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Note

The Chung–Feller theorem revisited

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Abstract

In this note, we provide a direct and elegant bijective proof of Chung–Feller theorem.

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Dyck paths are the most investigated objects related to the Catalan numbers C_n (see [2,6,5,8]). An n-Dyck path with k flaws is a path from $(0, 0)$ to $(2n, 0)$ with up $(1, 1)$ and down $(1, -1)$ steps having k down steps below the x-axis. Surprisingly, the number of n-Dyck paths with k flaws is independent of k which is the Chung–Feller theorem. In [1], the famous theorem was first proved by means of analytic method. The theorem was subsequently treated by more combinatorial methods in [7] (using cyclic permutation) and in [4] (using the Taylor expansions of generating functions). Recently, Eu et al. [3] proved a refinement of this result.

In this note, our purpose is to provide a direct and elegant bijective proof of Chung–Feller theorem. We utilize a simple bijection between n-Dyck paths with k flaws and n-Dyck paths with $k + 1$ flaws for $k = 0, 1, \ldots, n - 1$ to yield this result (Theorem 0.1).

Theorem 0.1 (*Chung–Feller*). *The number of n-Dyck paths with k flaws is the Catalan number C_n for $k = 0, 1, \ldots, n$.*

Proof. Let $\mathbb{D}_{n,k}$ be the set of n-Dyck paths with k flaws for $k = 0, 1, \ldots, n$. We will establish a bijection between $\mathbb{D}_{n,k}$ and $\mathbb{D}_{n,k+1}$ to prove this result (Fig. 1).

On the one hand, for a given path D in $\mathbb{D}_{n,k}$, let $D = BuAdC$, where u is the first up step above the x-axis and d is the first down step touching the x-axis after u. It is easy to see B is a path all below the x-axis, say with k_1 flaws for $k_1 \geqslant 0$, A is a path all above the x-axis (with 0 flaws), and C is the remaining path with $k - k_1$ flaws (A and B may be empty). Switch Bu with Ad to obtain $D' = AdBuC$. Since A, dBu, and C have 0, $k_1 + 1$, and $k - k_1$ flaws, respectively, then D' is a path in $\mathbb{D}_{n,k+1}$.

On the other hand, for a given path D' in $\mathbb{D}_{n,k+1}$, let $D' = AdBuC$, where d is the first down step below the x-axis and u is the first up step touching the x-axis after d. It is easy to see A is a path all above the x-axis (with 0 flaws), dBu is a path all below the x-axis, say with $k_1 + 1$ flaws for $k_1 \geqslant 0$, and C is the remaining path with $k - k_1$ flaws. Switch

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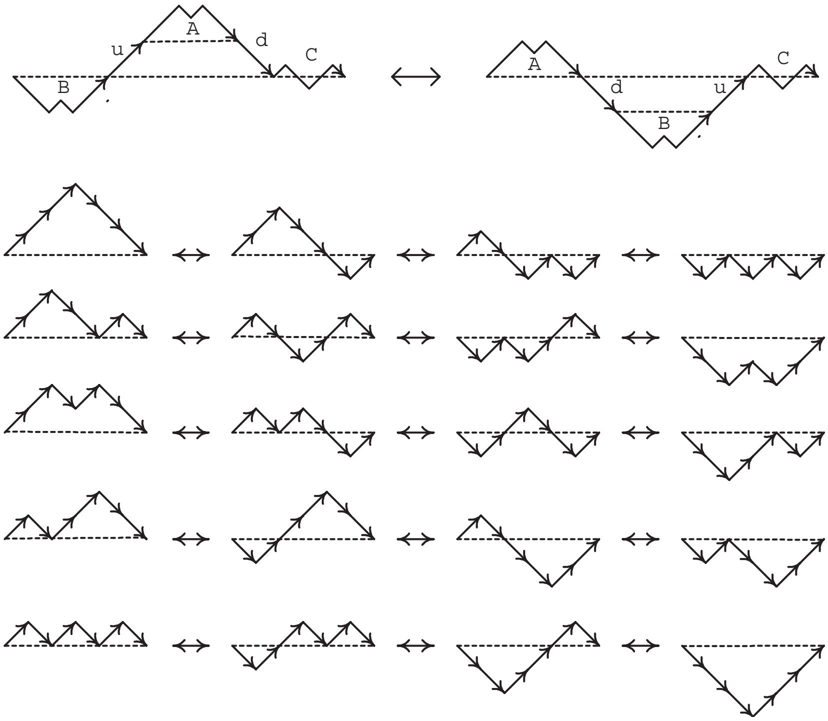

Fig. 1. 3-Dyck paths with k flaws and their corresponding 3-Dyck paths with $k+1$ flaws for $k = 0, 1, 2$.

Ad with Bu to obtain $D = BuAdC$. Since B, uAd, and C have k_1, 0, and $k - k_1$ flaws, respectively, then D is a path in $\mathbb{D}_{n,k}$. This completes the proof. $\square$

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