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BLOW-UP RESULTS AND ASYMPTOTIC BEHAVIOR OF THE EMDEN-FOWLER EQUATION $u'' = |u|^p$*

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Abstract In this article the author works with the ordinary differential equation $u'' = |u|^p$ for some $p > 0$ and obtains some interesting phenomena concerning blow-up, blow-up rate, life-span, stability, instability, zeros and critical points of solutions to this equation.

Key words Estimate, life-span, blow-up, blow-up rate, stability

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0 Introduction

In our articles [1–6] we studied the semilinear wave equation(s) $\Box u + f(u) = 0$ under some conditions, and found some interesting results on blow-up, blow-up rate and the estimates for the life-span of solutions, but no information on the singular set. Here we want to deal with the particular cases in lower dimensional wave equations. We hope that the experiences gained here will allow us to deal with more general lower dimensional cases later.

Consider one dimensional stationary, semilinear wave equation

$$\begin{cases} u'' - |u|^p = 0, \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{0.1}$$

From some calculations one can find that, for $p \in (0,1)$, equation (0.1) with $u_0 = u_1 = 0$ possesses infinitely many solutions, so the solutions of the above equation in general are not unique. It is clear that the functions $|u|^p, p \geq 1$, are locally Lipschitz, hence by the standard theory, the local existence of classical solutions is applicable to equation (0.1).

We discuss the problem (0.1) in three parts: $p > 1$, $p < 1$, and the singularity and regularity of solutions.

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Part A Estimates for the Life-Span of Solutions of (0.1) for $p>1$

We would in Section 1 deal with the estimates for the life-span of the solutions of (0.1); in Section 2 with the blow-up rate and blow-up constant; in Section 3 with the global existence, the critical point and the asymptotic behavior; in Section 4 with the null points (zeros) and triviality; in Section 5 with the stability and instability.

Notations and Fundamental Lemmas

For a given function u in this work we use the following abbreviations

$$a_u(t)=u(t)^2,\ E_u(0)=u_1^2-\frac{2}{p+1}\left|u_0\right|^p u_0,\ J_u\left(t\right)=a_u(t)^{-\frac{p-1}{4}}.$$

Definition A function $g:\mathbb{R}\to\mathbb{R}$ has a blow-up rate q means that g exists only in finite time, that is, there is a finite number T^* such that the followings are valid

$$\lim_{t\to T^*} g(t)^{-1}=0, \tag{0.2}$$

and there exists a non-zero $\beta\in\mathbb{R}$ with

$$\lim_{t\to T^*}\left(T^*-t\right)^q g(t)=\beta, \tag{0.3}$$

in this case β is called the blow-up constant of g.

As the solutions for equation (0.1) for $p>1$ is unique, we can rewrite $a_u(t)=a(t), J_u(t)=J(t)$ and $E_u(t)=E(t)$. After some elementary calculations we obtain the following lemma.

Lemma 1 Suppose that u is the solution of (0.1), then we have

$$E(t)=u'(t)^2-\frac{2}{p+1}\left|u(t)\right|^p u=E(0), \tag{0.4}$$

$$\left(p+3\right)u'(t)^2=(p+1)E(0)+a''(t), \tag{0.5}$$

$$J''(t)=\frac{p^2-1}{4}E(0)J(t)^{\frac{p+3}{p-1}}, \tag{0.6}$$

$$J'(t)^2=J'(0)^2-\frac{(p-1)^2}{4}E(0)J(0)^{\frac{2(p+1)}{p-1}}+\frac{(p-1)^2}{4}E(0)J(t)^{\frac{2(p+1)}{p-1}}, \tag{0.7}$$

and

$$a'(t)=a'(0)+2E(0)t+\frac{2\left(p+3\right)}{p+1}\int_0^t\left|u\right|^p u(r)\mathrm{d}r. \tag{0.8}$$

The following lemmas are easily to prove, so we omit the arguments.

Lemma 2 Suppose that r and s are real constants and $u\in C^2\left(\mathbb{R}\right)$ satisfies the inequality

$$u''+ru'+su\le 0,\quad u\ge 0,$$

$$u(0)=0,\quad u'(0)=0,$$

then u must be null, that is, $u\equiv 0$.

Lemma 3 If $g(t)$ and $h\left(t,r\right)$ are continuous with respected to their variables and the limit $\lim\limits_{t\to T}\int_0^{g(t)}h\left(t,r\right)\mathrm{d}r$ exists, then

$$\lim_{t\to T}\int_0^{g(t)}h\left(t,r\right)\mathrm{d}r=\int_0^{g(T)}h\left(T,r\right)\mathrm{d}r.$$

1 Estimates for the Life-Spans

To estimate the life-span of the solution of equation (0.1), we separate this section into three parts, $E(0) < 0$, $E(0) = 0$ and $E(0) > 0$. Here the life-span T of u means that u is the solution of problem (0.1) and the existence of time interval of u is only in $[0, T)$ so that problem (0.1) possesses the solution $u \in \bar{C}^2(0, T)$ and u make sense only in this interval.

1.1 Estimates for the Life-spans Under $E(0) \le 0$

In this subsection we deal with the case that $E(0) < 0$ and $E(0) = 0$, $a'(0) > 0$. The case that $E(0) = 0$ and $a'(0) \le 0$ will be considered in Section 3 and Section 4. We have the following result.

Theorem 4 If T is the life-span of the solution u of the problem (0.1) with $E(0) < 0$, then T is finite. Further, for $a'(0) \ge 0$ we have the estimate

$$T \le T_1^* := T_1^*(u_0, u_1, p) = \frac{2}{p-1} \int_0^{J(0)} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0) r^{\frac{2p+2}{p-1}}}}; \tag{1.1.1}$$

for $a'(0) < 0$,

$$T \le T_2^*(u_0, u_1, p) = \frac{2}{p-1} \left(\int_0^{\alpha} + \int_{J(0)}^{\alpha} \right) \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0) r^{\frac{2p+2}{p-1}}}}, \tag{1.1.2}$$

where $\alpha := \alpha\,(p, E(0)) = \left(\frac{2}{p+1} \frac{-1}{E(0)} \right)^{\frac{p-1}{2p+2}}$. Furthermore, if $E(0) = 0$ and $a'(0) > 0$, then

$$T \le T_3^*(u_0, u_1, p) := \frac{4}{p-1} \frac{a(0)}{a'(0)}. \tag{1.1.3}$$

Proof Under the circumstance, $E(0) < 0$, we know that $a(0) > 0$; as otherwise we get $a(0) = 0$, that is, $u_0 = 0$, then $E(0) = u_1^2 \ge 0$, this contradicts to $E(0) < 0$.

In this situation we separate the proof of this theorem into two cases, $a'(0) \ge 0$ and $a'(0) < 0$.

(i) $a'(0) \ge 0$. By (0.5) and (0.7) we find that

$$\begin{cases} a'(t) \ge a'(0) - (p+1)\,E(0)t & \forall t \ge 0, \\ a(t) \ge a(0) + a'(0)t - \frac{p+1}{2} E(0)t & \forall t \ge 0, \end{cases} \tag{1.1.4}$$

$$J'(t) = -\frac{p-1}{2} \sqrt{\frac{2}{p+1} + E(0) J(t)^{\frac{2p+2}{p-1}}} \le J'(0) \quad \forall t \ge 0, \tag{1.1.5}$$

and $J(t) \le a(0)^{-\frac{p-1}{4}} - \frac{p-1}{4} a(0)^{-\frac{p+3}{4}} a'(0)t, \forall t \ge 0$. Thus there exists a finite number $T_1^*(u_0, u_1, p) \le \frac{4}{p-1} \frac{a(0)}{a'(0)}$ such that $J\,(T_1^*\,(u_0, u_1, p)) = 0$ and that $a(t) \to \infty$ for $t \to T_1^*(u_0, u_1, p)$. This means that the life-span T of u is finite, that is, $T \le T_1^*(u_0, u_1, p)$.

Now we estimate this life-span $T_1^*(u_0, u_1, p)$. By (1.1.5) and the fact that $J\,(T_1^*(u_0, u_1, p)) = 0$ we find that

$$\int_{J(t)}^{J(0)} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0) r^{\frac{2p+2}{p-1}}}} = \frac{p-1}{2} t \quad \forall t \ge 0, \tag{1.1.6}$$

and hence we obtain the estimate (1.1.1).

(ii) $a'(0)<0$. By (1.1.4), $a'(0)<0$ and the convexity of a we can find a unique finite number $t_0=t_0(u_0,u_1,p)$ such that

$$\begin{cases} a'(t)<0 & \text{for } \ t\in(0,t_0)\,, \\ a'(t_0)=0, a'(t)>0 & \text{for } \ t>t_0, \end{cases} \tag{1.1.7}$$

and $a(t_0)>0$. If not, then $u(t_0)=0$, thus $E(0)=E(t_0)=u'(t_0)^2\geq 0$; yet this is a contradiction to $E(0)<0$. Thus we conclude that $a(t)>0, \forall t\geq 0, u'(t_0)=0, E(0)=-\frac{2}{p+1}u(t_0)^{p+1}$ and $J(t_0)^{\frac{2p+2}{p-1}}=\frac{2}{p+1}\frac{-1}{E(0)}$.

After arguments similar to the step (i), there exists a $T_2^*(u_0,u_1,p)$ such that the life-span T of u is bounded by $T_2^*(u_0,u_1,p)$, that is, $T\leq T_2^*(u_0,u_1,p)$. On the analogy of the above argumentation, using (1.1.7) and (0.7) we obtain that

$$J'(t)^2=-\frac{(p-1)^2}{4}E(0)\left(J\left(t_0(u_0,u_1,p)\right)^{\frac{2p+2}{p-1}}-J(t)^{\frac{2p+2}{p-1}}\right)\quad \forall t\geq t_0,$$

$$J'(t)^2=\frac{(p-1)^2}{4}E(0)\left(J(0)^{\frac{2p+2}{p-1}}-J(t)^{\frac{2p+2}{p-1}}\right)\quad \forall t\in[0,t_0]\,,$$

and

$$\begin{cases} J'(t)=-\dfrac{p-1}{2}\sqrt{\dfrac{2}{p+1}+E(0)J(t)^{\frac{2p+2}{p-1}}} & \forall t\geq t_0, \\ J'(t)=\dfrac{p-1}{2}\sqrt{\dfrac{2}{p+1}+E(0)J(t)^{\frac{2p+2}{p-1}}} & \forall t\in[0,t_0]\,, \end{cases} \tag{1.1.8}$$

where $t_0=t_0(u_0,u_1,p)$. Therefore we have

$$\begin{cases} \displaystyle\int_{J(t)}^{J(t_0)}\frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1}+E(0)r^{\frac{2p+2}{p-1}}}}=\frac{p-1}{2}\left(t-t_0\right) & \forall t\geq t_0, \\ \displaystyle\int_{J(0)}^{J(t_0)}\frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1}+E(0)r^{\frac{2p+2}{p-1}}}}=\frac{p-1}{2}t_0, \end{cases} \tag{1.1.9}$$

where $t_0=t_0(u_0,u_1,p)$. From (1.1.9) and the fact that $J\left(t_0(u_0,u_1,p)\right)^{\frac{2p+2}{p-1}}=\frac{2}{p+1}\frac{-1}{E(0)}$, $J(T_2^*(u_0,u_1,p))=0$, we conclude that

$$T_2^*(u_0,u_1,p)=t_0(u_0,u_1,p)+\frac{2}{p-1}\int_0^{\left(\frac{2}{p+1}\frac{-1}{E(0)}\right)^{\frac{p-1}{2p+2}}}\frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1}+E(0)r^{\frac{2p+2}{p-1}}}}. \tag{1.1.10}$$

This estimate (1.1.10) is equivalent to (1.1.2).

(iii) $E(0)=0$. Now we prove (1.1.3). By (0.6) and $E(0)=0$ we get that $J''(t)=0, \forall t\geq 0$. Since the positiveness of $a'(0)$, it follows that $J'(0)<0$ and $J(t)=a(0)^{-\frac{p-1}{4}}-\frac{p-1}{4}a(0)^{-\frac{p-1}{4}-1}a'(0)t, \forall t\geq 0$. Thus we conclude that

$$a(t)=a(0)^{\frac{p+3}{p-1}}\left(a(0)-\frac{p-1}{4}a'(0)t\right)^{-\frac{4}{p-1}}\quad \forall t\geq 0. \tag{1.1.11}$$

Therefore (1.1.3) is proved.

1.2 Estimates for the Life-spans Under $E(0) > 0$

In this subsection we consider the case $E(0) > 0$, and we have the following blow-up result.

Theorem 5 If T^* is the life-span of u which solves problem (0.1) with $E(0) > 0$, then T^* is finite. Further, we have

(i) for $u_0 > 0, a'(0) > 0$,

$$T^* \leq T_4^*(u_0, u_1, p) = \frac{2}{p-1}\int_0^{J(0)} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0)r^{\frac{2p+2}{p-1}}}}. \tag{1.2.1–1}$$

(ii) for $u_0 < 0, a'(0) > 0$,

$$T^* \leq T_5^*(u_0, u_1, p) := (S_1 + Z_1)(u_0, u_1, p) + \int_0^\infty \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1}r^{p+1}}}, \tag{1.2.1–2}$$

where

$$S_1(u_0, u_1, p) = \int_{-u_0}^{\left(\frac{p+1}{2}E\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1}r^{p+1}}},$$

$$Z_1(u_0, u_1, p) = \int_0^{\left(\frac{p+1}{2}E(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1}r^{p+1}}}.$$

(iii) for $u_1 > 0 = a'(0)$,

$$T^* \leq T_6^*(u_0, u_1, p) = \int_0^\infty \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1}r^{p+1}}}. \tag{1.3.1–1}$$

(iv) for $u_1 < 0 = a'(0)$,

$$T^* \leq T_7^* = (S_2 + Z_2)(u_0, u_1, p) + \int_0^\infty \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1}r^{p+1}}}, \tag{1.3.1–2}$$

where $S_2(u_0, u_1, p) = \int_{-u_0}^{\left(\frac{p+1}{2}u_1^2\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{u_1^2 - \frac{2}{p+1}r^{p+1}}}$ and $Z_2(u_0, u_1, p) = \int_0^{\left(\frac{p+1}{2}u_1^2\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{u_1^2 - \frac{2}{p+1}r^{p+1}}}$.

(v) for $u_1 > 0 > a'(0)$, $Z_3(u_0, u_1, p)$ given by

$$Z_3(u_0, u_1, p) = \int_0^{\sqrt{a(0)}} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1}r^{p+1}}} \tag{1.4.1}$$

is the zero of a. Further we have

$$T^* \leq T_8^*(u_0, u_1, p) := (Z_3 + T_6^*)(u_0, u_1, p). \tag{1.4.2}$$

(vi) for $u_1 < 0, a'(0) < 0$,

$$T^* \leq T_9^* = (2S_4 + Z_4)(u_0, u_1, p) + \int_0^\infty \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1}r^{p+1}}}, \tag{1.4.3}$$

where $S_4(u_0,u_1,p)=\int_0^{\left(\frac{p+1}{2}E(0)\right)^{\frac{1}{p+1}}}\frac{\mathrm{d}r}{\sqrt{E(0)-\frac{2}{p+1}r^{p+1}}}$ and $Z_4(u_0,u_1,p)=\int_0^{u_0}\frac{\mathrm{d}r}{\sqrt{E(0)+\frac{2}{p+1}r^{p+1}}}$.

Proof For the frequency we postpone the proof of (1.4.1) to Section 4 concerning the null point (zero) of u and to Section 3 for the critical point of u.

1) For $a'(0)>0$, by (0.6), we have

$$\begin{cases} kJ''(t)=(kJ(t))^q\,, \\ kJ(0)=k\,|u_0|^{-\frac{p-1}{2}}\,,kJ'(0)=\dfrac{1-p}{2}k\,|u_0|^{-\frac{p+3}{2}}\,u_0u_1, \end{cases}$$

where $k:=\left(\frac{p^2-1}{4}E(0)\right)^{\frac{p-1}{4}}$ and $q:=\frac{p+3}{p-1}$. Now we set

$$\tilde{E}(t):=k^2J'(t)^2-\frac{2}{q+1}\,(kJ(t))^{q+1}\,, \tag{1.2.2}$$

by some calculations we see that $\tilde{E}(t)$ is a constant and

$$\tilde{E}(t)=\frac{(p-1)^2}{16}k^2a(0)^{-\frac{p+3}{2}}\left(a'(0)^2-4E(0)a(0)\right)=\frac{(p-1)^2}{2p+2}k^2\mathrm{sign}u_0=\tilde{E}(0). \tag{1.2.3}$$

(i) $u_0>0$, then $u(t)>0, u'(t)>0$ and $a'(t)>0$ for $t\geq 0$. Together (0.7), (0.8), (1.2.2) and (1.2.3) we obtain that

$$J'(t)^2-\frac{2k^{q-1}}{q+1}J(t)^{q+1}=\frac{(p-1)^2}{2p+2},$$

$$a'(t)\geq a'(0)+2E(0)t>0\quad\forall t\geq 0, \tag{1.2.4}$$

$$J'(t)<0\quad\forall t\geq 0,$$

$$J'(t)=-\frac{p-1}{2}\sqrt{\frac{2}{p+1}+E(0)J(t)^{\frac{2p+2}{p-1}}}\quad\forall t\geq 0, \tag{1.2.5}$$

and

$$\int_{J(t)}^{J(0)}\frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1}+E(0)r^{(2p+2)/(p-1)}}}=\frac{p-1}{2}t\quad\forall t\geq 0. \tag{1.2.6}$$

By (1.2.5), there exists a finite number $T_4^*\left(u_0,u_1,p\right)$ such that $J\left(T_4^*\left(u_0,u_1,p\right)\right)=0$, and from (1.2.6) it follows the estimate (1.2.1–1).

(ii) $u_0<0, u_1<0$, by $u''=|u(t)|^p$, there exists $S_1\left(u_0,u_1,p\right):=S_1>0$ with $u'(t)<0$, $u(t)<0$ for $t\in(0,S_1)$ and $u'\left(S_1\right)=0, u'(t)>0$ for $t>S_1$; and also $Z_1(u_0,u_1,p):=Z_1>0$ with $u(t)<0$ in $\left(0,S_1+Z_1\right), u\left(S_1+Z_1\right)=0<u(t)$ for $t>S_1+Z_1$ and then $a'(t)>0$ for $t>S_1+Z_1$.

By (0.4) we have

$$u'=-\sqrt{E(0)-\frac{2}{p+1}\,|u\,(t)|^{p+1}}\qquad t\in[0,S_1]\,, \tag{1.2.7}$$

$$t=\int_{-u_0}^{-u(t)}\frac{\mathrm{d}r}{\sqrt{E(0)-\frac{2}{p+1}r^{p+1}}}\qquad t\in[0,S_1]\,, \tag{1.2.8}$$

$$S_1=\int_{-u_0}^{\left(\frac{p+1}{2}E\right)^{\frac{1}{p+1}}}\frac{\mathrm{d}r}{\sqrt{E(0)-\frac{2}{p+1}r^{p+1}}}\tag{1.2.9}$$

and also

$$u'=\sqrt{E(0)-\frac{2}{p+1}\left|u\left(t\right)\right|^{p+1}}\qquad t\geq S_1,$$

$$t-S_1=\int_{-u(t)}^{-u(S_1)}\frac{\mathrm{d}r}{\sqrt{E(0)-\frac{2}{p+1}r^{p+1}}}\qquad t\in\left[S_1,S_1+Z_1\right],\tag{1.2.10}$$

$$Z_1=\int_0^{\left(\frac{p+1}{2}E(0)\right)^{\frac{1}{p+1}}}\frac{\mathrm{d}r}{\sqrt{E(0)-\frac{2}{p+1}r^{p+1}}};$$

thus we get $a'(t)>0$ and $J'(t)<0,\forall t>S_1+Z_1$. We conclude that

$$\int_0^{u(t)}\frac{\mathrm{d}r}{\sqrt{E(0)+\frac{2}{p+1}r^{p+1}}}=t-\left(S_1+Z_1\right),\tag{1.2.11}$$

$$T^*=S_1+Z_1+\int_0^{\infty}\frac{\mathrm{d}r}{\sqrt{E(0)+\frac{2}{p+1}r^{p+1}}}.\tag{1.2.12}$$

By (1.2.5), there exists a finite number $T_5^*\left(u_0,u_1,p\right)$ such that $J\left(T_5^*\left(u_0,u_1,p\right)\right)=0$, and from (1.2.12) it follows easily the estimate (1.2.1–2).

2) From $a'(0)=0,\ u_0=0$ and $E(0)=u_1^2$, by (0.8) we obtain that $u'(t)=u_1+\int_0^t|u|^p\,(r)\mathrm{d}r$.

(iii) For $u_1>0$, then $u'(t)>0$, $J'(t)<0$, the estimate (1.3.1–1) could be easily obtained by (0.4).

(iv) $u_1<0$, there exists $S_2\left(u_0,u_1,p\right):=S_2>0$ with $u'(t)<0$, $u\left(t\right)<0$ for $t\in\left(0,S_2\right)$ and $u'\left(S_2\right)=0$, $u'(t)>0$ for $t>S_2$; and also $Z_2(u_0,u_1,p):=Z_2>0$ with $u(t)<0$ in $\left(0,S_2+Z_2\right)$, $u\left(S_2+Z_2\right)=0<u(t)$ for $t>S_2+Z_2$ and then $a'\left(t\right)>0$ for $t>S_2+Z_2$.

By (0.4),

$$u'=-\sqrt{u_1^2-\frac{2}{p+1}\left|u(t)\right|^{p+1}}\quad\text{for}\quad t\in\left[0,S_2\right],$$

$$t=\int_{-u_0}^{-u(t)}\frac{\mathrm{d}r}{\sqrt{u_1^2-\frac{2}{p+1}r^{p+1}}}\quad\text{for}\quad t\in\left[0,S_2\right],\tag{1.3.2}$$

$$S_2=\int_{-u_0}^{\left(\frac{p+1}{2}u_1^2\right)^{\frac{1}{p+1}}}\frac{\mathrm{d}r}{\sqrt{u_1^2-\frac{2}{p+1}r^{p+1}}},\tag{1.3.3}$$

also

$$u'=\sqrt{u_1^2-\frac{2}{p+1}\left|u(t)\right|^{p+1}}\quad\text{for}\quad t\in\left[S_2,S_2+Z_2\right],$$

$$t-S_2=\int_{-u(S_2)}^{-u(t)}\frac{-\mathrm{d}r}{\sqrt{u_1^2-\frac{2}{p+1}r^{p+1}}}\quad\text{for}\quad t\in\left[S_2,S_2+Z_2\right],\tag{1.3.4}$$

$$Z_2=\int_0^{\left(\frac{p+1}{2}u_1^2\right)^{\frac{1}{p+1}}}\frac{\mathrm{d}r}{\sqrt{u_1^2-\frac{2}{p+1}r^{p+1}}};\tag{1.3.5}$$

thus we get $a'(t) > 0$ and $J'(t) < 0, \forall t > S_2 + Z_2$. We conclude that

$$\int_0^{u(t)} \frac{\mathrm{d}r}{\sqrt{u_1^2 + \frac{2}{p+1} r^{p+1}}} = t - (S_2 + Z_2)\,, \tag{1.3.6}$$

$$T_7^* = S_2 + Z_2 + \int_0^{\infty} \frac{\mathrm{d}r}{\sqrt{u_1^2 + \frac{2}{p+1} r^{p+1}}}.$$

3) For $a'(0) < 0$, by (0.1) we expect that $a'(t) \geq 0$ for large t.

(v) For $u_1 > 0$, then $u'(t) \geq u_1$, $u(t) \geq u_1 t + u_0 > 0$ for large t ; thus $a'(t) > 0$ for large t. There exists $Z_3 := Z_3(u_0, u_1, p)$ with $u\,(Z_3) = 0$, $u(t) < 0$ for $t \in (0, Z_3)$; $u\,(t) > 0$ for $t > Z_3$. By (0.4), then $u' = \sqrt{u_1^2 - \frac{2}{p+1} |u(t)|^{p+1}}$,

$$\int_0^{-u_0} \frac{\mathrm{d}r}{\sqrt{u_1^2 - \frac{2}{p+1} r^{p+1}}} = Z_3, \tag{1.4.4}$$

and this is equivalent to (1.4.1) . Using (1.3.1–1) we obtain (1.4.2)

(vi) $u_1 < 0$, there exists $Z_4(u_0, u_1, p) := Z_4 > 0$ with $u'(t) < 0$, $u\,(t) > 0$ for $t \in (0, Z_4)$ and $u\,(Z_4) = 0$, $u(t) < 0$ for $Z_4 < t < Z_4 + S_4$; $u'\,(Z_4 + S_4) = 0, u'(t) > 0$ for $t > Z_4 + S_4$; $u(t) < 0$ for $Z_4 < t < Z_4 + S_4 + Z_5$ and $u(t) > 0$ for $t > Z_4 + S_4 + Z_5$.

$$u'(t) = -\sqrt{E(0) + \frac{2}{p+1} |u\,(t)|^{p+1}} \quad \text{for} \quad t \in [0, Z_4]\,,$$

$$t = -\int_{u_0}^{u(t)} \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1} r^{p+1}}} \quad \text{for} \quad t \in [0, Z_4]\,, \tag{1.4.5}$$

$$Z_4 = \int_0^{u_0} \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1} r^{p+1}}}, \tag{1.4.6}$$

and

$$u'(t) = -\sqrt{E(0) - \frac{2}{p+1} |u\,(t)|^{p+1}} \quad \text{for} \quad t \in [Z_4, Z_4 + S_4]\,,$$

$$t = \int_0^{-u(Z_4+t)} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1} r^{p+1}}} \quad \text{for} \quad t \in [0, S_4]\,, \tag{1.4.7}$$

$$S_4 = \int_0^{\left(\frac{p+1}{2} E(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1} r^{p+1}}}, \tag{1.4.8}$$

and also

$$u'(t) = \sqrt{E(0) - \frac{2}{p+1} |u\,(t)|^{p+1}} \quad \text{for} \quad t \in [Z_4 + S_4, Z_4 + S_4 + Z_5]\,,$$

$$t - (Z_4 + S_4) = \int_{-u(t)}^{\left(\frac{p+1}{2} E(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1} r^{p+1}}},$$

$$Z_5 = \int_0^{\left(\frac{p+1}{2} E(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1} r^{p+1}}} = S_4;$$

thus we get $a'(t) > 0$ and $J'(t) < 0, \forall t > Z_4 + 2S_4$. We conclude that

$$\int_0^{u(t)} \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1} r^{p+1}}} = t - (Z_4 + 2S_4)\,,$$

$$T_8^* = 2S_4 + Z_4 + \int_0^{\infty} \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1} r^{p+1}}}.$$

1.3 Some Properties Concerning the Life-span $T_1^*\,(u_0, u_1, p)$

In principle, $T_1^*(u_0, u_1, p)$ depends on u_0, u_1 and p. Set $c_{k,p} := \frac{(p+1)u_1^2}{2u_0^{p+1}}$, then $T_1^*(u_0, u_1, p) = \frac{\sqrt{2p+2}}{p-1} u_0^{-\frac{p-1}{2}} \cdot (1 - c_{k,p})^{-\frac{p-1}{2p+2}} \cdot \int_0^{(1-c_{k,p})^{\frac{p-1}{2p+2}}} \frac{\mathrm{d}r}{\sqrt{1-r^{\frac{2p+2}{p-1}}}}$ and $\lim\limits_{p\to\infty} T_1^*(u_0, u_1, p) = 0, \lim\limits_{p\to 1} T_1^*(u_0, u_1, p) = \infty$. For convenience, we consider the case $u_1 = 0, T_1^*(u_0, 0, p) = \frac{\sqrt{\pi}}{\sqrt{2p+2}} u_0^{-\frac{p-1}{2}} \frac{\Gamma\left(\frac{p-1}{2p+2}\right)}{\Gamma\left(\frac{p}{p+1}\right)}$.

Using Maple we obtain the graphs of $T_1^*\,(u_0, 0, p)$ below:

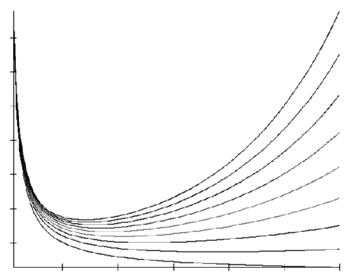

Fig.1 Graphs of $T_1^*(u_0, 0, p), u_0 \leq 1$

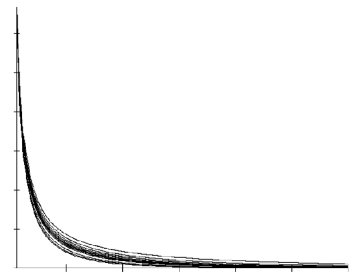

Fig.2 Graphs of $T_1^*(u_0, 0, p), u_0 > 1$

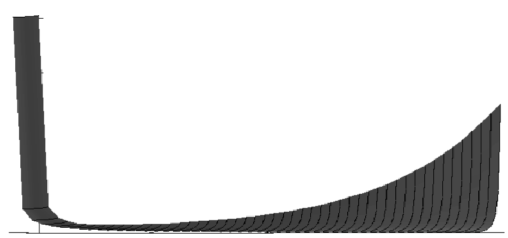

Fig.3 Graphs of $T_1^*(u_0, 0, p)$

The above pictures show the properties of $T_1^*\,(u_0, 0, p)$:

(1) there exists a constant u_0^* such that $T_1^*\,(u_0, 0, p)$ is monotone decreasing in p for $u_0 \in [u_0^*, 1)$;

(2) there is a p_0 such that $T_1^*\,(u_0, 0, p)$ is decreasing in $(1, p_0)$ and increasing in (p_0, ∞) provided $u_0 \in [0, u_0^*)$;

(3) $T_1^*\,(u_0, 0, p)$ is differentiable in its variables and

(4) for $u_0 > 1$ the life-span $T_1^*\,(u_0, 0, p)$ is decreasing in p.

We now show the validity of statements (3) and (4) by using the monotonicity of $T_1^*\,(1, 0, p)$ for $u_0 \neq 0$. To prove (1) and (2) we must show the existence of u_0^* with $\frac{\partial}{\partial p} T_1^*\,(u_0, 0, p) \leq 0$ for

$1 > u_0 \geq u_0^*$, that is,

$$0 \leq \frac{p-1}{p+1}(p+3)\int_0^1 \left(1-r^{2\frac{p+1}{p-1}}\right)^{-1/2} \mathrm{d}r + 4\int_0^1 \left(1-r^{2\frac{p+1}{p-1}}\right)^{-3/2} r^{2\frac{p+1}{p-1}} \ln r \mathrm{d}r$$
$$+(p-1)^2(\ln u_0)\int_0^1 \left(1-r^{2\frac{p+1}{p-1}}\right)^{-1/2} \mathrm{d}r,$$

thus the existence of u_0^* can be obtained provided

$$\frac{p-1}{p+1}(p+3)\left(r^{2\frac{p+1}{p-1}}-1\right) - 4\ln r > 0, \forall r > 1.$$

After some calculations it is easy to get the above assertion.

To grasp the property of the life-span $T_1^*(u_0, u_1, p)$ is very difficult, but for fixed initial data we want to know how the life-span varies with p, so now we consider the life-span $T_1^*(0.6, 0.2, p)$ and list the following tables as below.

p	$T_1^*(0.6, 0.2, p)$
1.001	2001. 5
1.004	501. 42
1.008	251. 42
1.012	168. 08

p	$T_1^*(0.6, 0.2, p)$
2	3. 4135
2.5	2. 7698
3	2. 4659
3.6497	2. 2644

After some computations we get

$$T_1^*(u_0, u_1, p) = \frac{\sqrt{2p+2}}{p-1}\left(u_0^{p+1} - \frac{p+1}{2}u_1^2\right)^{-\frac{p-1}{2p+2}} \int_0^{\left(1-\frac{p+1}{2u_0^{p+1}}u_1^2\right)^{\frac{p-1}{2p+2}}} \frac{\mathrm{d}r}{\sqrt{1-r^{\frac{2p+2}{p-1}}}}.$$

By the experience in studying the life span $T_1^*(u_0, 0, p)$, we consider the properties of the life-span $T_1^*(u_0, u_1, p)$ with $a'(0) \geq 0$ in three cases:

Case 1 $0 < u_0^{p+1} - (p+1)u_1^2/2 < 1$. In this situation we find that

(i) for fixed u_1,

(5) there exists a constant u_0^* depending on u_1 such that $T_1^*(u_0, u_1, p)$ is monotone decreasing in p for $u_0 \geq u_0^*$,

(6) there is a p_0 such that $T_1^*(u_0, u_1, p)$ decreases in $(1, p_0)$ and increases in (p_0, ∞) provided $u_0 \in [0, u_0^*)$;

(ii) for fixed u_0, the life-span $T_1^*(u_0, u_1, p)$ decreases in u_1^2.

Case 2 $u_0^{p+1} - (p+1)u_1^2/2 > 1$. The life-span $T_1^*(u_0, u_1, p)$ decreases in p.

Case 3 $u_0^{p+1} - (p+1)u_1^2/2 = 1$. On the surface

$$\left\{(u_0, u_1, p) \in \mathbb{R}^3 \,\middle|\, u_0^{p+1} - (p+1)u_1^2/2 = 1,\ p > 1\right\},$$

we find that

$$T_1^*(u_0, u_1, p) = T_1^*(u_0, p) = \frac{\sqrt{2p+2}}{p-1}\int_0^{u_0^{-(p-1)/2}} \frac{1}{\sqrt{1-r^{2(p+1)/(p-1)}}}\mathrm{d}r,$$

and $T_1^*(u_0, p)$ is monotone decreasing both in u_0 and p.

2 Blow-up Rate and Blow-up Constant

In this section we study the blow-up rate and the blow-up constant for a, a' and a'' under the conditions in section 1. We have got the following results

Theorem 6 If u is the solution of problem (0.1) with one of the following properties:

(i) $E(0) < 0$,

(ii) $E(0) = 0, a'(0) > 0$,

(iii) $E(0) > 0$.

Then the blow-up rate of a is $4/(p-1)$, and the blow-up constant K_1 of a is $\sqrt[p-1]{4(p-1)^{-4}(p+1)^2}$, that is, for $m \in \mathbb{N}, m \in [1,9]$,

$$\lim_{t \to T_m^{*-}} (T_m^* - t)^{\frac{4}{p-1}} a(t) = 2^{\frac{2}{p-1}} (p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{4}{p-1}}. \tag{2.1.1}$$

The blow-up rate of a' is $(p+3)/(p-1)$, and the blow-up constant K_2 of a' is $2^{\frac{2p}{p-1}}(p+1)^{\frac{2}{p-1}}(p-1)^{-\frac{p+3}{p-1}}$, that is, for $m \in \mathbb{N}, m \in [1,9]$,

$$\lim_{t \to T_m^{*-}} (T_m^* - t)^{\frac{p+3}{p-1}} a'(t) = 2^{\frac{2p}{p-1}} (p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{p+3}{p-1}}. \tag{2.1.2}$$

The blow-up rate of a'' is $(2p+2)/(p-1)$ and the blow-up constant K_3 of a'' is $2^{\frac{2p}{p-1}} (p+1)^{\frac{8}{p-1}}$ $(p-1)^{-\frac{2p+8}{p-1}} (p+3)$, that is, $m \in \mathbb{N}, m \in [1,9]$,

$$\lim_{t \to T_m^{*-}} a''(t) (T_m^* - t)^{\frac{2p+2}{p-1}} = 2^{\frac{2p}{p-1}} (p+3)(p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{2p+2}{p-1}}. \tag{2.1.3}$$

Proof i) Under this condition, $E(0) < 0,\ a'(0) \geq 0$, by (1.1.1), (1.1.6) and Lemma 4, we get

$$\int_0^{J(t)} \frac{1}{T_1^* - t} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0) r^{\frac{2p+2}{p-1}}}} = \frac{p-1}{2}, \quad \forall t \geq 0, \tag{2.1.4}$$

$$\lim_{t \to T_1^{*-}} \sqrt{\frac{p+1}{2}} \frac{J(t)}{T_1^* - t} = \frac{p-1}{2}. \tag{2.1.5}$$

This equality (2.1.5) is equivalent to (2.1.1) for $m = 1$.

For $E(0) < 0,\ a'(0) < 0$, by (1.1.9) and (1.1.2), we have also

$$\int_0^{J(t)} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0) r^{\frac{2p+2}{p-1}}}} = \frac{p-1}{2} (T_2^* - t), \quad \forall t \geq t_0. \tag{2.1.6}$$

Through Lemma 4 and (2.1.6), therefore we get (2.1.1) for $m = 2$.

Seeing (1.1.5) and (1.1.8) we find

$$\lim_{t \to T_m^{*-}} J'(t) = -\frac{p-1}{\sqrt{2p+2}}, \tag{2.1.7}$$

$$\lim_{t \to T_m^{*-}} a'(t) (T_m^* - t)^{\frac{p+3}{p-1}} = 2^{\frac{2p}{p-1}} (p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{p+3}{p-1}}, \tag{2.1.8}$$

$$\lim_{t \to T_m^{*-}} u'(t)^2 (T_m^* - t)^{\frac{2p+2}{p-1}} = 2^{\frac{2p}{p-1}} (p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{2p+2}{p-1}}, \tag{2.1.9}$$

for $m=1,2$. Using (0.5) and (2.1.9), we obtain, for $m=1,2$,

$$\lim_{t\to T_m^{*-}} a''(t)\,(T_m^*-t)^{\frac{2p+2}{p-1}} = (p+3)\lim_{t\to T_m^{*-}} u'(t)^2\,(T_m^*-t)^{\frac{2p+2}{p-1}}\,. \tag{2.1.10}$$

Thus, (2.1.10) and (2.1.3) are equivalent.

ii) For $E(0)=0,\ a'(0)>0$, by (1.1.11), we get, for $m=1,2$,

$$a(t)=a(0)^{\frac{p+3}{p-1}}\left(\frac{p-1}{4}a'(0)\right)^{-\frac{4}{p-1}}\cdot(T_3^*-t)^{-\frac{4}{p-1}}\,,\quad \forall t\ge 0. \tag{2.1.11}$$

Therefore the estimates (2.1.1), (2.1.2) and (2.1.3) for $m=3$ follow from (2.1.11).

iii) For $E(0)>0$, the estimates (2.1.1), (2.1.2) and (2.1.3) for $m\in[4,9]$ are similar to the above arguments (i) in the proof of Theorem 6.

Now we consider the property of the blow-up constants K_1, K_2 and K_3. We have

$$K_1\,(p)=2^{\frac{2}{p-1}}(p+1)^{\frac{2}{p-1}}(p-1)^{-\frac{4}{p-1}},$$

$$K_2\,(p)=2^{\frac{2p}{p-1}}(p+1)^{\frac{2}{p-1}}(p-1)^{-\frac{p+3}{p-1}},$$

$$K_3\,(p)=2^{\frac{2p}{p-1}}\,(p+3)\,(p+1)^{\frac{2}{p-1}}\,(p-1)^{-\frac{2p+2}{p-1}}.$$

Using Maple we have the graphs of K_1, K_2 and K_3 below.

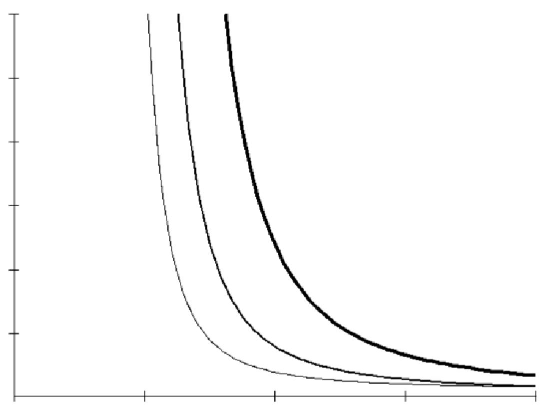

Fig.4 Graphs of $K_1(p)$ in thin, $K_2(p)$ in medium, $K_3(p)$ in thick

We see that the graphs $K_i\,(p)\,, i=1,2,3$, are all decreasing in p, and $K_i\,(p)$ tends to 1 as p tends to infinity. The monotonicity of these functions can be obtained after showing the following inequalities

$$\frac{p-1}{p+1}-2\le \ln(2p+2)-2\ln(p-1),\ \ \forall p>1,$$

$$\frac{2p-2}{p+1}+4\ln(p-1)\le 2\ln 2+2\ln(p+1)+p+3,\ \ \forall p>1,$$

$$\frac{(p-1)^2}{p+3}+\frac{2p-2}{p+1}+4\ln(p-1)\le 2(\ln 2)+2\ln(p+1)+2p+2,\ \ \forall p>1.$$

These inequalities are easy to check, so we omit the arguments.

3 Global Existence and Critical Point

In this section we study the following cases that $E(0) = 0 > a'(0)$ and $E(0) > 0$. Here we take the global existence of the solutions of the problem (0.1) in the following sense:

$$J(t) > 0,\ a'(t)^{-2} > 0,\ a''(t)^{-2} > 0, \quad \forall t \in [0, T]\,,$$

where T is the time till that u exists, in other words, in any finite time u does not blow up in C^2 senses, even though u blows up in a finite time in some sense, for example, C^k or L^k for some $k \geq 3$.

By [5, p.151] every positive proper solution of problem (0.1) has the asymptotic form

$$u(t) \sim ct^{-2/(p-1)}.$$

This result can happen and be explained below only in the case that $E(0) = 0$ and $a'(0) < 0$. Under the condition it is easily to see that $J(t) > 0, \forall t \in (0, T)$, and

$$a(t) = a(0)^{\frac{p+3}{p-1}} \left(a(0) - \frac{p-1}{4} a'(0)t\right)^{\frac{-4}{p-1}}, \quad \forall t \in (0, T),$$

$$a'(t)^{-2} = a(0)^{\frac{-2p-6}{p-1}} a'(0)^{-2} \left(a(0) - \frac{p-1}{4} a'(0)t\right)^{\frac{2p+6}{p-1}} > 0, \quad \forall t \in (0, T),$$

$$a''(t)^{-2} = \frac{16}{(p+3)^2} a(0)^{\frac{-2p-6}{p-1}} a'(0)^{-4} \left(a(0) - \frac{p-1}{4} a'(0)t\right)^{\frac{4p+4}{p-1}} > 0, \quad \forall t \in (0, T).$$

Hence we find $\lim\limits_{t\to\infty} a(t) = 0$, $\lim\limits_{t\to\infty} a'(t) = 0$, $\lim\limits_{t\to\infty} a''(t) = 0$ and

$$\lim_{t\to\infty} t^{\frac{4}{p-1}} a(t) = a(0)^{\frac{p+3}{p-1}} \left(\frac{p-1}{-4} a'(0)\right)^{-\frac{4}{p-1}}, \tag{3.1}$$

$$\lim_{t\to\infty} t^{\frac{p+3}{p-1}} a'(t) = a(0)^{\frac{p+3}{p-1}} a'(0) \left(\frac{p-1}{-4} a'(0)\right)^{-\frac{p+3}{p-1}}, \tag{3.2}$$

$$\lim_{t\to\infty} t^{\frac{2p+2}{p-1}} a''(t) = \frac{p+3}{4} a(0)^{\frac{p+3}{p-1}} a'(0)^2 \left(\frac{p-1}{-4} a'(0)\right)^{-\frac{2p+2}{p-1}}. \tag{3.3}$$

Proposition 1 Suppose that u is the solution of the problem (0.1) with $E(0) = 0$ and $a'(0) < 0$, then u can be defined globally and the estimates (3.1), (3.2) and (3.3) are valid.

Under the situation that $E(0) > 0$ we have results concerning zeros of u'.

Theorem 7 Suppose u is the solution of problem (0.1) with $E(0) > 0$. Then

(i) for $a'(0) > 0 > u_0$ we have the zero of u',

$$S_1 := S_1(u_0, u_1, p) = \int_{-u_0}^{\left(\frac{p+1}{2}E(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1} r^{p+1}}}; \tag{3.4}$$

(ii) for $u_1 < 0 = a'(0)$, we have the critical point of u,

$$S_2 := S_2(u_0, u_1, p) = \int_{-u_0}^{\left(\frac{p+1}{2}u_1^2\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{u_1^2 - \frac{2}{p+1} r^{p+1}}}; \tag{3.5}$$

(iii) for $u_1 < 0$ and $a'(0) < 0$, we have the critical point of u,

$$S_3 := S_3(u_0, u_1, p) = S_4 + Z_4, \tag{3.6}$$

where $S_4 = \int_0^{\left(\frac{p+1}{2}E(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E(0)-\frac{2}{p+1}r^{p+1}}}$ and $Z_4 = \int_0^{u_0} \frac{\mathrm{d}r}{\sqrt{E(0)+\frac{2}{p+1}r^{p+1}}}$.

Further, for $i = 1, 2, 3$,

$$\lim_{t\to S_i} (S_i - t)^{-1} u'(t) = -\left(\frac{p+1}{2}E(0)\right)^{\frac{p}{p+1}}. \tag{3.7}$$

Proof The proof of these three cases are similar, we only mention the case (i). For $u_0 < 0, u_1 < 0$, by $u'' = |u(t)|^p$, there exists $S_1 > 0$ with $u'(t) < 0$, $u(t) < 0$ for $t \in (0, S_1)$ and $u'(S_1) = 0$, $u'(t) > 0$ for $t > S_1$; suppose not, then $u(t) < 0, u'(t) < 0$ and

$$\begin{aligned} t &= \int_{-u_0}^{-u(t)} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1}r^{p+1}}} \le \int_0^{\left(\frac{p+1}{2}E(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1}r^{p+1}}} \\ &= \left(\frac{p+1}{2}E(0)\right)^{\frac{1}{p+1}} E(0)^{-\frac{1}{2}}\beta\left(\frac{1}{2}, \frac{1}{p+1}\right) < \infty, \quad \forall t \ge 0; \end{aligned}$$

yet, this is unreasonable. Thus there exists $S_1 > 0$ with $u'(t) < 0$, $u(t) < 0$ for $t \in (0, S_1)$ and $u'(S_1) = 0$, $u'(t) > 0$ for $t > S_1$. By (0.4) we obtain that

$$t = \int_{-u_0}^{-u(t)} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1}r^{p+1}}} \qquad t \in [0, S_1],$$

$$S_1 = \int_{-u_0}^{\left(\frac{p+1}{2}E\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1}r^{p+1}}}$$

We conclude the estimates (3.4) and (3.7) for $i = 1$.

4 Existence of Zero and Triviality

In this section we discuss the triviality of the solution for the problem (0.1) under the cases that $E(0) = 0 = a'(0)$ and $E(0) > 0$.

Proposition 2 If u is the solution of the problem (0.1) with $p > 1, E(0) = 0$ and $a'(0) = 0$, then u must be null.

Proof For $E(0) = 0 = a'(0)$, $u_0 = 0 = u_1$, herein the supremum below exists

$$t_1 := \sup\{\alpha : a(t) \le 1\ \forall t \in [0, \alpha]\},$$

and then $(p+1)u'(t)^2 = 2|u(t)|^{p+1} \ge 0$,

$$a''(t) = (p+3)\, u'(t)^2 = 2\frac{p+3}{p+1}|u(t)|^{p+1} = 2\frac{p+3}{p+1}a(t)^{\frac{p+1}{2}}.$$

By Lemma 2 we conclude that $a''(t) \le (p+3)\, a(t), a(t) \equiv 0 \equiv u(t)$ in $[0, t_1]$.

Continue these steps we get the assertion of this result.

For the case that $E(0) > 0$, we have the result on zero.

Theorem 8 Suppose that u solves problem (0.1) with $E(0) > 0$, then

(i) for $u_0 < 0 < a'(0)$,

$$S_1 + Z_1(u_0, u_1, p) = S_1 + \int_0^{\left(\frac{p+1}{2}E(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1}r^{p+1}}} \tag{4.1}$$

is the zero of u.

(ii) for $u_1 < 0 = a'(0)$,

$$S_2 + Z_2(u_0, u_1, p) := S_2 + \int_0^{\left(\frac{p+1}{2}u_1^2\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{u_1^2 - \frac{2}{p+1}r^{p+1}}} \tag{4.2}$$

is the zero of u.

(iii) for $u_1 > 0 > a'(0)$, $Z_3(u_0, u_1, p)$ given by

$$Z_3(u_0, u_1, p) = \int_0^{\sqrt{a(0)}} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1}r^{p+1}}} \tag{4.3}$$

is the zero of u.

(iv) for $u_1 < 0,\ a'(0) < 0$,

$$Z_4 := Z_4(u_0, u_1, p) = \int_0^{u_0} \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1}r^{p+1}}} \tag{4.4}$$

and $2S_4 + Z_4$ are zeros of u.

Proof (i) and (ii). For $u_0 < 0, u_1 < 0$, in Section 3, there exists $S_1 > 0$ with $u'(t) < 0$ $u(t) < 0$ for $t \in (0, S_1)$ and $u'(S_1) = 0$, $u'(t) > 0$ for $t > S_1$; we will show that there exists $Z_1 := Z_1(u_0, u_1, p) > 0$ with $u(t) < 0$ in $(0, S_1 + Z_1)$ and $u(S_1 + Z_1) = 0 < u(t)$ for $t > S_1 + Z_1$; suppose that the zero of u does not exist, then $u(t) < 0$ for all positive t. By (0.4) we obtain that

$$t - S_1 = \int_{-u(t)}^{-u(S_1)} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1}r^{p+1}}} \le \int_0^{\left(\frac{p+1}{2}E(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1}r^{p+1}}}, \quad t \ge S_1, \tag{4.5}$$

yet this is unreasonable for large t. Thus, we get the conclusion (4.1) by using (4.5) and the fact that $|u(S_1)|^{p+1} = \frac{p+1}{2}E(0)$. By the similar way, for $u_0 = 0 > u_1$ one can obtain (4.2).

(iii) For $u_1 > 0 > a'(0)$, then $u'(t) \ge u_1$, $u(t) \ge u_1 t + u_0 > 0$ for large t ; thus there exists $Z_3 := Z_3(u_0, u_1, p)$ with $u(Z_3) = 0$, $u(t) < 0$ for $t \in (0, Z_3)$; $u(t) > 0$ for $t > Z_3$. By (0.4) , then

$$u' = \sqrt{E(0) - \frac{2}{p+1}|u(t)|^{p+1}},$$

and (4.3) is obtained.

(iv) For $a'(0) < 0$ and $u_1 < 0$, there exists $Z_4 := Z_4(u_0, u_1, p) > 0$ with $u(t) > 0$, $u'(t) < 0$ for $t \in (0, Z_4)$ and $u(Z_4) = 0$; suppose that $u(t) > 0$ for all $t \ge 0$, then there exists $s_* > 0$ with

$u'(t) < 0 = u'(s_*)$ and $u'(t) > 0$ for $t > s_*$; otherwise $u'(t) < 0$, then by (0.4),

$$t = -\int_{u_0}^{u(t)} \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1}r^{p+1}}} \leq \int_0^{\infty} \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1}r^{p+1}}} < \infty,$$

for each t, but it is unreasonable for large t. Even thought such a s_* exists, using (0.4) again,

$$t - s_* = \int_{u(s_*)}^{u(t)} \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1}r^{p+1}}} \leq \int_0^{\infty} \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1}r^{p+1}}} < \infty;$$

thus the zero of u exists. (4.4) can be obtained by (0.4) and the fact $u(t) > 0, u'(t) < 0$ for $t \in (0, Z_4)$ since that $t = -\int_{u_0}^{u(t)} \frac{\mathrm{d}r}{\sqrt{E(0)+\frac{2}{p+1}r^{p+1}}}$ for $t \in [0, Z_4]$.

By (ii) there exists another zero of u, since $u(Z_4) = 0 > u'(Z_4)$. After some computations, one can see that $Z_4 + 2S_4$ is also a zero of u.

Property concerning $Z_1(u_0, u_1, p)$ and $Z_4(u_0, u_1, p)$

As the analysis concerning the zeros is very complex, we merely discuss $Z_1(p, E)$ and $Z_4(u_0, u_1, p)$. We have $Z_1(p, E) = 2^{-\frac{1}{p+1}}\sqrt{\pi}(p+1)^{\frac{-p}{p+1}}E^{\frac{1-p}{2p+2}}\frac{\Gamma\left(\frac{1}{p+1}\right)}{\Gamma\left(\frac{p+3}{2p+2}\right)}$ and its graph is below:

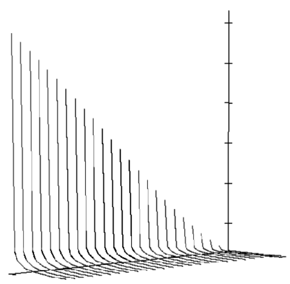

Fig.5 Graphs of $Z_1(p,E)$ for $E \in (0,1)$

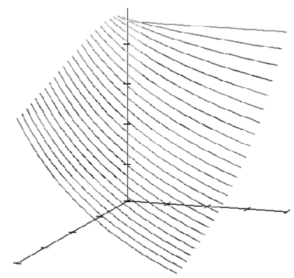

Fig.6 Graphs of $Z_1(p,E)$ for $E \in [1,3]$

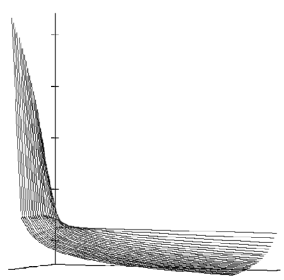

Fig.7 Graphs of $Z_1(p,E)$ for $E \in (0,10)$

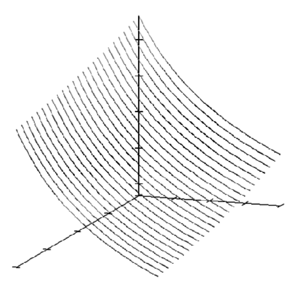

Fig.8 Graphs of $Z_1(p,E)$ for $E \in [1,3]$

The above pictures (Figures 5–8) show the properties of $Z_1(p, E)$:

(1) there exists a constant E^* $\left(\doteq \mathrm{e}^{-\Psi(0.5)+\Psi(1)-1} \sim 1.471\,5\right)$ such that $Z_1(p, E)$ is monotone decreasing in p for $E > E^*$;

(2) there is a p_0 such that $Z_1(p, E)$ is decreasing in $(1, p_0)$ and increasing in (p_0, ∞) provided $E \in [0, E^*)$;

(3) $Z_1(p, E)$ is differentiable in its variables and

(4) $Z_1(p,E)$ is decreasing in E.

The phenomena of (1) and (2) are caused by the fact that $\frac{\partial}{\partial p}\frac{Z_1(p,E)}{\sqrt{\pi}} = 2^{-\frac{1}{p+1}}\cdot\frac{E^{\frac{1-p}{2p+2}}}{(p+1)^2(p+1)^{\frac{p}{p+1}}}\cdot\frac{\Gamma\left(\frac{1}{p+1}\right)}{\Gamma\left(\frac{p+3}{2p+2}\right)}B(p,E)>0$ for $B(p,E)>0$, where $B(p,E)=\ln\frac{2}{(p+1)E}-\Psi\left(\frac{1}{p+1}\right)+\Psi\left(\frac{p+3}{2p+2}\right)-p, \Psi(r)=\frac{\int_0^\infty \mathrm{e}^{-t}\ln t\cdot t^{r-1}\mathrm{d}t}{\int_0^\infty \mathrm{e}^{-t}t^{r-1}\mathrm{d}t}$ and $B(p,E)<0$ for $E>E^*$.

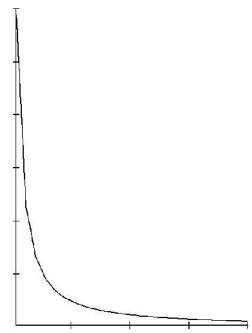

Fig.9 Graphs of $B(p,E)=0$

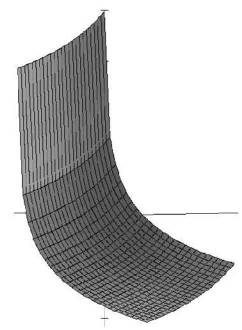

Fig.10 Graphs of $B(p,E)$

As the analysis on $Z_4(u_0,u_1,p)$ is very complex, we merely discuss it on the curve

$$K:=\left\{(u_0,p):u_0=\left(\frac{4}{p+1}\right)^{\frac{-1}{p+1}},\ p>1\right\}$$

and by (4.2) we have on K,

$$Z_4(u_0,u_1,p)=\left(\frac{p+1}{2}\right)^{\frac{1}{p+1}}E(0)^{\frac{1-p}{2p+2}}\int_0^{\left(\frac{p+1}{2}E(0)\right)^{\frac{-1}{p+1}}\sqrt{a(0)}}\frac{\mathrm{d}r}{\sqrt{1+r^{p+1}}},$$

$$Z_4(u_0,-1,p)=\left(\frac{p+1}{2}\right)^{\frac{1}{p+1}}\left(\frac{p+1}{2}-u_0^{p+1}\right)^{\frac{1-p}{2p+2}}\int_0^{\left(\frac{p+1}{2}-u_0^{p+1}\right)^{\frac{-1}{p+1}}u_0}\frac{\mathrm{d}r}{\sqrt{1+r^{p+1}}}.$$

On the curve K,

$$Z_4(u_0,-1,p)=2^{-\frac{2-p}{p+1}}(p+1)^{\frac{3-p}{2p+2}}\int_0^1\frac{\mathrm{d}r}{\sqrt{1+r^{p+1}}},$$

and we have the following table:

p	$Z_4(u_0,-1,p)$	p	$Z_4(u_0,-1,p)$	p	$Z_4(u_0,-1,p)$	p	$Z_4(u_0,-1,p)$
1	0.88137	2	1.0924	3	1.1024	13	0.65503
1.1	0.91976	2.1	1.0987	4	1.0547	14	0.63001
1.2	0.95321	2.2	1.1035	5	0.99391	15	0.60730
1.3	0.98218	2.3	1.1068	6	0.93410	16	0.58659
1.4	1.0924	2.4	1.1089	7	0.87928	17	0.56764
1.5	1.0284	2.5	1.1099	8	0.83024	18	0.55022
1.6	1.0464	2.6	1.110	9	0.78669	19	0.53415
1.7	1.0615	2.7	1.1091	10	0.74801	20	0.51927
1.8	1.074	2.8	1.1075	11	0.71356	21	0.50546
1.9	1.0842	2.9	1.1053	12	0.68274	22	0.49259

By the above data one can see that there exists p_1 such that $Z_4(u_0,-1,p)$ is increasing in p for $p\in(1,p_1)$ and decreasing in p for $p\geq p_1$.

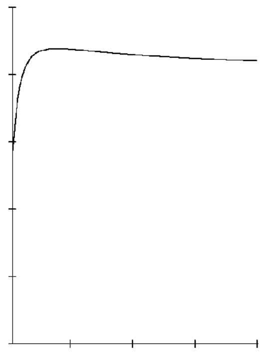

Fig.11 Graphs of K

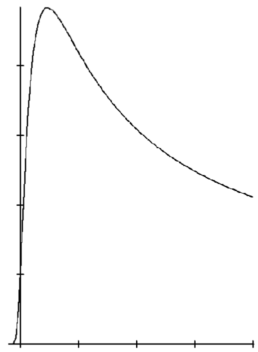

Fig.12 Graphs of $Z_4(u_0,-1,p)$ in circle dots, Graph of $2^{-\frac{2-p}{p+1}}(p+1)^{\frac{3-p}{2p+2}}$ in line

5 Stability and Instability

We now consider the applications of the above theorems to the stability theory for the problem

$$\begin{cases} u''(t)=|u(t)|^p\,, \\ u(0)=\varepsilon_1, u'(0)=\varepsilon_2. \end{cases} \tag{$*$}$$

We say the problem $(*)$ is stable under condition F, if any nontrivial global solution $u\in C^2(\mathbb{R}^+)$ of $(*)$ under the condition F satisfies

$$\|u\|_{C^2}\to 0 \ \text{ for } \ |\varepsilon_1|+|\varepsilon_2|\to 0.$$

According to the Theorems 4–8 we have the following result.

Corollary The problem $(*)$ with $p>1$ is stable under $E_u(0)=0, \varepsilon_1\varepsilon_2<0$ and unstable under the one of the following conditions:

(i) $E_u(0)<0$,

(ii) $E_u(0)=0<\varepsilon_1\varepsilon_2$,

(iii) $E_u(0)>0$.

Part B Null, Critical Point of Solutions of the Equation (0.1) Under $p<1$

Before the study of the properties of solutions for the differential equation (0.1), we collect some results on the situation that $E_u(0)=0$.

(i) For $u_0>0$ and $u_1>0$, we have

$$u(t)=\left(u_0^{\frac{1-p}{2}}+\frac{1-p}{2}\sqrt{\frac{2}{p+1}}t\right)^{\frac{2}{1-p}}$$

and $t^{\frac{2}{p-1}}u(t)\to\left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}}$ as $t\to\infty$.

(ii) For $u_0 > 0$ and $u_1 < 0$, the solutions of (0.1) can be given as

$$u_c(t) = \begin{cases} \left(u_0^{\frac{1-p}{2}} - \frac{1-p}{2}\sqrt{\frac{2}{p+1}}t\right)^{\frac{2}{1-p}} & t \in [0, T_0], \\ 0 & t \in [T_0, T_0 + c], \\ \pm\left(\frac{(1-p)^2}{2p+2}\right)^{\frac{1}{1-p}} (t - T_0 - c)^{\frac{2}{1-p}} & t \geq T_0 + c, \end{cases}$$

where c is any positive real number and $T_0 = \frac{2}{1-p}\sqrt{\frac{p+1}{2}u_0^{1-p}}$, and also $t^{\frac{2}{p-1}}u(t) \to \left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}}$ as $t \to \infty$.

(iii) For $u_0 = 0 = u_1$, the solutions of (0.1) can be given as

$$u_c(t) = \begin{cases} 0 & t \in [0, c], \\ \left(\frac{(1-p)^2}{2p+2}\right)^{\frac{1}{1-p}} (t - c)^{\frac{2}{1-p}} & t \geq c, \end{cases}$$

where c is any positive real number and also $t^{\frac{2}{p-1}}u\,(t) \to \left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}}$ as $t \to \infty$.

We would in Section 6 deal with the estimates for the critical point of the solutions of (0.1) with $E_u(0) > 0$, in Section 7 with zero of the solutions of (0.1) with $E_u(0) > 0$, in Section 8 with the critical point of the solutions for equation (0.1) with $E_u(0) < 0$.

6 Critical Point of Solutions for Equation (0.1) with $E_u(0) > 0$

Under the situation that $E_u(0) > 0 > u_1$ we have results concerning zeros of u'.

Theorem 9 Suppose u is a solution of problem (0.1) with $E_u(0) > 0 > u_1$. Then

(i) for $u_0 < 0$, we have the zero of u',

$$S_{u1} := S_{u1}(u_0, u_1, p) = \int_{-u_0}^{\left(\frac{p+1}{2}E_u(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E_u(0) - \frac{2}{p+1}r^{p+1}}}; \tag{6.1}$$

(ii) for $u_0 = 0$, we have the critical point of u,

$$S_{u2} := S_{u2}(u_0, u_1, p) = \int_{-u_0}^{\left(\frac{p+1}{2}u_1^2\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{u_1^2 - \frac{2}{p+1}r^{p+1}}}; \tag{6.2}$$

(iii) for $u_0 > 0$ we have the critical point of u,

$$S_{u3} := S_{u3}(u_0, u_1, p) = S_{u4} + Z_{u4}, \tag{6.3}$$

where $S_{u4} = \int_0^{\left(\frac{p+1}{2}E_u(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E_u(0) - \frac{2}{p+1}r^{p+1}}}$ and $Z_{u4} = \int_0^{u_0} \frac{\mathrm{d}r}{\sqrt{E_u(0) + \frac{2}{p+1}r^{p+1}}}$.

Further, for $i = 1, 2, 3$,

$$\lim_{t \to S_{ui}^-} (S_{ui} - t)^{-1} u'(t) = -\left(\frac{p+1}{2}E_u(0)\right)^{\frac{p}{p+1}}. \tag{6.4}$$

Proof The proof of these three cases are similar, we only mention the case (i). For $u_0 < 0, u_1 < 0$, by $u'' = |u(t)|^p$, there exists $S_{u1} > 0$ with $u'(t) < 0$, $u(t) < 0$ for $t \in (0, S_{u1})$ and $u'(S_{u1}) = 0$, $u'(t) > 0$ for $t > S_{u1}$; suppose not, then $u(t) < 0, u'(t) < 0$ and

$$t = \int_{-u_0}^{-u(t)} \frac{\mathrm{d}r}{\sqrt{E_u(0) - \frac{2}{p+1} r^{p+1}}} \le \int_0^{\left(\frac{p+1}{2}E_u(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E_u(0) - \frac{2}{p+1} r^{p+1}}}$$
$$= \left(\frac{p+1}{2}E_u(0)\right)^{\frac{1}{p+1}} E_u(0)^{-\frac{1}{2}} \beta\left(\frac{1}{2}, \frac{1}{p+1}\right) < \infty, \quad \forall t \ge 0,$$

yet, this is unreasonable. Thus there exists $S_{u1} > 0$ with $u'(t) < 0$, $u(t) < 0$ for $t \in (0, S_{u1})$ and $u'(S_{u1}) = 0$, $u'(t) > 0$ for $t > S_{u1}$. By (0.4) we obtain that

$$t = \int_{-u_0}^{-u(t)} \frac{\mathrm{d}r}{\sqrt{E_u(0) - \frac{2}{p+1} r^{p+1}}}, \qquad t \in [0, S_{u1}],$$

$$S_{u1} = \int_{-u_0}^{\left(\frac{p+1}{2}E_u(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E_u(0) - \frac{2}{p+1} r^{p+1}}}.$$

We conclude the estimates (3.4) and (3.7) for $i = 1$.

7 Zero of Solutions for Equation (0.1) with $E_u(0) > 0$

For the case $E_u(0) > 0$, we have the result on zero. In this section we discuss the case $E_u(0) > 0$ and we have the following result concerning the null point (zero) and asymptotic behavior at infinity of the solutions for equation (0.1):

Theorem 10 Suppose that T^* is the life-span of u (in sense of C^3) of a solution of problem (0.1) with $E_u(0) > 0$. Then for

(1) $u_0 > 0$ and $u_1 < 0$, there exists a constant Z_{u0} such that $T^* \le Z_{u0}$ and $\lim\limits_{t \to Z_{u0}^-} u(t) = 0$, $\lim\limits_{t \to Z_{u0}^-} u'(t) = -\sqrt{E_u(0)}$ and $\lim\limits_{t \to Z_{u0}^-} u'''(t)^{-1} = 0$. Moreover,

$$Z_{u0} = \int_0^{u_0} \frac{\mathrm{d}r}{\sqrt{E_u(0) + \frac{2}{p+1} r^{p+1}}}, \tag{7.1}$$

$$\lim_{t \to Z_{u0}^-} u'''(t) \left(Z_{u0} - t\right)^{1-p} = -pE_u(0)^{\frac{p}{2}} \tag{7.2}$$

and $2S_4 + Z_{u0}$ also is zero of u.

(2) $u_0 > 0$ and $u_1 \ge 0$,

$$\lim_{t \to \infty} u(t) t^{-\frac{2}{1-p}} = \left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}}. \tag{7.3}$$

(3) $u_0 < 0$ and $u_1 > 0$, there exists a constant Z_{u1} such that $T^* \le Z_{u1}$ and $\lim\limits_{t \to Z_{u1}^-} u(t) = 0$, $\lim\limits_{t \to Z_{u1}^-} u'(t) = \sqrt{E_u(0)}$ and also $\lim\limits_{t \to Z_{u1}^-} u'''(t)^{-1} = 0$. Moreover,

$$Z_{u1} = \int_0^{u_0} \frac{\mathrm{d}r}{\sqrt{E_u(0) - \frac{2}{p+1} r^{p+1}}}, \tag{7.4}$$

$$\lim_{t\to Z_{u1}^-} u'''(t)\left(Z_{u1}-t\right)^{1-p} = pE_u(0)^{\frac{p}{2}}. \tag{7.5}$$

(4) $u_0 \le 0$ and $u_1 < 0$,

$$S_{u1}+Z_{u2}(u_0,u_1,p) = S_{u1} + \int_0^{\left(\frac{p+1}{2}E_u(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E_u(0)-\frac{2}{p+1}r^{p+1}}}$$

is zero of u and

$$\lim_{t\to\infty} u(t)t^{-\frac{2}{1-p}} = -\left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}}. \tag{7.6}$$

Proof (1) For $u_0>0$ and $u_1<0$, there exists $Z_{u0}:=Z_{u0}(u_0,u_1,p)>0$ with $u(t)>0$, $u'(t)<0$ for $t\in(0,Z_{u0})$ and $u\left(Z_{u0}\right)=0$; suppose that $u\left(t\right)>0$ for all $t\ge 0$, then there exists $S_{u4}>0$ with $u'(t)<0=u'\left(S_{u4}\right)$ and $u'(t)>0$ for $t>S_{u4}$; otherwise $u'(t)<0$, then by (0.4) , $t=-\int_{u_0}^{u(t)}\frac{\mathrm{d}r}{\sqrt{E_u(0)+\frac{2}{p+1}r^{p+1}}} \le \int_0^\infty \frac{\mathrm{d}r}{\sqrt{E_u(0)+\frac{2}{p+1}r^{p+1}}}<\infty$ for each t, but it is unreasonable for large t. Even thought such an S_{u4} exists, by using (0.4) again,

$$t-S_{u4} = \int_{u(S_{u4})}^{u(t)} \frac{\mathrm{d}r}{\sqrt{E_u(0)+\frac{2}{p+1}r^{p+1}}} \le \int_0^\infty \frac{\mathrm{d}r}{\sqrt{E_u(0)+\frac{2}{p+1}r^{p+1}}} < \infty;$$

thus zero of u exists.

By (4) there exists another zero of u, since $u\left(Z_{u0}\right)=0>u'\left(Z_{u0}\right)$. After some computations, one can see that $Z_{u0}+2S_{u4}$ is also a zero of u. After some calculations we obtain

$$u'(t) = -\sqrt{E_u(0)+\frac{2}{p+1}\left|u\right|^p u(t)} \le -\sqrt{\frac{2}{p+1}\left|u\right|(t)^{p+1}}, \quad \forall t\in[0,T^*)\,, \tag{7.7}$$

$$u(t) \le \left(u_0^{\frac{1-p}{2}} - \frac{1-p}{2}\sqrt{\frac{2}{p+1}}t\right)^{\frac{2}{1-p}}, \quad \forall t\in[0,T^*)\,;$$

thus there exists a constant Z_{u0} such that $T^*\le Z_{u0}$ and $\lim\limits_{t\to Z_{u0}} u(t)=0$. By (7.7) and Lemma 3, we conclude that $\lim\limits_{t\to Z_{u0}^-} u'(t) = -\sqrt{E_u(0)}$ and

$$t = \int_{u(t)}^{u_0} \frac{\mathrm{d}r}{\sqrt{E_u(0)+\frac{2}{p+1}r^{p+1}}}, \quad \forall t\in[0,T^*)\,,$$

$$Z_{u0} = \lim_{t\to Z_{u0}} \int_{u(t)}^{u_0} \frac{\mathrm{d}r}{\sqrt{E_u(0)+\frac{2}{p+1}r^{p+1}}} = \int_0^{u_0} \frac{\mathrm{d}r}{\sqrt{E_u(0)+\frac{2}{p+1}r^{p+1}}},$$

$$\lim_{t\to Z_{u0}^-} u'''(t)\left(Z_{u0}-t\right)^{1-p} = p\lim_{t\to Z_{u0}^-}\left(\frac{u(t)}{Z_{u0}-t}\right)^{p-1} u'\left(t\right) = -pE_u(0)^{\frac{p}{2}}.$$

Therefore (7.1) and (7.2) are proved.

(2) For $u_0>0$ and $u_1\ge 0$ we have

$$u'(t) = \sqrt{E_u(0)+\frac{2}{p+1}u(t)^{p+1}} \ge \sqrt{\frac{2}{p+1}u(t)^{p+1}}, \quad \forall t\ge 0,$$

$$u(t)^{\frac{1-p}{2}} \geq u_0^{\frac{1-p}{2}} + \frac{1-p}{2}\sqrt{\frac{2}{p+1}}t, \quad \forall t \geq 0. \tag{7.8}$$

On the other hand, for $K_{0u} := \left(\frac{p+1}{2}E_u(0)\right)^{\frac{1}{p+1}}$,

$$u'(t) \leq \sqrt{\frac{2}{p+1}}\left(u\left(t\right)+K_{0u}\right)^{\frac{p+1}{2}}, \quad \forall t \geq 0,$$

$$\left(u(t)+K_{0u}\right)^{\frac{1-p}{2}} \leq \left(u_0+K_{0u}\right)^{\frac{1-p}{2}} + \frac{1-p}{2}\sqrt{\frac{2}{p+1}}t, \quad \forall t \geq 0. \tag{7.9}$$

From (7.8) and (7.9), the estimate (7.3) follows.

(3) For $u_0 < 0$ and $u_1 > 0$; suppose that there is no null point of u, then for $v = -u > 0$ and $v' = -\sqrt{E_u(0) - \frac{2}{p+1}v(t)^{p+1}}$, after some calculations, we obtain

$$t = \int_{v(t)}^{v(0)} \frac{\mathrm{d}r}{\sqrt{E_u(0) - \frac{2}{p+1}r^{p+1}}}, \quad \forall t \geq 0; \tag{7.10}$$

here leads a contradiction; thus there exists a constant Z_{u1} such that $T^* \leq Z_{u1}$ and $\lim\limits_{t \to Z_{u1}} u(t) = 0$. By (7.10) and Lemma 3 we obtain (7.4) and (7.5).

(4) For $u_0 < 0$ and $u_1 < 0$, in Section 6, there exists $S_{u1} > 0$ with $u'(t) < 0$, $u(t) < 0$ for $t \in (0, S_{u1})$ and $u'(S_{u1}) = 0$, $u'(t) > 0$ for $t > S_{u1}$; we will show that there exists $Z_{u2} := Z_{u2}(u_0, u_1, p) > 0$ with $u(t) < 0$ in $(0, S_{u1} + Z_{u2})$ and $u(S_{u1} + Z_{u2}) = 0 < u(t)$ for $t > S_{u1} + Z_{u2}$; suppose that the zero of u does not exist, then $u(t) < 0$ for all positive t. By (0.4) we obtain that

$$t - S_{u1} = \int_{-u(t)}^{-u(S_{u1})} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1}r^{p+1}}} \leq \int_0^{\left(\frac{p+1}{2}E(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E(0) - \frac{2}{p+1}r^{p+1}}}, \quad t \geq S_{u1}, \tag{7.11}$$

yet this is unreasonable for large t. Thus, we get the conclusion (7.6) by using (7.11) and the fact that $|u(S_{u1})|^{p+1} = \frac{p+1}{2}E(0)$.

Graph of $Z_{u2}(u_0, u_1, p)$ We have

$$Z_{u2}(p, E_u) = \int_0^{\left(\frac{p+1}{2}E_u(0)\right)^{\frac{1}{p+1}}} \frac{\mathrm{d}r}{\sqrt{E_u(0) - \frac{2}{p+1}r^{p+1}}},$$

and its graph below:

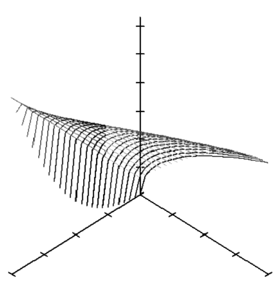

Fig.13 Graphs of $Z_{u2}(p, E_u)$ for $E_u \in (0, 1)$

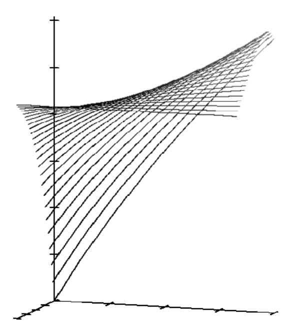

Fig.14 Graphs of $Z_{u2}(p, E_u)$ for $E_u \in [1, 3]$

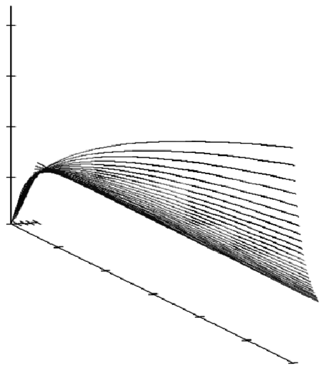

Fig.15 Graphs of $Z_{u2}(p, E_u), E_u \in (0, 30)$

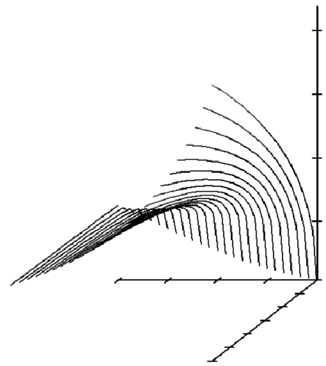

Fig.16 Graphs of $Z_{u2}(p, E_u), E_u(0, 30)$

The above pictures (Figures 13–16) show the properties of $Z_{u2}\,(p, E_u)$:

(1) there exists a constant E^* such that $Z_{u2}\,(p, E_u)$ is monotone decreasing in p for $E_u > E^*$;

(2) there is a p_0 such that $Z_{u2}\,(p, E_u)$ is decreasing in $(1, p_0)$ and increasing in (p_0, ∞) provided $E_u \in [0, E^*)$;

(3) $Z_{u2}\,(p, E)$ is differentiable in its variables and

(4) $Z_{u2}\,(p, E)$ is concave down in E.

8 Critical Point of Solutions for the Equation (0.1) under $E_u(0) < 0$

In this section we discuss the case $E_u(0) < 0$. Similar to the above arguments proving Theorem 10 we have the following result on critical point and asymptotic behavior at infinity of the solutions for the equation (0.1):

Theorem 11 Suppose that u is a solution of problem (0.1) with $E_u(0) < 0$. Then for (1) $u_1 \geq 0$,

$$\lim_{t\to\infty} u(t)t^{-\frac{2}{1-p}} = \left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}} := AZ\,(p)\,; \tag{8.1}$$

(2) $u_1 < 0$, there exists a constant Z_{2u} such that $\lim\limits_{t\to Z_{2u}} u'(t) = 0$ and

$$Z_{2u} = \sqrt[p+1]{\frac{p+1}{2}}\,(-E_u(0))^{\frac{1-p}{2p+2}} \int_1^{\left(\frac{p+1}{-2}E_u(0)\right)^{\frac{-1}{p+1}}u_0} \frac{\mathrm{d}r}{\sqrt{r^{p+1}-1}}; \tag{8.2}$$

Remark Under $u_1 < 0$, (8.1) is also valid.

Proof (1) For $u_1 > 0$, after some calculations we get that

$$u'(t) = \sqrt{E_u(0) + \frac{2}{p+1}u(t)^{p+1}} \leq \sqrt{\frac{2}{p+1}u(t)^{p+1}}, \quad \forall t \geq 0, \tag{8.5}$$

$$u(t) \leq \left(u_0^{\frac{1-p}{2}} + AZ\,(p)\,t\right)^{\frac{2}{1-p}}, \quad \forall t \geq 0. \tag{8.6}$$

By (8.5) we have that $u'(t) \geq \sqrt{\frac{2}{p+1}\,(u(t) - K_u)^{p+1}}, \forall t \geq 0$, where $K_u := \left(\frac{p+1}{2}\,(-E_u(0))\right)^{\frac{1}{p+1}}$ and then

$$(u(t) - K_u)^{\frac{1-p}{2}} \leq (u_0 - K_u)^{\frac{1-p}{2}} + \frac{1-p}{2}\sqrt{\frac{2}{p+1}}t, \quad \forall t \geq 0. \tag{8.7}$$

Together with (8.6) and (8.7) , we obtain (8.1) .

(2) For $u_0 > 0$, $u_1 < 0$; suppose that $u'(t) < 0$ for all $t \geq 0$, then we have

$$u'(t) = -\sqrt{E_u(0) + \frac{2}{p+1} u(t)^{p+1}} \leq -\sqrt{\frac{2}{p+1}} \left(u\left(t\right) - K_u\right)^{\frac{p+1}{2}}, \tag{8.8}$$

$$\left(u(t) - K_u\right)^{\frac{1-p}{2}} \leq \left(u_0 - K_u\right)^{\frac{1-p}{2}} - \frac{1-p}{2}\sqrt{\frac{2}{p+1}}t; \tag{8.9}$$

yet, this creates a contradictions for large t; thus there exists a constant Z_{2u} such that

$$u\left(Z_{2u}\right) = \left(-\frac{p+1}{2} E_u(0)\right)^{\frac{1}{p+1}} \tag{8.10}$$

and $\lim\limits_{t \to Z_{2u}} u'(t) = 0$. By (8.8) , (8.10) and Lemma 3 we conclude that

$$t = \int_{u(t)}^{u_0} \frac{\mathrm{d}r}{\sqrt{E_u(0) + \frac{2}{p+1} r^{p+1}}}, \quad \forall t \in [0, T^*)\,,$$

$$Z_2 = \lim_{t \to Z_{2u}} \int_{u(t)}^{u_0} \frac{\mathrm{d}r}{\sqrt{E_u(0) + \frac{2}{p+1} r^{p+1}}} = \int_{\left(-\frac{p+1}{2} E_u(0)\right)^{\frac{1}{p+1}}}^{u_0} \frac{\mathrm{d}r}{\sqrt{E_u(0) + \frac{2}{p+1} r^{p+1}}}. \tag{8.11}$$

The estimates (8.11) and (8.2) are equivalent.

Property of $AZ(p)$

We have seen that $AZ\left(p\right) = \left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}}$ and the graph using Maple

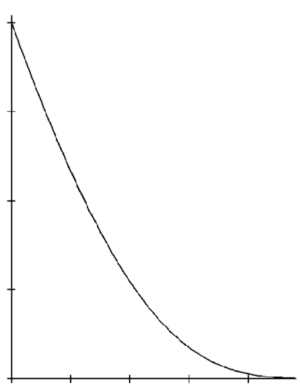

Fig.17 Graphs of $AZ(p), p \in [0, 0.6]$

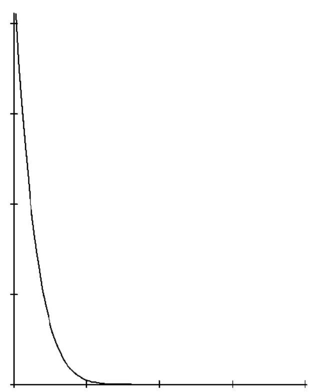

Fig.18 Graphs of $AZ(p), p \in [0.6, 1]$

By the graph we find that $AZ\left(p\right)$ is decreasing in p, since that

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}} = \frac{\sqrt{2}}{1-p}\sqrt{\frac{1}{1+p}}\left(\sqrt{\frac{2}{p+1}}\frac{1-p}{2}\right)^{\frac{2}{1-p}-1}\left(\ln\sqrt{\frac{2}{p+1}}\left(\frac{1-p}{2}\right) - \frac{p+3}{2(p+1)}\right)$$

and then $\frac{\mathrm{d}AZ(p)}{\mathrm{d}p} \leq 0$ for all $p \in (0, 1)$.

Part C Regularity of Solutions to Problem (0.1) with $p>1$ and the Blow-up Constants of $u^{(n)}$

In this part we study the regularity of the solution u of the nonlinear equation (0.1) as $p>1$ and also the blow-up behavior of $u^{(n)}$.

We would in Section 9 deal with the regularity of solution to equation (0.1) with $p\in\mathbb{N}$, in Section 10 with the regularity of solution to equation (0.1) for $p\in Q-N$, in Section 11 with the blow-up rate and blow-up constant for $u^{(n)}$.

9 Regularity of Solution to the Equation (0.1), $p\in\mathbb{N}$

In this section we study the regularity of the solution u of the nonlinear equation (0.1) with $p\in\mathbb{N}$, we have the following results:

Theorem 12 Suppose that $p\in\mathbb{N}$ and u is the nontrivial solution of the problem (0.1) with life-span T^*, then u is non-negative in (Z^*,T^*) for some positive real constant Z^* and $u\in C^q\left(0,T^*\right)$ for any $q\in\mathbb{N}$ and also

$$u^{(2n)}=\sum_{i=0}^{[(\frac{C_{n0}}{p+1})]}E_{n,i}u^{C_{n,i}}, \tag{9.1}$$

$$u^{(2n+1)}=\sum_{i=0}^{[(\frac{C_{n0}}{p+1})]}E_{n,i}C_{n,i}u^{C_{n,i}-1}u'=\sum_{i=0}^{[(\frac{C_{n0}}{p+1})]}O_{n,i}u^{C_{n,i}-1}u' \tag{9.2}$$

in (Z^*,T^*), for positive integer n, where $\left[\left(\frac{C_{n0}}{p+1}\right)\right]$ denotes the Gaussian integer number of $\frac{C_{n0}}{p+1}$,

$$C_{n,i}=(n-i)\left(p+1\right)-2n+1,O_{n,i}=E_{n,i}C_{n,i},\ E_{00}=1$$

and

$$\begin{aligned}E_{n0}&=O_{(n-1)0}\left[\left(\frac{2}{p+1}\left(C_{(n-1)0}-1\right)+1\right)\right]\\&=E_{(n-1)0}C_{(n-1)0}\left[\left(\frac{2}{p+1}\left(C_{(n-1)0}-1\right)+1\right)\right],\\E_{n(n-1)}&=O_{(n-1)(n-2)}\left(C_{(n-1)(n-2)}-1\right)E(0)\\&=E_{(n-1)(n-2)}C_{(n-1)(n-2)}\left(C_{(n-1)(n-2)}-1\right)E(0),\end{aligned}$$

$$\begin{aligned}E_{nk}&=O_{(n-1)(k-1)}\left(C_{(n-1)(k-1)}-1\right)E(0)+O_{(n-1)k}\left[\left(\frac{2}{p+1}\left(C_{(n-1)k}-1\right)+1\right)\right]\\&=E_{(n-1)(k-1)}C_{(n-1)(k-1)}\left(C_{(n-1)(k-1)}-1\right)E(0)\\&\quad+E_{(n-1)k}C_{(n-1)k}\left[\left(\frac{2}{p+1}\left(C_{(n-1)k}-1\right)+1\right)\right],\end{aligned}$$

for positive integer k and $0<k<n$.

Proof To show the positivity of u we consider the following cases:

(1) For $E(0)\le 0$ or $E(0)>0$, $u_0>0$, $u_1\ge 0$ or $E(0)>0$, $u_0\ge 0$, $u_1>0$, then $u(t)\ge 0$ for all $t\ge 0$, we set $Z^*=0$.

(2) For $E(0)>0, u_1<0$ and $u_0<0$, by Theorem 8 then $u(t)>0$ for all $t>S_1+Z_1$, we set $Z^*=S_1+Z_1$.

(3) For $E(0)>0$, $u_1<0$ and $u_0=0$, by Theorem 8 then $u(t)>0$ for all $t>S_2+Z_2$, we set $Z^*=S_2+Z_2$.

(4) For $E(0)>0$, $u_1<0$ and $u_0>0$, by Theorem 8 then $u(t)>0$ for all $t>2S_4+Z_4$, we set $Z^*=2S_4+Z_4$.

(5) For $E(0)>0$, $u_1>0$ and $u_0<0$, by Theorem 8 then $u(t)>0$ for all $t>Z_3$, we set $Z^*=Z_3$.

Now let v_n be the n-th derivative of u; that is, $v_n:=u^{(n)}$, then $v_0^n=u^n$, $v_0=u$, $v_1=u'$, $v_2=u''$, $v_1^2=(u')^2$. To prove (9.1), by mathematical induction, for $n=1$, we have

$$v_2=\sum_{i=0}^{[(\frac{C_{10}}{p+1})]} E_{1\ i}u^{C_{1\ i}}=E_{10}u^{C_{10}}=v_0^p$$

and

$$C_{00}=(0-0)(p+1)-2\times 0+1=1, C_{10}=p,$$
$$E_{10}=E_{00}C_{00}\left[\left(\frac{2}{p+1}(C_{00}-1)+1\right)\right]=1.$$

For $n\in\mathbb{N}$, $v_{2n}=\sum\limits_{i=0}^{[(\frac{C_{n0}}{p+1})]}E_{n,i}\cdot v_0^{C_{ni}}$, by (0.4) we obtain

$$v_{2n+1}=\sum_{i=0}^{[(\frac{C_{n0}}{p+1})]}E_{n,i}C_{n,i}\cdot v_0^{C_{n,i}-1}\cdot v_1,$$

$$v_{2n+2}=\sum_{i=0}^{[(\frac{C_{n0}}{p+1})]}E_{n,i}C_{n,i}v_0^{C_{n,i}-1}v_2+\sum_{i=0}^{[(\frac{C_{n0}}{p+1})]}E_{n,i}C_{n,i}(C_{n,i}-1)v_0^{C_{n,i}-2}v_1^2,$$

$$\begin{aligned}
v_{2n+2}=&\sum_{i=0}^{[(\frac{C_{n0}}{p+1})]}O_{n,i}\left\{\left[\left(\frac{2}{p+1}(C_{n,i}-1)+1\right)\right]v_0^{C_{n,i}+p-1}+(C_{n,i}-1)E(0)v_0^{C_{n,i}-2}\right\}\\
=&\sum_{i=0}^{[(\frac{C_{n0}}{p+1})]}O_{n,i}\left\{\left[\left(\frac{2}{p+1}(C_{n,i}-1)+1\right)\right]v_0^{C_{n+1,i}}+(C_{n,i}-1)E(0)v_0^{C_{n+1,i+1}}\right\}\\
=&O_{n0}\cdot\left[\left(\frac{2}{p+1}(C_{n0}-1)+1\right)\right]v_0^{C_{n+1,0}}+O_{n0}\cdot(C_{n0}-1)\cdot E(0)v_0^{C_{n+1,1}}\\
&+O_{n1}\cdot\left[\left(\frac{2}{p+1}(C_{n1}-1)+1\right)\right]v_0^{C_{n+1,1}}+O_{n1}\cdot(C_{n1}-1)\cdot E(0)v_0^{C_{(n+1)2}}\\
&+O_{n2}\cdot\left[\left(\frac{2}{p+1}(C_{n2}-1)+1\right)\right]v_0^{C_{(n+1)2}}\\
&+\cdots+O_{n[(\frac{C_{n0}}{p+1})]}\cdot\left(C_{n[(\frac{C_{n0}}{p+1})]}-1\right)\cdot E(0)v_0^{C_{(n+1)([(\frac{C_{n0}}{p+1})]+1)}}.
\end{aligned}$$

Hence

$$v_{2n+2}=\sum_{i=0}^{\left[\left(\frac{C_{(n+1)0}}{p+1}\right)\right]}E_{n+1,i}\cdot v_0^{C_{n+1,i}},$$

which completes the induction steps and we obtain (9.1). Using (9.1), we get (9.2).

The properties concerning $u^{(n)}$

In this section we want to draw the graphs of $u^{(n)}$. Drawing the graphs of the $u^{(n)}$ are not easy, so in this section we choose a spacial index $p = 2$.

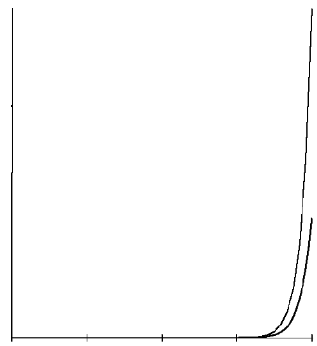

Fig.19 Graphs of u, for $u'' = u^2, (u_0, u_1) = (-1, 1)$ in dots; $(-1, -1)$ in medium line

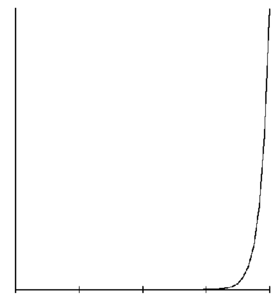

Fig.20 Graphs of $u, u'' = u_2$, $u(0) = 0, u'(0) = -1$

We consider only the properties of the solution u to the case that $E(0) = 0$ for the equation $u'' = u^2, u(0) = 1, u'(0) = \sqrt{\frac{2}{3}}$. This equation can be solved explicitly: $u(t) = \frac{6}{\left(\sqrt{6}-t\right)^2}$, and it affords the graphs of u, u', u'', $u^{(3)}$and $u^{(4)}$ as below.

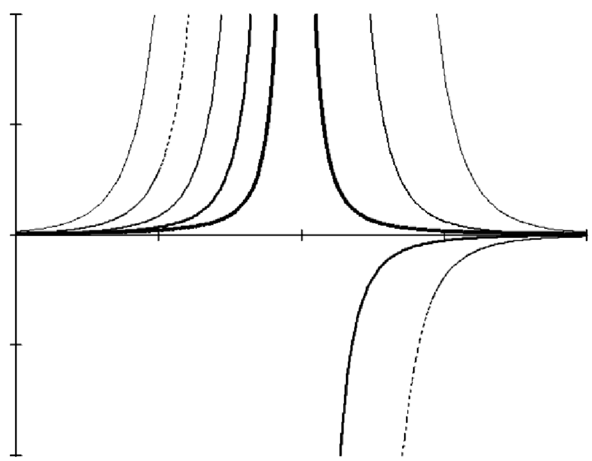

Fig.21 Graphs of u in thick solid, u' in medium dots, u'' in thin solid, $u^{(3)}$ in thin dash, $u^{(4)}$ in thin dots

With the help of graphing with maple we find that the n-th derivative $u^{(n)}$ is smooth and that the blow-up rate of $u^{(n)}$ is increasing in n. Here we do not give rigorous proof, we will illustrate this in Section 11.

10 Regularity of Solution to the Equation (0.1), $p \in \mathbb{Q} - \mathbb{N}$

According to the preceding section we obtain that the solution $u \in C^q\left(0, T^*\right)$ of (0.1) with $p \in \mathbb{N}$ for any $q \in \mathbb{N}$. In this section we reconsider equation (0.1) with $p \in \mathbb{Q} - \mathbb{N}$.

Obviously, if we obviate the null and critical point, of u, then we have the following results:

Except the null or critical points of u, $u^{(q)}$ are also differentiable in $(0,T^*)$ for any $q\in\mathbb{N}$. We have

Theorem 13 If $p\in\mathbb{Q}-\mathbb{N}$, $p\geq 1$ and u is the nontrivial solution of problem (0.1) with one of the following properties:

(1) $E(0)>0, u_1\geq 0, u_0\geq 0$,

(2) $E(0)\leq 0, u_1\geq 0$, then u is non-negative and $\in C^q(0,T^*)$ for any $q\in\mathbb{N}$. Further, (9.1) and (9.2) are also valid.

Proof Under (1) or (2), u is non-negative; according to the same procedures given in the proof of Theorem 12, we obtain the assertion.

Similarly, by the same arguments above, we have also the result:

Theorem 14 If $1<p\in\mathbb{Q}-\mathbb{N}$ and u is the nontrivial solution of problem (0.1) with one of the following properties:

(1) $E(0)<0$, $u_1<0$,

(2) $E(0)>0$, $u_1<0<u_0$,

(3) $E(0)>0$, $u_1<0, u_0\leq 0$,

(4) $E(0)>0$, $u_1>0, u_0<0$.

Then u is nonnegative in (Z^*,T^*) for some positive real constant Z^* and $u\in C^{[(p)]+2}(0,T^*)$, where $[(p)]$ means Gaussian integer number of p. Further, for $t\in(Z^*,T^*)$, (9.1) and (9.2) are both true, particularly for $n\leq\left[\left(\frac{p}{2}\right)\right]+1$, we have

$$u^{(2n)}(t)=\sum_{i=0}^{n-1}E_{n,i}u^{C_{n,i}}(t), \tag{10.1}$$

$$u^{(2n+1)}(t)=\sum_{i=0}^{n-1}E_{n,i}C_{n,i}u^{C_{n,i}-1}(t)u'(t)=\sum_{i=0}^{n-1}O_{n,i}u^{C_{n,i}-1}(t)u'(t). \tag{10.2}$$

Proof Same as the proof of Theorem 12, under either one of (1), (2), (3) or (4), u is non-negative after Z^*, we obtain also the equalities (10.1) and (10.2). By Theorems 4, 7 and 8, we know that u has critical point (Figure 22). If $z:=z(u_0,u_1,p)$ is the null point of u, then $\lim_{t\to z^-}u^{-c_{n,i}}(t)=0$ for $C_{n,i}<0$.

Hence, we would find the range of n with $C_{n,i}\geq 0$ as $i=n-1$, and then $u^{(2n)}$ exists only in this situation. Here $C_{n,i}=(p+1)(n-i)-2n+1$. Let

$$C_{n,n-1}=(p+1)(n-(n-1))-2n+1\geq 0,$$

then we get $n\leq\frac{p}{2}+1$. As n is an integer, we have $n\leq\left[\left(\frac{p}{2}\right)\right]+1$.

Now $u^{(2n)}$ exists for $n\leq\left[\left(\frac{p}{2}\right)\right]+1$ in the case of $a'(0)<0$; thus we obtain that $u\in C^{[(p)]+2}(0.T)$.

Here we want to draw the graphs of $u^{(n)}$ for $p\in\mathbb{Q}-\mathbb{N}$, but it is not easy, so we choose a special index $p=\frac{7}{3}$. We consider the properties of the solution u to the case that $E(0)>0$ for the equation

$$u''=u^{\frac{7}{3}}, u(0)=-1,\ u'(0)=1.$$

Because the solution of the above equation cannot be solved explicitly, so we solve it numerically. We have the graphs of u, u', u'', $u^{(3)}$ $u^{(4)}$ and $u^{(5)}$ as below.

By Theorem 4, we know that $u \in C^4(0,T)$. With the help of graph with maple, we find $t_0 \sim 1.4$ of the null point of u (Figure 22) and $u^{(5)}$ goes to infinity as t approaches to 1.4 (Figure 23). Hence we know that $u^{(5)}(t)$ does not exist for $t = t_0$ by the graph. The blow-up rate of $u^{(n)}$ is increasing in n. It will be illustrated in the next section.

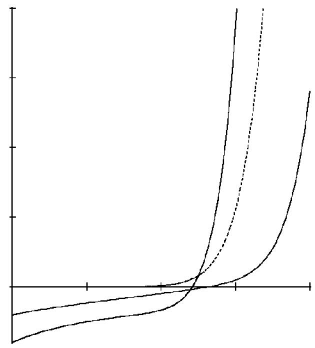

Fig.22 Graphs of u in solid, u' in dash, u'' in dots

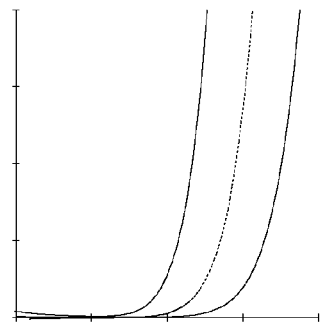

Fig.23 Graphs of $u^{(3)}$ in solid, $u^{(4)}$ in dash, $u^{(5)}$ in dots

11 The Blow-up Rate and Blow-up Constant for $u^{(n)}$

Finding out the blow-up rate and blow-up constant of $u^{(n)}$ of equation (0.1), given as follows is our main result. We have the following results:

Theorem 15 If u is the solution of the problem (0.1) with one of the following properties:

(i) $E(0) < 0$,

(ii) $E(0) = 0,\ a'(0) > 0$,

(iii) $E(0) > 0$,

then the blow-up rate of $u^{(2n)}$ is $\frac{2}{p-1} + 2n$, and the blow-up constant of $u^{(2n)}$ is $\left|E_{n0}\left(\frac{\sqrt{2(P+1)}}{p-1}\right)^{\frac{2}{p-1}+2n}\right|$; that is, for $n \in \mathbb{N}$, $m \in \{1,2,3,4,5,6\}$,

$$\begin{aligned}\lim_{t\to T_m^*} u^{(2n)}(t)\,(T_m^*-t)^{\frac{2}{p-1}+2n} &= (\pm 1)^{C_{n0}}\,E_{n0}\left(\frac{\sqrt{2\,(P+1)}}{p-1}\right)^{\frac{2}{p-1}+2n}\\ &:= K_{2n}.\end{aligned} \tag{11.1}$$

The blow-up rate of $u^{(2n+1)}$ is $\frac{2}{p-1} + 2n + 1$, and the blow-up constant of $u^{(2n+1)}$ is $\left|E_{n0}C_{n0}\sqrt{\frac{2}{p+1}}\left(\frac{\sqrt{2(P+1)}}{p-1}\right)^{\frac{2}{p-1}+2n+1}\right|$; that is, for $n \in \mathbb{N}$, $m \in \{1,2,3,4,5,6\}$,

$$\begin{aligned}\lim_{t\to T_m^*} u^{(2n+1)}(t)\,(T_m^*-t)^{\frac{2}{p-1}+2n} &= (\pm)^{C_{n0}}\,E_{n0}C_{n0}\sqrt{\frac{2}{p+1}}\left(\frac{\sqrt{2\,(P+1)}}{p-1}\right)^{\frac{2}{p-1}+2n+1}\\ &:= K_{2n+1},\end{aligned} \tag{11.2}$$

where

$$C_{n0} = (p-1)n+1,\ E_{n0} = \prod_{i=0}^{n-1}\left[\frac{2\,(p-1)^2\,i^2 + (p-1)i}{p+1} + (p-1)i + 1\right].$$

Proof According to Theorem 12, u is positive in the neighborhood of the life-span T_m^*. We show the result only in this case. Under condition (i), $E(0) < 0, a'(0) \geq 0$, by (1.1.6) and (1.1.1), we get

$$\int_0^{J(t)} \frac{1}{T_1^* - t} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0) r^{\frac{2p+2}{p-1}}}} = \frac{p-1}{2} \quad \forall t \geq 0. \tag{11.3}$$

From Lemma 3 and (1.1.6), we obtain

$$\lim_{t \to T_1^*} \sqrt{\frac{p+1}{2}} \frac{J(t)}{T_1^* - t} = \frac{p-1}{2};$$

in other words,

$$\lim_{t \to T_1^*} a(t) (T_1^* - t)^{\frac{4}{p-1}} = \left(\frac{2}{(p-1)\sqrt{\frac{2}{p+1}}} \right)^{\frac{4}{p-1}}, \tag{11.4}$$

and then

$$\lim_{t \to T_1^*} u(t) (T_1^* - t)^{\frac{2}{p-1}} = \pm \left(\frac{2}{(p-1)\sqrt{\frac{2}{p+1}}} \right)^{\frac{2}{p-1}}. \tag{11.5}$$

Here $C_{n,i} = p + (n-1-i)(p+1) - 2(n-1)$, hence we have $C_{n,i} > C_{n,j}$ as $i < j$. From (10.1) and (11.5), we obtain

$$\lim_{t \to T_1^*} u^{(2n)}(t) (T_1^* - t)^{\frac{2}{p-1} \times C_{n0}} = (\pm 1)^{C_{n0}} E_{n0} \left(\frac{2}{(p-1)\sqrt{\frac{2}{p+1}}} \right)^{\frac{2}{p-1} \times C_{n0}}.$$

Since $\frac{2}{p-1} \times C_{n0} = \frac{2}{p-1} + 2n$, so we get (11.1) for $m = 1$.

By (1.1.5), we find that

$$\lim_{t \to T_1^*} J'(t) = -\frac{p-1}{\sqrt{2p+2}}, \tag{11.6}$$

and

$$\frac{2\sqrt{2}}{\sqrt{p+1}} = \lim_{t \to T_1^*} \left(a(t) (T_1^* - t)^{\frac{4}{p-1}} \right)^{-\frac{p-1}{4} - 1} \cdot \lim_{t \to T_1^*} a'(t) (T_1^* - t)^{\frac{4}{p-1} \times \frac{p+3}{4}}.$$

By (11.4) and (10.2) we obtain that

$$\lim_{t \to T_1^*} u'(t) (T_1^* - t)^{\frac{2}{p-1} + 1} = \pm \sqrt{\frac{2}{p+1}} \left(\frac{2}{(p-1)\sqrt{\frac{2}{p+1}}} \right)^{\frac{2}{p-1} + 1}, \tag{11.7}$$

and

$$\begin{aligned}
&\lim_{t \to T_1^*} u^{(2n+1)}(t) (T_1^* - t)^{\frac{2}{p-1} C_{n0} + 1} \\
= &\lim_{t \to T_1^*} \sum_{i=0}^{n-1} E_{n,i} C_{n,i} u^{C_{n,i} - 1}(t) \cdot u'(t) \cdot (T_1^* - t)^{\frac{2}{p-1} C_{n0} + 1} \\
= &\lim_{t \to T_1^*} E_{n0} C_{n0} u^{C_{n0} - 1}(t) \cdot u'(t) \cdot (T_1^* - t)^{\frac{2}{p-1} C_{n0} + 1}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to T_1^*} E_{n0}C_{n0}u^{C_{n0}-1}(t)\cdot (T_1^*-t)^{\frac{2}{p-1}C_{n0}-1}\cdot u'\cdot (T_1^*-t)^{\frac{2}{p-1}+1}\\
&= (\pm)^{C_{n0}}\, E_{n0}C_{n0}\sqrt{\frac{2}{p+1}}\left(\frac{2}{(p-1)\sqrt{\frac{2}{p+1}}}\right)^{\frac{2}{p-1}C_{n0}+1};
\end{aligned}
$$

thus (11.2) for $m=1$ is proved.

For $E(0)<0,\ a'(0)<0$, by (1.1.9) we have

$$
\int_0^{J(t)} \frac{\mathrm{d}r}{(T_2^*-t)\sqrt{\frac{2}{p+1}+E(0)r^{\frac{2p+2}{p-1}}}} = \frac{p-1}{2},\quad \forall t\ge t_0. \tag{11.8}
$$

Using Lemma 3, (11.8) and (10.1), therefore we gain the estimate (11.1) for $m=2$, and by (1.1.8) we get the estimate (11.2) for $m=2$.

Under (ii), $E(0)=0, a'(0)>0$, we have

$$
a(t)=a(0)^{\frac{p+3}{p-1}}\left(\frac{p-1}{4}a'(0)\,(T_3^*-t)\right)^{-\frac{4}{p-1}},\quad \forall t\ge 0. \tag{11.9}
$$

In view of (11.9) and (10.1), we get the estimate (11.1) for $m=3$. Also, we have $J'(t)=J'(0), \forall t\ge 0$, and $\lim\limits_{t\to T_1^*} a(t)^{-\frac{p-1}{4}-1}a'(t) = -\frac{p-1}{4}a(0)^{-\frac{p-1}{4}-1}a'(0)$. By (11.9) and (10.2), the estimate (11.2) for $m=3$ is completely proved.

Under (iii), the proofs of estimates (11.1) and (11.2) for $m=4,5,6$, are similar to the above arguments, we omit them.

Theorem 16 If u is the solution of the problem (0.1) with $E(0)>0$ and $a'(0)<0$. Then we have

$$
\lim_{t\to z^-(u_0,u_1,p)} u^{(2n)}(t)\,(z(u_0,u_1,p)-t)^{-C_{n,n-1}} = (\pm 1)^{C_{n,n-1}}\,E_{n(n-1)}E(0)^{\frac{C_{n,n-1}}{2}} \tag{11.10}
$$

and

$$
\lim_{t\to z^-(u_0,u_1,p)} u^{(2n+1)}(t)\,(z(u_0,u_1,p)-t)^{-C_{n,n-1}+1} = E_{n(n-1)}C_{n(n-1)}E(0)^{C_{n,n-1}-1}, \tag{11.11}
$$

for $n\in\mathbb{N}$, where z is the null point (zero) of u and

$$
C_{n,n-1}=p-2n+2,\ E_{n,n-1}=\prod_{i=0}^{n-1}(p-2i+2)\,(p-2i+1)\,E(0)^{n-1}.
$$

Proof For $E(0)>0$ and $a'(0)<0$, we have

$$
\begin{aligned}
&\lim_{t\to z^-(u_0,u_1,p)} u^{(2n)}(t)\,(z(u_0,u_1,p)-t)^{-C_{n,n-1}}\\
&= \lim_{t\to z^-(u_0,u_1,p)} \sum_{i=0}^{n-1} E_{n,i}u^{C_{n,i}}(t)\,(z\,(u_0,u_1,p)-t)^{-C_{n,n-1}}\\
&= \lim_{t\to z^-(u_0,u_1,p)} E_{n,n-1}u^{C_{n,n-1}}(t)\,(z\,(u_0,u_1,p)-t)^{-C_{n,n-1}}\\
&= (\pm 1)^{C_{n,n-1}}\,E_{n,n-1}E(0)^{\frac{C_{n,n-1}}{2}}.
\end{aligned}
$$

Therefore, (11.10) is proved.

From (10.2), we obtain that

$$
\begin{aligned}
&\lim_{t\to z^-(u_0,u_1,p)} u^{(2n+1)}(t)\,(z(u_0,u_1,p)-t)^{-C_{n,n-1}+1} \\
=&\lim_{t\to z^-(u_0,u_1,p)} \sum_{i=0}^{n-1} E_{ni}C_{n,i}u^{C_{n,i}-1}(t)u'(t)\,(z(u_0,u_1,p)-t)^{-C_{n,n-1}+1} \\
=&\lim_{t\to z^-(u_0,u_1,p)} E_{n,n-1}C_{n,n-1}u^{C_{n,n-1}-1}(t)\,u'(t)\,(z(u_0,u_1,p)-t)^{-C_{n,n-1}+1} \\
=&\,E_{n(n-1)}C_{n,n-1,}E(0)^{C_{n,n-1}}.
\end{aligned}
$$

Thus, (11.11) is proved.

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References

1 Li M R. Estimates for The Life-span of positive solutions of some semilinear wave equations in 2-dimensional bounded domains $\Box u = u^p$. Proceedings of the Workshop on Differential Equations VII. pp119–134, May 1999, National Chung-Hsing Uni Taichung, Taiwan

2 Li M R. Estimates for The Life-span of Solutions of Semilinear Wave Equations. (CPAA, preprint)

3 Li M R, Tsai L Y. Existence and nonexistence of global solutions of some system of semilinear wave equations. Nonlinear Analysis, 2003, **54**: 1397–1415

4 Li M R, Tsai L Y. On a system of nonlinear wave equations. Taiwanese Journal of Mathematics, 2003, **7**(4): 557–573

5 Li M R. On the semilinear wave equations. Taiwanese J Math, 1998, **2**(3): 329–345

6 Li M R. Estimates for the life-span of solutions for semilinear wave equations. Proceedings of the Workshop on Differential Equations V.pp129–138, Jan.10–11,1997, National Tsing -Hua Uni Hsinchu, Taiwan

7 Bellman R. Stability Theory for Nonlinear Ordinary Differential Equations. New York: McGraw-Hill, 1953