) | u \le z \le v \text{ a.e. in } Q_T\}.$$

Let

$$\ell(h,\chi)=\int_{Q_T}\Big(\sum_{i,j=1}^{N} a_{ij}\frac{\partial h}{\partial x_j}\frac{\partial \chi}{\partial x_i}+\sum_{i=1}^{N} b_i\frac{\partial h}{\partial x_i}\chi\Big)dxdt$$

Definition. A function $u : Q_T \longrightarrow R^1$ of W_0 is called a weak solution of (2.1) if the following conditions are fulfilled:

(i) $u(x,0) = \varphi(x)$ in the sense that $\lim_{t\to 0} \|u(\cdot,t) - \varphi\|_2 = 0$.

(ii) $< \frac{\partial u}{\partial t}, \chi > + \ell(u,\chi) = \int_{Q_T} g(x,t,u,F(u))\chi dxdt$ for all $\chi \in V_0$.

Here $< \cdot,\cdot >$ denote the scalar product of elements from V^* and V.

We impose the following hypotheses which will be used later.

(A0) there exist ϕ and $\psi \in W$ such that

$$\begin{aligned} &\frac{\partial \phi}{\partial t} - L\phi \ge g(x,t,\phi,F(\phi)) \text{ in } Q_T \\ &\phi(x,0) \ge \varphi(x), \quad x \in \Omega \qquad (2.2) \\ &\phi \ge 0 \text{ on } \Gamma_T \end{aligned}$$

and

$$
\begin{aligned}
&\frac{\partial \psi}{\partial t} - L\psi \le g(x,t,\psi,F(\psi)) \text{ in } Q_T \\
&\psi(x,0) \le \varphi(x), \quad x \in \Omega \qquad (2.3)\\
&\psi \le 0 \text{ on } \Gamma_T
\end{aligned}
$$

Note that (2.2) and (2.3) will be understood in the weak sense. And ϕ and ψ are called weak upper and weak lower solution of (2.1) respectively.

(A1) $g(x,t,u,\eta) : Q_T \times R \times R \to R$ is of Caratheodory type and there is a positive constant γ such that

$$|g(\cdot,\cdot,u,\eta_1) - g(\cdot,\cdot,v,\eta_2)| \le \gamma[|u-v| + |\eta_1 - \eta_2|] \quad \text{for } u,v,\eta_1,\eta_2 \in R.$$

(A2) $g(x,t,u,\eta)$ is monotone nondecreasing with respect to η for each $(x,t) \in Q_T$ and $u \in R$.

(A3) F is Lipschitz continuous on $L^2(Q^+)$, that is, there exists a positive constant $\delta > 0$ such that $\|F(u) - F(v)\|_2 \le \delta \|u - v\|_2$ for $u, v \in L^2(Q^+)$, where $Q^+ = \{(x,t) \in L^2(Q_T) |\ u(x,t) \ge v(x,t) \text{ a.e. in } Q_T\}$.

(A4) F is monotone nondecreasing in u, *i.e.*, $u \le v$ in Q_T implies $F(u) \le F(v)$ in Q_T.

Theorem 2.1. (comparison): *Let* $(A0)$, $(A1)$ *and* $(A3)$ *be satisfied for a pair of functions* ϕ, $\psi \in W$ *with respect to the order interval* $I = [\inf(\phi,\psi), \sup(\phi,\psi)]$. *Then* $\psi \le \phi$ *a.e. in* Q_T.

Proof. Due to hyopthesis (A0), the functions ϕ and ψ fulfill inequalities (2.2) and (2.3), respectively, in the weak sense. Hence we have

$$
\begin{aligned}
&< \frac{\partial \phi}{\partial t}, \chi > + \ell(\phi,\chi) \ge \int_{Q_T} g(x,t,\phi,F(\phi))\chi dxdt \\
&< \frac{\partial \psi}{\partial t}, \chi > + \ell(\psi,\chi) \le \int_{Q_T} g(x,t,\psi,F(\psi))\chi dxdt
\end{aligned}
$$

for all test functions $\chi \in V_0, \chi \geq 0$ and

$$\psi(x,0) \leq \varphi(x) \leq \phi(x,0), \quad x \in \Omega$$
$$\psi(x,t) \leq 0 \leq \phi(x,t) \text{ on } \Gamma_T$$

Let $w = \psi - \phi$ in Q_T. Then

$$(2.4) < \frac{\partial w}{\partial t}, \chi > + \ell(w,\chi) \leq \int_{Q_T} [g(x,t,\psi,F(\psi)) - g(x,t,\phi,F(\phi))]\chi dxdt,$$

and

$$(2.5) \qquad w(x,t) \leq 0 \quad \text{on} \quad \Gamma_T \cup (\Omega \times \{0\}).$$

Let $w^+ = \max_{Q_T}(w,0)$. It is clear that $w^+ \in V_0$ and $w^+ \geq 0$ in Q_T.

Put $\chi = w^+$ in (1.4) and we see that by (2.5)

$$(2.6) \qquad < \frac{\partial w}{\partial t}, w^+ >= \frac{1}{2}\int_\Omega [w^+(\cdot,T)]^2 dx$$

and that, from [2], we get

$$(2.7) \qquad \ell(w,w^+) = \ell(w^+,w^+),$$

Let $Q^+ = \{(x,t) \in Q_T | \psi(x,t) \geq \phi(x,t)\}$. By using (A1), Hölder inequality, and (A3), we have

$$(2.8) \qquad \begin{aligned} &\textstyle\int_{Q_T}[g(x,t,\psi,F(\psi)) - g(x,t,\phi,F(\phi))]w^+ dxdt \\ &\leq \textstyle\int_{Q_T} \gamma[|\psi-\phi| + |F(\psi)-F(\phi)|]w^+ dxdt \\ &\leq \gamma \textstyle\int_{Q_T}(w^+)^2 dxdt + \gamma \int_{Q_T}(F(\psi)-F(\phi))w^+ dxdt \\ &\leq \gamma \textstyle\int_{Q_T}(w^+)^2 dxdt + \gamma[(\int_{Q^+}|F(\psi) \\ &\quad - F(\phi)|^2 dxdt)^{1/2}(\int_{Q^+}(w^+)^2 dxdt)^{1/2}] \\ &\leq \gamma(1+\delta)\textstyle\int_{Q_T}(w^+)^2 dxdt \end{aligned}$$

With (2.6), (2.7) and (2.8), we get the inequality

$$\frac{1}{2}\int_\Omega (w^+(\cdot,T))^2dx + \ell(w^+,w^+) \leq \gamma(1+\delta)\int_{Q_T}(w^+)^2dxdt \tag{2.9}$$

By ellipticity and Hölder inequality, we can choose sufficiently large $\lambda > 0$ so that

$$\ell(w^+,w^+) + \lambda\int_{Q_T}(w^+)^2dxdt \geq 0.$$

Hence from (2.9), we obtain

$$\int_\Omega (w^+(\cdot,T))^2dx \leq c\int_{Q_T}(w^+)^2dxdt, \tag{2.10}$$

where $c = 2[\lambda + \gamma(1+\delta)]$.

Let $y(s) = \int_{Q_s}(w^+)^2dxdt$, $0 \leq s \leq T$. Then by (2.10), we have

$$\frac{dy}{ds}(T) \leq cy(T),$$

where $y(0) = 0$ and $y(T) \geq 0$. Thus $y(T) = 0$. This means $w^+ = 0$ a.e. in Q_T which follows that $\psi \leq \phi$ a.e. in Q_T.

Corollary 2.2. *Let $c(x,t)$ be a real bounded measurable function in Q_T. Instead of replacing L by $L_c = L - c$ in Theorem 2.1, we obtain the same comparison result.*

proof. Let $\ell_c(h,\chi) = \int_{Q_T}\left(\sum_{i,j=1}^N a_{ij}\frac{\partial h}{\partial x_j}\frac{\partial \chi}{\partial x_i} + \sum_{i=1}^N b_i\frac{\partial h}{\partial x_i}\chi + ch\chi\right)dxdt$. We then have $\ell_c(w,w^+) = \ell_c(w^+,w^+)$. By following the same argument as in the proof of Theorem 2.1, we get the conclusion.

Now we shall use monotone scheme to construct the solution of (2.1) by using lemma.1.1.

Theorem 2.3. (Existence and Uniqueness) : *Let* (A0)-(A4) *be satisfied for a pair of functions* $\phi,\ \psi \in W$ *with respect to* $I = [\inf(\phi,\psi), \sup(\phi,\psi)]$. *Then the initial-boundary value problem* (2.1) *has a unique solution* $u \in W$ *with* $\psi \le u \le \phi$ *in* Q_T.

Proof. Let $H(x,t,u,\eta) = g(x,t,u,\eta) + Mu$, where M is a large constant such that H is monotone nondecreasing with respect to u. In fact, M just needs to be bigger than γ. Consider the iterative scheme

$$\begin{aligned} &\frac{\partial u^{i+1}}{\partial t} - Lu^{i+1} + Mu^{i+1} = H(x,t,u^i,F(u^i)) \quad \text{in} \quad Q_T, \\ &u^{i+1}(x,0) = \varphi(x) \quad \text{in} \quad \Omega, \\ &u^{i+1}(x,t) = 0 \quad \text{on} \quad \Gamma_T,\ i = 0,1,2,\ldots. \end{aligned} \tag{2.11}$$

Taking $v^0 = \phi$ as an initial function in (2.11), a sequence $\{v^i\}_{i=0}^{\infty}$ is generated.

We shall claim that $v^{i+1} \le v^i$ in Q_T for $i = 0,1,2,\ldots$. In fact, let $w^1 = v^1 - v^0$ in Q_T, then we have

$$\frac{\partial w^1}{\partial t} - Lw^1 + Mw^1 = g(x,t,\phi,F(\phi)) - (\frac{\partial \phi}{\partial t} - L\phi) \le 0,$$

and $w^1(x,0) \le 0$ in Ω, $w^1 \le 0$ on Γ_T.

Hence by Corollary 2.2, we have $w^1 \le 0$ in Q_T. That is, $v^1 \le v^0$ in Q_T.

Assume that $v^n \le v^{n-1}$ hold in Q_T for $n > 1$. Then, from (2.11), we have the initial-boundary value problem for $w^{n+1} = v^{n+1} - v^n$

$$\frac{\partial w^{n+1}}{\partial t} - Lw^{n+1} + Mw^{n+1} = H(x,t,v^n,F(v^n)) - H(x,t,v^{n-1},F(v^{n-1})) \le 0$$

with $w^{n+1}(x,0) = 0$ in Ω and $w^{n+1} = 0$ on Γ_T.

By Corollary 2.2 again, we get $w^{n+1} \le 0$ in Q_T. That is, $v^{n+1} \le v^n$ in Q_T.

Therefore, by induction, we obtain $v^{i+1} \le v^i$ in Q_T for all $i \ge 0$.

When applying the iterative scheme (2.11) with $z^0=\psi$, we can similarly obtain a sequence $\{z^i\}_{i=1}^{\infty}$, $z^0=\psi$ such that $z^i\le z^{i+1}$ in Q_T, $z^{i+1}(x,0)=\varphi(x)$ in Ω and $z^{i+1}=0$ on Γ_T for $i\ge 0$.

Next we claim that $z^i\le v^i$ in Q_T for $i\ge 0$. In fact, by Theorem 2.1, $z^0=\psi\le\phi=v^0$ in Q_T. Let $w^1=z^1-v^1$ in Q_T. By (A2) and (A4), w^1 satisfies the linear initial-boundary value problem

$$\frac{\partial w^1}{\partial t}-Lw^1+Mw^1=H(x,t,z^0,F(z^0))-H(x,t,v^0,F(v^0))\le 0 \text{ in } Q_T$$

with $w^1(x,0)=0$ in Ω and $w^1=0$ on Γ_T.

Applying Corollary 2.2, we get $w^1\le 0$ in Q_T, That is, $z^1\le v^1$ in Q_T.

By induction argument, due to (A2) and (A4), we have shown that $z^i\le v^i$ in Q_T for all $i\ge 0$.

Therefore, we get two monotone sequences $\{v^i\}$ and $\{z^i\}$ such that

$$\psi=z^0\le z^1\le z^2\le\cdots\le v^2\le v^1\le v^0=\phi \text{ in } Q_T.$$

According to Lemma 1.1, we have the estimates

$$\begin{aligned}\|v^{i+1}\|_W^2&\le c\Big(\|\varphi\|_{L^2(\Omega)}^2+\|H(\cdot,\cdot,v^i,F(v^i))\|_{L^2(Q_T)}^2\Big)\\ \|z^{i+1}\|_W^2&\le c\Big(\|\varphi\|_{L^2(\Omega)}^2+\|H(\cdot,\cdot,z^i,F(z^i))\|_{L^2(Q_T)}^2\Big)\end{aligned}\tag{2.12}$$

Since $\{v^i\}$ and $\{z^i\}$ are situated in the order interval $[\psi,\phi]$, the right-hand sides of (2.12) are bounded for all $i\ge 0$, thus the sequence $\{v^i\}$ and $\{z^i\}$ are bounded in W. Since each bounded sequence in a reflexive Banach space W has a weakly convergent subsequence ([11]), there exist subsequence of $\{v^i\}$ and $\{z^i\}$ which are weakly convergent in W. Let v and z be the weak limits of the subsequence of $\{v^i\}$ and $\{z^i\}$ respectively. Moreover, the compact embedding $W\subseteq L^2(Q_T)$ implies the convergence of some subsequences of $\{v^i\}$ and $\{z^i\}$ in $L^2(Q_T)$. Because of the monotonicity of $\{v^i\}$ and $\{z^i\}$, the sequences $\{v^i\}$ and $\{z^i\}$ must converge in $L^2(Q_T)$ to v and z, respectively.

From [4], we get the weak convergence in W of the whole sequences $\{v^i\}$ and $\{z^i\}$ to v and z, respectively. Hence v and z are the solutions of the initial-boundary value problem (2.1). By Theorem 2.1, $v = z$ in Q_T. Hence the sequences $\{v^i\}$ and $\{z^i\}$ are convergent to the same limit solution v of (2.1).

The uniqueness of the solution of (2.1) is also quaranteed by Theorem 2.1.

Remarks. (1) If in (A2), $g(x,t,u,\eta)$ is monotone nonincreasing with respect to η, and at the same time in (A4), F is monotone nonincreasing, then we can have the same result of Theorem 2.3.

(2) The above conclusion also holds with mixed boundary conditions $\frac{\partial u}{\partial n} + d(x,t)u = 0$ on Γ_T, where $\frac{\partial u}{\partial n}$ is the outward normal derivative of u on Γ_T and $d(x,t) \geq 0$ on Γ_T by considering the new bilinear form $\ell(h,\chi) + \int_{\Gamma_T} dh\chi ds dt$.

3. Existence for systems. In this section, we shall consider the nonlinear initial-boundary value problem for the system

$$
\begin{aligned}
&\frac{\partial u_k}{\partial t} - L_k u_k = g_k(x,t,u,F(u)) \text{ in } Q_T, \\
&u_k(x,0) = \varphi_k(x) \text{ in } \Omega, \\
&u_k(x,t) = 0 \text{ on } \Gamma_T,
\end{aligned}
\tag{3.1}
$$

for $k = 1,\ldots,n$, where $u \in R^n$. Here $F(u) = (F_1(u_1),\ \ldots, F_n(u_n))$ is an integral operator, for example, it is of the form : $F_i(u_i)(x,t) = \int_\Omega k_i(x,y)u_i(y,t)dy$.

Assume that the operator L_k is uniformly elliptic in Q_T and the coefficients a_{ij}^k and b_i^k are real bounded measurable in Q_T, and that $\varphi_k \in L^2(\Omega)$.

Let

$$\ell_k(h,\chi)=\int_{Q^T}\Big(\sum_{i,j=1}^{N}a_{ij}^k\frac{\partial h}{\partial x_j}\frac{\partial \chi}{\partial x_i}+\sum_{i=1}^{N}b_{ij}^k\frac{\partial h}{\partial x_i}\chi\Big)dxdt,$$

Definition. A function $u:Q_T\to R^n$ in W_0^n is called a weak solution of (3.1) if the following conditions are fulfilled:

(i) $u(x,0)=\varphi(x)$ in the sense that $\lim_{t\to 0}\|u_k(\cdot,t)-\varphi_k\|_2=0,\quad k=1,\ldots,n.$

(ii) $<\frac{\partial u_k}{\partial t},\chi>+\ell_k(u_k,\chi)=\int_{Q_T}g_k\,(x,t,u,F(u))\chi dxdt$, for all $\chi\in V_0$, $k=1,\ldots,n.$

Let

$$[u,v]=\{z\in(L^2(Q_T))^n|u_k\le z_k\le v_k \text{ a.e. in } Q_T \text{ for } k=1,\ldots,n\}.$$

For $u,v\in R^n$, let

$G_k\ (x,t,u_k,[v]_k,F_k(u_k),[F(v)]_k)=g_k(x,t,u_k,[v]_k,F_k(u_k),\ [F(v)]_k)-[M]_k[v]_k$ for some $n\times n$ matrix M, where $[\cdot]_k[\cdot]_k$ is the scalar product in R^{n-1}.

Herafter we give the following assumptions which will be used later.

(B0) There exist vectors ϕ and $\psi\in W^n$ such that

$$\begin{aligned}&\frac{\partial\phi_k}{\partial t}-L_k\phi_k-[M]_k[\phi]_k\ge G_k(x,t,\phi_k,[\psi]_k,F_k(\phi_k),[F(\psi)]_k) \text{ in } Q_T,\\ &\phi_k(x,0)\ge\varphi_k(x) \text{ in } \Omega,\\ &\phi_k\ge 0 \text{ on } \Gamma_T\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}&\frac{\partial\psi_k}{\partial t}-L_k\psi_k-[M]_k[\psi]_k\le G_k(x,t,\psi_k,[\phi]_k,F_k(\psi_k),[F(\phi)]_k) \text{ in } Q_T,\\ &\psi_k(x,0)\le\varphi_k(x) \text{ in } \Omega,\end{aligned}\tag{3.3}$$

$\psi_k \geq 0$ on Γ_T

for a $n \times n$ matrix $M = (M_{ij}) \geq 0$ such that $G_k(\cdot, \cdot, u, \cdot)$ is monotone nonincreasing in u_i, for $i \neq k$.

(B1) $g_k(x, t, u, p)$, $k = 1, 2, \ldots, n$, are of Caratheodory type and there exist some nonnegative constants γ_{ki}, $1 \leq i \leq n$ such that

$$g_k(\cdot, \cdot, u, p) - g_k(\cdot, \cdot, v, q) \leq \sum_{i=1}^{n} \gamma_{ki}|u_i - v_i| - \sum_{i=1, i\neq k}^{n} \gamma_{ki}(p_i - q_i) + \gamma_{kk}|p_k - q_k|,$$

holds for $u, v \in R^n$ and $p, q \in R^n$.

(B2) $g_k(\cdot, \cdot, \cdot, \eta)$ are monotone nondecreasing with respect to η_k and monotone nonincreasing with respect to η_i, $i \neq k$.

(B3) Each F_i, $i = 1, 2, \ldots, n$, is Lipschitz continuous on $L^2(Q^+)$, that is, there exists constant δ such that $\|F_i(u) - F_i(v)\|_2 \leq \delta\|u - v\|_2$, for $u, v \in L^2(Q^+)$, here $Q^+ = \{(x, t) \in L^2(Q_T) | u(x, t) \geq v(x, t)$ a.e. in $Q_T\}$.

(B4) Each F_i, $i = 1, 2, \ldots, n$, is monotone nondecreasing in u, i.e., $u \leq v$ in Q_T implies $F_i(u) \leq F_i(v)$ in Q_T.

Theorem 3.1. (comparison): *If* (B0), (B1) *and* (B3) *are satisfied for a pair of functions* $\phi, \psi \in W^n$ *with respect to the order interval* $I = [\inf(\phi, \psi), \sup(\phi, \psi)]$. *Then* $\psi \leq \phi$ *in* Q_T.

Proof. Let $w_k = \psi_k - \phi_k$ in Q_T. By (B0), for all test functions $\chi \in V_0$, $\chi \geq 0$, we have

$$< \frac{\partial w_k}{\partial t}, \chi > + \ell_k(w_k, \chi) - \int_{Q_T} [M]_k [w]_k \chi dx dt \tag{3.4}$$

$$\leq \int_{Q_T} \{G_k(x, t, \psi_k, [\phi]_k, F_k(\psi_k), [F(\phi)]_k) - G_k(x, t, \phi_k, [\psi]_k, F_k(\phi_k), [F(\psi)]_k)\} \chi dx dt$$

Let $w_k^+ = \max_{Q^T}(w_k, 0)$. And put $\chi = w_k^+$ in (3.4). We see that

$$\text{(3.5)} \qquad \text{(i)} \qquad < \frac{\partial w_k}{\partial t}, w_k^+ >= \frac{1}{2}\int_\Omega (w_k^+(\cdot, T))^2 dx,$$

since $w_k(x,0) \le 0$ in Ω.

$$\text{(3.6)} \qquad \text{(ii)} \qquad \int_{Q_T} [M]_k[w]_k w_k^+ dxdt \le \int_{Q_T} [M]_k[w^+]_k w_k^+ dxdt,$$

$$\text{(3.7)} \qquad \text{(iii)} \qquad \ell_k(w_k, w_k^+) = \ell_k(w_k^+, w_k^+),$$

(iv) The last integral in (3.4)

$$\begin{aligned}
&\le \int_{Q_T} \gamma_{kk}|\psi_k - \phi_k| w_k^+ dxdt + \int_{Q_T} \gamma_{kk}|F_k(\psi_k) - F_k(\phi_k)| w_k^+ dxdt \\
&\quad + \int_{Q_T} ([M]_k[\psi - \phi]_k + [\gamma]_k[|\phi - \psi|]_k) w_k^+ dxdt \\
&\quad - \int_{Q_T} [\gamma]_k[F(\phi) - F(\psi)]_k w_k^+ dxdt \\
&\le \int_{Q_T} \gamma_{kk}(w_k^+)^2 dxdt + \int_{Q_T} [M+\gamma]_k[w^+]_k w_k^+ dxdt \\
\text{(3.8)} &\quad + K_1 \int_{Q_T} (w_k^+)^2 dxdt + \frac{\delta}{2}\sum_{i=1}^n \gamma_{ki}\Big(\int_{Q_T} (w_i^+)^2 dxdt + \int_{Q_T} (w_k^+)^2 dxdt\Big)
\end{aligned}$$

for some constant K_1, here we have used (B1), Hölder inequality and (B3). Combining (3.5), (3.6), (3.7) and (3.8), we get the estimate

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (w_k^+(\cdot, T))^2 dx + \ell_k(w_k^+, w_k^+) \\
\le\ & \int_{Q_T} \{(\gamma_{kk} + K_1)(w_k^+)^2 + [M]_k[w^+]_k w_k^+\} dxdt \\
&+ \sum_{i=1}^n \frac{1}{2}\gamma_{ki} \int_{Q_T} \{(w_i^+)^2 + (w_k^+)^2\} dxdt
\end{aligned}$$

By ellipticity and Hölder inequality, there is a positive large constant λ such that

$$\ell_k(w_k^+, w_k^+) + \lambda \int_{Q_T} (w_k^+)^2 dxdt \ge 0 \text{ for } k = 1, \ldots, n.$$

Hence we obtain

$$\int_\Omega (w_k^+(\cdot,T))^2 dx \le c\int_{Q_T}\sum_{i=1}^n (w_i^+)^2 dxdt \tag{3.9}$$

for some constant c.

Summing up all the inequalities in (3.9) from $k=1$ to $k=n$, we get

$$\frac{dy(T)}{ds} \le cy(T)$$

where we set $y(s)=\int_{Q_s}\sum_{j=1}^n (w_j^+)^2 dxdt$, for $0\le s\le T$.

Since $y(0)=0$ and $y(T)\ge 0$ for $T\ge 0$, we then get $y(T)=0$. This means $w_k^+=0$ in Q_T, and then it follows that $\psi\le\phi$ in Q_T.

Corollary 3.2. *Let $c^k(x,t)$ be real bounded measurable functions in Q_T for $k=1,\dots,n$. Replacing L_k by $L_k^c=L_k-c^k(x,t)$ in Theorem* 3.1. *We obtain the same conclusion of Theorem* 3.1.

Now, we shall prove the following existence result by using the above comparison theorem.

Theorem 3.3. (Existence):*Let* (B0)-(B4) *be satisfied for a pair of functions $\phi,\psi\in W^n$ with respect to $I=[\inf(\psi,\phi),\sup(\psi,\phi)]$. Then the initial-boundary value problem* (3.1) *has a solution $u\in W^n$ with $\psi\le u\le\phi$ in Q_T.*

Proof. Let

$H_k(x,t,u_k,[v]_k,F_k(u_k),[F(v)]_k)=G_k(x,t,u_k,[v]_k,F_k(u_k),[F(v)]_k)+M_{kk}u_k$, where M_{kk} is a sufficiently large constant such that H is monotone nondecreasing with respective to u_k withn I.

Consider the following initial boundary value problems iteratively

$$\begin{aligned} &\frac{\partial u_k^{j+1}}{\partial t} - L_k u_k^{j+1} + M_{kk} u_k^{j+1} - [M]_k [u^{j+1}]_k \\ (3.10.a) \quad &= H_k(x,t,u_k^j,[v^j]_k,F_k(u_k^j),[F(v^j)]_k) \text{ in } Q_T, \\ &\frac{\partial v_k^{j+1}}{\partial t} - L_k v_k^{j+1} + M_{kk} v_k^{j+1} - [M]_k [v^{j+1}]_k \\ (3.10.b) \quad &= H_k(x,t,v_k^j,[u^j]_k,F_k(v_k^j),[F(u^j)]_k) \text{ in } Q_T, \end{aligned}$$

with

$$\begin{aligned} &u_k^{j+1}(x,0) = v_k^{j+1}(x,0) = \varphi_k(x) \text{ in } \Omega, \\ &u_k^{j+1} = v_k^{j+1} = 0 \text{ on } \Gamma_T, \quad k = 1,\ldots,n. \end{aligned}$$

Take $u^0 = \phi$ and $v^0 = \psi$ in (3.10). By Lemma 1.1, the existence and uniqueness of the solutions are guaranteed. By the similar arguments as in Theorem 2.3 and by Corollary 3.2, we obtain that

$$\psi = v^0 \le v^1 \le v^2 \le \cdots \le u^2 \le u^1 \le u^0 = \phi.$$

As in the arguments of Theorem 2.3, we see that the sequences $\{u^j\}$ and $\{v^j\}$ converge weakly in W^n to some function u and v, respectively. Hence the limiting process can be carried out in the weak formulation of (3.1). It follows that the limits u and v are the solutions of

$$(3.11) \quad \begin{aligned} &\frac{\partial u_k}{\partial t} - L_k u_k - [M]_k [u]_k = G_k(x,t,u_k,[v]_k,F_k(u_k),[F(v)]_k \text{ in } Q_T, \\ &\frac{\partial v_k}{\partial t} - L_k v_k - [M]_k [v]_k = G_k(x,t,v_k,[u]_k,F_k(v_k),[F(u)]_k \text{ in } Q_T, \\ &u_k(x,0) = v_k(x,0) = \varphi_k(x) \text{ in } \Omega, \end{aligned}$$

and $u_k = v_k = 0$ on Γ_T.

By Theorem 3.1, we have $u \equiv v$ in Q_T. Therefore, from (3.11), u is a solution of (3.1) and $\psi \leq u \leq \phi$ in Q_T.

Theorem 3.4. (Uniqueness):*Assume that all assumptions of theorem* 3.3 *are satisfied except that assumption* (B1) *is replaced by the following condition:*

(B1*) *$g_k(x,t,u,p)$, $k = 1,2,\ldots,n$, are of Caratheodory type and satisfy the inequality*

$$\begin{aligned}&g_k(\cdot,\cdot,u_k,[u]_k,p_k,[p]_k) - g_k(\cdot,\cdot,v_k,[v]_k,q_k,[q]_k)\\&\leq \gamma_{kk}(|u_k - v_k| + |p_k - q_k|) + [\gamma]_k([u-v]_k - [p-q]_k),\end{aligned}$$

for $u, v \in R^n$, $p, q \in R^n$ and for some nonnegative constants γ_{ki}, $1 \leq i \leq n$.

Then there exists a unique solution of (3.1).

Proof. By doing the same arguments as in proof of Theorem 2.1.

Example 3.5. Consider the following initial-boundary value problem:

$$\text{(3.12)}\quad \begin{aligned}&\frac{\partial u_1}{\partial t} - \Delta u_1 = \cos u_1 \sin u_2 + \int_\Omega a \cdot u_1(x,t)dx - \int_\Omega b \cdot u_2(x,t)dx \text{ in } Q_T,\\&\frac{\partial u_2}{\partial t} - \Delta u_2 = \sin u_1 \cos u_2 - \int_\Omega a \cdot u_1(x,t)dx + \int_\Omega b \cdot u_2(x,t)dx \text{ in } Q_T,\\&u_1(x,0) = \varphi_1(x), u_2(x,0) = \varphi_2(x) \text{ in } \Omega\\&u_1(x,t) = u_2(x,t) = 0 \text{ on } \Gamma_T \text{ in } \Omega,\end{aligned}$$

here a and b are some nonnegative constants.

Assume that $0 \leq \varphi_k(x) \leq L$ in Ω.

Set $\phi_k = -\psi_k = \varepsilon e^{\lambda t}$, $k = 1,2$, where ε and λ are chosen such that $\varepsilon \geq \max\{1, L\}$ and $\lambda \geq 3 + (a+b)Vol(\Omega)$. All assumptions of Theorem 3.3 are satisfied. Hence there is a solution u of (3.12) lying between $-\varepsilon e^{\lambda t}$ and $\varepsilon e^{\lambda t}$.

References

1. R. A. Adams, Sobolev Spaces, Academic Press, London, (1975).

2. S. Carl, *Lösung semilinear parabolischer Rand-Anfangswert problem durch monotone Iteration im Sobolevraum* $W_2^{1,1}(Q_T)$, Z. Anal. Anwendungen, **5**(1986), 237-251.

3. S. Carl, *A monotone iterative scheme for nonlinear reaction-diffusion systems having nonmonotone reaction terms*, J. Math. Anal. Appl., **134**(1988), 81-93.

4. H. Gajewski, K. Gröger and K. Zacharias, Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen, Akademie-Verlag, Berlin, 1974.

5. J. L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, Berlin, 1971.

6. C. V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York, 1992.

7. V. P. Politjukov, *On the theory of upper and lower solutions and the solvability of quasilinear integro-differential equations*, Math. USSR. Sbornik. **35**(1979), 499-507.

8. L. Y. Tsai, *On integro-differential equations of parabolic type*, Bull. Inst. Math. Acad. Sinica, (1981), 311-320.

9. L. Y. Tsai, *Existence of solutions for parabolic integro-differential systems*, Sino-Japanese. Joint seminar on PDE, Academic, Sinica, (1990).

10. L. Y. Tsai, *Comparison and stability results for parabolic integro-differential equations*, Proc. International Math. Conference '94, World Scientific, 203-218, 1996.

11. E. Zeidler, Nonlinear Functional Analysis and its Application, Vol. II/A. Spring-Verlag, New York, 1986.

Department of Mathematical Science, National Chengchi University, Taipei, Taiwan.