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A New Approach Solving a Two-Node Closed Queueing Network

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Abstract

In this paper, a general approach to solve a closed queueing network of two nodes in which the service times are of the phase type is considered. The Laplace-Stieltjes Transforms of service time distributions satisfying a system of equations is presented first. According to state balance equations, the stationary probabilities on the unboundary states can be shown as a linear combination of Kronecker product-forms. Each component of these products can be expressed in terms of roots of an associated characteristic polynomial. Furthermore, a procedure for solving stationary probabilities can be presented. The complexity of this approach that is independent of the number of customers in the system is proved. Consequently, the computational complexity for a general closed queueing network can be reduced.

Keywords: Closed Queueing Networks, Matrix-Geometric Solutions, Kronecker Product-Forms.

1. Introduction

Closed queueing systems and related product-form solution techniques have been widely used as a model in the performance analysis of computer systems, communication systems and other complex systems for over thirty years. Gordon and Newell [8] studied a closed queueing system containing M nodes and finite number of customers, N. The service time is exponentially distributed. Let $\mathbf{n} = (n_1, n_2; \dots, n_M)$ be the state vector of system where n_i is the number of customers (including those in service) in the i-th node. The state space of the system is denoted by $S_N = \{\mathbf{n} | n_1 + n_2 + \cdots + n_M = N\}$. The probability of state $\mathbf{n}$ can be expressed in terms of product-forms

$$p(\mathbf{n}) = p(n_1, n_2, \dots, n_M) = g_M(N)^{-1} \prod_{i=1}^{M} f_i(n_i) \tag{1.1}$$

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where the function $f_i(n_i)$ depends on the characteristics of the i-th node. $g_M(N)^{-1}$ is so called a normalization constant chosen to make the sum of all feasible states probabilities $p(\mathbf{n})$ equal to one. Therefore, it yields

$$g_M(N) = \sum_{\mathbf{n}\in S_N} \prod_{i=1}^{M} f_i(n_i). \tag{1.2}$$

Although a product-form of closed queueing networks are easily expressed, calculating the normalization constant with a large number of nodes in a closed system can be highly intricate due to all possible combinations of non-negative integers of N. Many algorithms have been tried to solve a normalization constant and performance evaluation for closed systems, such as the convolution algorithm [4], the mean value analysis algorithm (MVA) [12], the recursion by chain algorithm (RECAL) [5], but most of those researches were focused on exponential service times. Nevertheless, most practical queueing problems lead to non-exponential service systems. Here, we present approximation method as a model to deal with closed systems with service time of each node of phase-type distributions.

Baynat and Dallery [1] presented a unified view of two product-form approximation techniques namely the aggregation method and Marie's method [10]. According the partition scheme proposed in their paper, the network can be analyzed using one of these product-form approximation methods. Each noded (subsystem) in the network will be analyzed in isolation using either the aggregation technique or Marie's method. In general, if each node consists of exponential service stations either in tandem or in parallel which is not impossible for a proper modelling, we extend the product-form solution technique to be of the vector type. As a result, it greatly reduces the computational complexity for calculating the stationary probabilities in the network. We shall explain the basic idea in this paper and illustrate it by simple examples.

The solution technique is based on a novel approach proposed by Luh [9] to solve a two-node multi-server queueing system where both interarrival times and service times are of the phase-type denoted by $PH/PH/c_1 \to /PH/c_2$. The first PH denotes the phase-type interarrival time distribution while the second and third ones indicate the service time distributions with respect to servers at nodes 1 and 2. It is simply a model of a series of two queues with c_1 and c_2 servers respectively. Unlike exponential tandem queueing models, each service time in this system follows a phase-type distribution. The system is constructed with an infinite buffer and multiple servers in each node. He showed that the stationary distribution of the number of customers in the system when all servers are busy is a linear combination of vector product-forms.

In order to present a basic solution scheme that is used to solve a closed queueing network, we are concerned with a two-node closed queueing system with a single server at each node, denoted by $-/PH/1 \rightarrow /PH/1$ shown in Figure 1 where the service times are identically distributed of the phase-type. Because it is a closed system and there is no arrival, we leave out the first PH to differentiate the symbol used for an open system. We solve it by using the result on the Matrix-geometric solution of a quasi-birth-and-death (QBD) process with a countable number of phases in each level. Readers may be referred to [3] for a general discussion of QBD models. The result yields a new expression of the stationary distribution which can be written as a linear combination of Kronecker product-forms and may be used to compute other performance measures, such as the delay probability, the moments of the queue size distribution and the waiting distribution. To the best of our knowledge, it is a completely new approach to solve a closed queueing network.

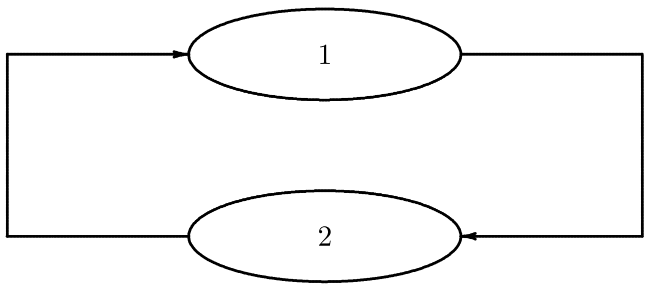

Figure 1. A two-node closed queueing network.

The key to the analysis is the identification of an important subspace of bounded solutions derived from the system of homogeneous vector-valued Wiener-Hopf equations [8] associated with the semi-Markov chain. In particular, the linear equations in the boundary probabilities obtained from the transform method are shown to correspond to a basis of the shift operator on this subspace. The goal of the present paper is to investigate whether it is possible to generalize this approach to a closed queueing network. As a result, it can be exploited to show that the equilibrium probabilities can be written as a finite linear combination of Kronecker product-forms.

2. The model

We consider a two-node $-/PH/1 \rightarrow /PH/1$ closed queueing system containing finite number of customers, N. Each node has one server and an associated buffer where customers may wait before receiving services. Assume the largest buffer size at one of two nodes is N. The service time of k-th node is independent and identically distributed (i.i.d.) with a random variable subjecting to an irreducible phase-type distribution $PH(\boldsymbol{\beta}_k, \mathbf{S}_k)$ with J_k phases and service rate μ_k, $k = 1, 2$. Phase-type distributions may be interpreted in the following. Suppose the time for service at node 1 involves performance m types of tasks, $m = 1, 2, \ldots, J_1$, where each task takes an exponentially distributed amount of time. Define $\boldsymbol{\beta}_1$=(β_1, β_2 ... β_{J_1}) to be the probabilities that time is started in a possible phase in order to complete its service. Customers are severed under the First-come First-served discipline (FCFS). Denote by $\mathbf{e}'_k$ the column vector of all entries equal to 1. Define $\boldsymbol{\gamma}_k$ to be a J_k- column vector such that $\mathbf{S}_k\mathbf{e}'_k + \boldsymbol{\gamma}_k = 0$. The phase-type is said to be irreducible if $\boldsymbol{\gamma}_k\boldsymbol{\beta}_k + \mathbf{S}_k$ is irreducible, or equivalently $-\boldsymbol{\beta}_k\mathbf{S}_k^{-1} > 0$.

The phase type distribution is a good approximation for general distribution. It consists in a mixture of exponential distributions. By grouping the states into sets, it can be seen that we have a QBD process with a very regular structure. We thus propose a matrix-geometric formulation of this problem and we use the results developed for geometric solutions of stochastic models. Let $F_k(x)$ be the distribution function of phase type distribution, and its Laplace-Stieltjes Transforms (LST) is represented by $S_k^*(s)$. Then, it is known that

$$F_k(x) = 1 - \boldsymbol{\beta}_k exp(\mathbf{S}_k x)\mathbf{e}'_k = -\sum_{n=1}^{\infty} \boldsymbol{\beta}_k \mathbf{S}_k^n \mathbf{e}'_k \frac{x^n}{n!}, \tag{2.1}$$

and $S_k^*(s)$ can be written as

$$S_k^*(s) = \boldsymbol{\beta}_k(s\mathbf{I}_k - \mathbf{S}_k)^{-1}\boldsymbol{\gamma}_k, \tag{2.2}$$

where $\mathbf{I}_k$ is the identity matrix with the same dimension of $\mathbf{S}_k$.

Then we have

$$S_1^*(s) = \boldsymbol{\beta}_1(s\mathbf{I}_1 - \mathbf{S}_1)^{-1}\boldsymbol{\gamma}_1, \text{ and}$$
$$S_2^*(s) = \boldsymbol{\beta}_2(s\mathbf{I}_2 - \mathbf{S}_2)^{-1}\boldsymbol{\gamma}_2.$$

By definition of LST of $S_1^*(s)$ and $\boldsymbol{\gamma}_1$, the mean service time of server 1 is given by

$$\begin{aligned}\mu_1^{-1} &= -\frac{d}{d\ s} S_1^*(s)|_{s=0} = -\boldsymbol{\beta}_1(s\mathbf{I}_1 - \mathbf{S}_1)^{-2}(-1)\boldsymbol{\gamma}_1|_{s=0} \\ &= \boldsymbol{\beta}_1\mathbf{S}_1^{-2}(-\mathbf{S}_1\mathbf{e}_1') \\ &= -\boldsymbol{\beta}_1\mathbf{S}_1^{-1}\mathbf{e}_1'.\end{aligned}$$

Similarly, we have $\mu_2^{-1} = -\boldsymbol{\beta}_2\mathbf{S}_2^{-1}\mathbf{e}_2'$.

The number of customers already present in the k-th node (including those in service) is denoted by n_k, $k = 1, 2$, and $n_1 + n_2 = N$. The system state is $\mathbf{n} = (n_1, n_2; i_1, i_2)$ where i_k is the phase of the service process at the k-th node. The index i_k is interpreted to be zero if the corresponding server is idle.

According to the previous studies for an open network in Luh [9], we draw a proposition that the two-node $-/PH/1 \to /PH/1$ closed system have the following properties.

Proposition 1. *Suppose $\mu_1 \geq \mu_2$ in a $-/PH/1 \to /PH/1$ system. We solve the system of equations for h, s_1 and s_2:*

$$\begin{cases} S_1^*(s_1) = h^{-1} \\ S_2^*(s_2) = h \\ s_1 + s_2 = 0. \end{cases} \tag{2.3}$$

Given the solutions of h, s_1, s_2, we can construct a solution basis for the stationary probabilities of unboundary states.

The system of equations (2.3) has at least two solutions one of which is $(h, s_1, s_2) = (1, 0, 0)$. Suppose other solutions are $(h, s_1, s_2) = (\eta, \omega_1, \omega_2)$. We shall provide arguments why the proposition holds true in the following sections.

3. State balance equations

Without loss of generality, we assume $\mu_1 \geq \mu_2$ in the system. We arrange the states (n_1, n_2, i_1, i_2) in lexicographic order and partition of the state space according to the number of customers at node 1, n_1, i.e.,

$$\mathcal{L}_m = \{(n_1, n_2, i_1, i_2) | n_1 = m, n_1 + n_2 = N\}, \quad m = 0, 1, 2, \ldots, N. \tag{3.1}$$

For fixed n_1, n_2 the states can be lexicographically in according with phases i_1, i_2. The state space can be organized into three groups:

$$\mathcal{L}_0 = \{(0, N, 0, 1), (0, N, 0, 2), \cdots, (0, N, 0, J_2)\},$$

$$\mathcal{L}_N = \{(N,0,1,0),(N,0,2,0),\cdots,(N,0,J_1,0)\},$$

and

$$\begin{aligned}\mathcal{L}_m = \{&(m,N-m,1,1),\ (m,N-m,1,2),\ \cdots,\ (m,N-m,1,J_2);\\ &(m,N-m,2,1),\ (m,N-m,2,2),\ \cdots,\ (m,N-m,2,J_2);\\ &\quad\vdots \qquad\qquad \vdots \qquad\qquad \cdots \qquad\qquad \vdots\\ &(m,N-m,J_1,1),(m,N-m,J_1,2),\cdots,(m,N-m,J_1,J_2)\}\\ &\text{where } 1\le m\le N-1. \end{aligned} \tag{3.2}$$

$\mathcal{L}_0$ and $\mathcal{L}_N$ are defined as sets of boundary states. $\mathcal{L}_m$, $1 \le m \le N-1$, are defined as sets of unboundary states. Denote by $\boldsymbol{\pi}$ the stationary probability in row-vector partitioned corresponding to $\mathcal{L}_m$:

$$\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_N) \tag{3.3}$$

where $\boldsymbol{\pi}_m$ is a stationary probability when $n_1 = m$. $\boldsymbol{\pi}_0$ is a row vector of size J_2, $\boldsymbol{\pi}_N$ of size J_1 and $\boldsymbol{\pi}_m$ of size J_1J_2, for $m \ne 0, N$. Define $\mathbf{Q}$ to be the matrix of transition rates according to the arrangement of $\mathcal{L}_m$. Then $\mathbf{Q}$ is of the block-tridiagonal form and written as

$$\mathbf{Q} = \begin{array}{c} \\ \mathcal{L}_0 \\ \mathcal{L}_1 \\ \mathcal{L}_2 \\ \vdots \\ \mathcal{L}_{N-2} \\ \mathcal{L}_{N-1} \\ \mathcal{L}_N \end{array} \begin{array}{c} \begin{matrix} \mathcal{L}_0 & \mathcal{L}_1 & \mathcal{L}_2 & \cdots & \mathcal{L}_{N-2} & \mathcal{L}_{N-1} & \mathcal{L}_N \end{matrix} \\ \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_0 & & & & & \\ \mathbf{C}_0 & \mathbf{B} & \mathbf{A} & & & & \\ & \mathbf{C} & \mathbf{B} & \mathbf{A} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \mathbf{C} & \mathbf{B} & \mathbf{A} & \\ & & & & \mathbf{C} & \mathbf{B} & \mathbf{A}_1 \\ & & & & & \mathbf{C}_1 & \mathbf{B}_1 \end{bmatrix} \end{array}, \tag{3.4}$$

where $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{C}_0$, $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{C}_1$, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are submatrices with proper dimensions.

The submatrices could be written in Kronecker product and Kronecker sum defined in Bellman [2], which were denoted by $\otimes$ and $\oplus$, respectively. Kronecker product and Kronecker sum were used to simplify the representation of state balance equations in the system by many researchers, for example [11] and [6]. Here, the submatrices $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{C}_0$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{B}_1$, and $\mathbf{C}_1$ are given below:

$$\begin{aligned} &\mathbf{A}_0 = \boldsymbol{\beta}_1 \otimes \boldsymbol{\gamma}_2\boldsymbol{\beta}_2, && \mathbf{B}_0 = \mathbf{S}_2, && \mathbf{C}_0 = \boldsymbol{\gamma}_1 \otimes \mathbf{I}_2,\\ &\mathbf{A} = \mathbf{I}_1 \otimes \boldsymbol{\gamma}_2\boldsymbol{\beta}_2, && \mathbf{B} = \mathbf{S}_1 \oplus \mathbf{S}_2, && \mathbf{C} = \boldsymbol{\gamma}_1\boldsymbol{\beta}_1 \otimes \mathbf{I}_2,\\ &\mathbf{A}_1 = \mathbf{I}_1 \otimes \boldsymbol{\gamma}_2, && \mathbf{B}_1 = \mathbf{S}_1, && \mathbf{C}_1 = \boldsymbol{\gamma}_1\boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2. \end{aligned} \tag{3.5}$$

To combine the state balance equations $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$ and the normalization condition $\boldsymbol{\pi}\mathbf{e}' = 1$, we obtain the following equations:

$$\boldsymbol{\pi}_0\mathbf{B}_0 + \boldsymbol{\pi}_1\mathbf{C}_0 = \mathbf{0} \tag{3.6}$$

$$\boldsymbol{\pi}_0\mathbf{A}_0 + \boldsymbol{\pi}_1\mathbf{B} + \boldsymbol{\pi}_2\mathbf{C} = \mathbf{0} \tag{3.7}$$

$$\boldsymbol{\pi}_{m-1}\mathbf{A} + \boldsymbol{\pi}_m\mathbf{B} + \boldsymbol{\pi}_{m+1}\mathbf{C} = \mathbf{0} \qquad 2 \le m \le N-2 \tag{3.8}$$

$$\boldsymbol{\pi}_{N-2}\mathbf{A} + \boldsymbol{\pi}_{N-1}\mathbf{B} + \boldsymbol{\pi}_N\mathbf{C}_1 = \mathbf{0} \tag{3.9}$$

$$\boldsymbol{\pi}_{N-1}\mathbf{A}_1 + \boldsymbol{\pi}_N\mathbf{B}_1 = \mathbf{0} \tag{3.10}$$

$$\boldsymbol{\pi}\mathbf{e}' = 1 \tag{3.11}$$

It is easy to rewrite the balance equations by plugging (3.5) into equations (3.6)~(3.10):

$$\boldsymbol{\pi}_0\mathbf{S}_2 = \boldsymbol{\pi}_1(\mathbf{S}_1\mathbf{e}_1' \otimes \mathbf{I}_2) \tag{3.12}$$

$$\boldsymbol{\pi}_1(\mathbf{S}_1 \oplus \mathbf{S}_2) = \boldsymbol{\pi}_0(\boldsymbol{\beta}_1 \otimes \mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2) + \boldsymbol{\pi}_2(\mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1 \otimes \mathbf{I}_2) \tag{3.13}$$

For $2 \le m \le N-2$

$$\boldsymbol{\pi}_m(\mathbf{S}_1 \oplus \mathbf{S}_2) = \boldsymbol{\pi}_{m-1}(\mathbf{I}_1 \otimes \mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2) + \boldsymbol{\pi}_{m+1}(\mathbf{S}_1\mathbf{e}_1'\boldsymbol{\beta}_1 \otimes \mathbf{I}_2) \tag{3.14}$$

$$\boldsymbol{\pi}_{N-1}(\mathbf{S}_1 \oplus \mathbf{S}_2) = \boldsymbol{\pi}_{N-2}(\mathbf{I}_1 \otimes \mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2) + \boldsymbol{\pi}_N(\mathbf{S}_1\mathbf{e}_1' \otimes \boldsymbol{\beta}_2) \tag{3.15}$$

$$\boldsymbol{\pi}_N\mathbf{S}_1 = \boldsymbol{\pi}_{N-1}(\mathbf{I}_1 \otimes \mathbf{S}_2\mathbf{e}_2') \tag{3.16}$$

In sequel, our goal is to make use of Kronecker product-form solutions in expression of the unboundary state stationary probability. To aim at this objective, we define $\mathbf{w}_m$ as

$$\mathbf{w}_m = \eta^m(\mathbf{u}_1 \otimes \mathbf{u}_2) \tag{3.17}$$

where

$$\mathbf{u}_1 \in C^{J_1},\ \mathbf{u}_2 \in C^{J_2},\ \eta \ne 1.$$

We shall require that $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfy the normalization condition, i.e.,

$$\mathbf{u}_1\mathbf{e}_1' = \mathbf{u}_2\mathbf{e}_2' = 1.$$

First, we solve system of equations (2.3) for h, s_1 and s_2. Afterwards, we construct each $\mathbf{u}_i$ where the derivation is given below.

Theorem 1. *For a fixed k and a positive h, there exists a number ω such that $S_k^*(\omega) = \frac{1}{h}$. Define a row vector*

$$\mathbf{u} = b\boldsymbol{\beta}_k(\mathbf{S}_k - \omega\mathbf{I}_k)^{-1}$$

where b is a constant. Then, it results in

$$\mathbf{u}(h\boldsymbol{\gamma}_k\boldsymbol{\beta}_k + \mathbf{S}_k) = \omega\mathbf{u}. \tag{3.18}$$

The proof was discussed in [13].

Here, we give a simple formula to calculate b.

Corollary 1. *If $\mathbf{u}\mathbf{e}_k' = 1$, then $b = h\omega/(1-h)$ where $h \neq 1$.*

Proof. Since $\mathbf{u}\boldsymbol{\gamma}_k = -bS_k^*(\omega)$, $\boldsymbol{\beta}_k\mathbf{e}_k' = 1$ and $\mathbf{u}\mathbf{S}_k\mathbf{e}_k' = bS_k^*(\omega)$, we have both sides of (3.18) multiplied by $\mathbf{e}'$, equivalently,

$$\mathbf{u}(h\boldsymbol{\gamma}_k\boldsymbol{\beta}_k + \mathbf{S}_k)\mathbf{e}' = \omega \tag{3.19}$$

$$\Rightarrow -bh^{-1}\omega + bh^{-1} = \omega \tag{3.20}$$

implying $b = h\omega/(1-h)$.

(2.3) may produces as many as $J_1 + J_2 - 1$ solutions in total. Suppose one of the solutions is $(\eta, \omega_1, \omega_2)$, $\eta \neq 1$. Let

$$\mathbf{u}_1 = b_1\boldsymbol{\beta}_1(\mathbf{S}_1 - \omega_1\mathbf{I}_1)^{-1} \text{ and } \mathbf{u}_2 = b_2\boldsymbol{\beta}_2(\mathbf{S}_2 - \omega_2\mathbf{I}_2)^{-1} \tag{3.21}$$

where b_1, b_2 are constants satisfying $\mathbf{u}_1\mathbf{e}_1' = \mathbf{u}_2\mathbf{e}_2' = 1$. According to Corollary 1, we obtain

$$b_1 = \frac{\omega_1\eta}{1-\eta}, \qquad b_2 = \frac{\omega_2}{\eta - 1}, \qquad \text{for } \eta \neq 1.$$

From (3.21), we can easily to drive

$$\mathbf{u}_1\mathbf{S}_1 = b_1\boldsymbol{\beta}_1 + \omega_1\mathbf{u}_1 \qquad \mathbf{u}_2\mathbf{S}_2 = b_2\boldsymbol{\beta}_2 + \omega_2\mathbf{u}_2. \tag{3.22}$$

and

$$\mathbf{u}_1\mathbf{S}_1\mathbf{e}_1' = b_1S_1^*(w_1) = \frac{b_1}{\eta}, \qquad \mathbf{u}_2\mathbf{S}_2\mathbf{e}_2' = b_2S_2^*(w_2) = b_2\eta. \tag{3.23}$$

In what follows, we want to show $\mathbf{w}_m$, $1 \leq m \leq N-1$, satisfies equation (3.14). Inserting (3.17) into (3.14) in which $\boldsymbol{\pi}_m$ is substituted by $\mathbf{w}_m$ and dividing by η^m, the left-hand of the equation (3.14) becomes

$$
\begin{aligned}
&(\mathbf{u}_1 \otimes \mathbf{u}_2)(\mathbf{S}_1 \oplus \mathbf{S}_2) \\
&= \mathbf{u}_1\mathbf{S}_1 \otimes \mathbf{u}_2 + \mathbf{u}_1 \otimes \mathbf{u}_2\mathbf{S}_2 && (3.24) \\
&= b_1\boldsymbol{\beta}_1 \otimes \mathbf{u}_2 + \mathbf{u}_1 \otimes b_2\boldsymbol{\beta}_2 + (\omega_1 + \omega_2)(\mathbf{u}_1 \otimes \mathbf{u}_2) && (3.25) \\
&= b_1\boldsymbol{\beta}_1 \otimes \mathbf{u}_2 + b_2\mathbf{u}_1 \otimes \boldsymbol{\beta}_2 && (3.26)
\end{aligned}
$$

Equation (3.25) follows from equation (3.24) by substitution of (3.22).

Inserting (3.17) into (3.14) and dividing by η^m, the right side of the equation (3.14) becomes

$$
\begin{aligned}
&\frac{1}{\eta}(\mathbf{u}_1 \otimes \mathbf{u}_2)(\mathbf{I}_1 \otimes \mathbf{S}_2\mathbf{e}_2'\boldsymbol{\beta}_2) + \eta(\mathbf{u}_1 \otimes \mathbf{u}_2)(\mathbf{S}_1\mathbf{e}'\boldsymbol{\beta}_1 \otimes \mathbf{I}_2) && (3.27) \\
&= \frac{1}{\eta}(b_2\eta\mathbf{u}_1 \otimes \boldsymbol{\beta}_2) + \eta(\frac{b_1}{\eta}\boldsymbol{\beta}_1 \otimes \mathbf{u}_2) && (3.28)
\end{aligned}
$$

The equation (3.28) is equal to (3.26). Hence it balances the equation (3.14).

Now any linear combination of $\mathbf{w}_m$, $2 \le m \le N-1$, obviously satisfies the state balance equations for boundary conditions. Since we have obtained $J_1 + J_2 - 1$ solutions of $(\eta, \omega_1, \omega_2)$, denote each of them by i. Define

$$
\mathbf{w}_m(i) = \eta^m(i)[\mathbf{u}_1(i) \otimes \mathbf{u}_2(i)]. \tag{3.29}
$$

Lemma 1. $\mathbf{w}_m$ *is a complex solution of general equations* (3.14) *if*

$$
\sum_{m,i} |\mathbf{w}_m(i)| < +\infty
$$

Since (3.29) satisfies (3.14) and $J_1 + J_2 - 1$ is finite, the proof is immediately clear by adopting the stability assumption of this system.

We assume unboundary probabilities have linear combination of $\mathbf{w}_m(i)$ which has the following form:

$$
\boldsymbol{\pi}_m = \sum_{i=1}^{J_1+J_2-1} c_i\mathbf{w}_m(i). \tag{3.30}
$$

where $1 \le m \le N-1$ and c_i are the coefficients associated with $\mathbf{w}_m(i)$. For each coefficient c_i, $\boldsymbol{\pi}_m$ satisfies equation (3.14). Inserting $\boldsymbol{\pi}_m$ into remaining equations (3.12), (3.13), (3.15) and (3.16). By adding the equation of normalization condition, those equations form a linear homogeneous system of equations of c_i. Since the system is stable, at least one of the coefficient c_i must be nonzero. Thus, for an appropriate choice of c_i, we can solve boundary probabilities.

4. Solution to the Boundary Probabilities

A procedure for solving boundary stationary probabilities $\boldsymbol{\pi}_0$ and $\boldsymbol{\pi}_N$ can be summaried as follows. Define

$$\boldsymbol{\pi}_0 = (\pi_{0,1}, \pi_{0,2}, \cdots, \pi_{0,J_2}) \quad \text{and} \quad \boldsymbol{\pi}_N = (\pi_{N,1}, \pi_{N,2}, \cdots, \pi_{N,J_1}). \tag{4.1}$$

Consider the equations (3.12), (3.13), (3.15) and (3.16). The number of linear equations are J_2, J_1J_2, J_1J_2 and J_1, respectively. However, the total number of unknowns including c_i, $\boldsymbol{\pi}_0$ and $\boldsymbol{\pi}_N$ are $2J_1 + 2J_2 - 1$. Because those equations form a linear homogeneous system of equations, we should substituted one of them by the normalization condition equation (3.11). Inserting (3.30) and (4.1) into the linear homogeneous system of equations, we now have a system of linear nonhomogeneous equations. Total number of equations are $J_1 + J_2 + J_1J_2$ which is greater than unknowns.

Since the solution of this problem exists, we use the least square algorithm [7, p.248] to get a good approximated solution. Therefore the unknown quantities will be determined.

We describe the algorithm of solving stationary probabilities of $-/PH/1 \to /PH/1$ system with $\mu_1 \geq \mu_2$ as follows.

Step 1 Solve system of equation (2.3).

Step 2 Given solutions $(\eta, \omega_1, \omega_2)$ where $\eta \neq 1$,

1. compute $\mathbf{w}_m(i)$ defined in (3.29).
2. let $\boldsymbol{\pi}_m$ be a linear combination of $\mathbf{w}_m(i)$

Step 3 Set a linear nonhomogeneous system consisting of equations (3.12), (3.13), (3.15), (3.16) and (3.11).

Step 4 Solve linear nonhomogeneous system and obtain coefficients of $\boldsymbol{\pi}_m$ and boundary stationary probabilities.

Step 5 Substituting coefficients c_i, $i = 1, \ldots, J_1 + J_1$ to (3.30) and obtain unboundary stationary probabilities $\boldsymbol{\pi}_m$, $1 \leq m \leq N - 1$.

Remarks. Since all the stationary probabilities for unboundary states are expressed in terms of Kronecker product-forms, instead of solving (3.6)$\sim$ (3.11) we can solve (3.12)$\sim$

(3.16) and omit equation (3.14) . It is important to note that no matter how many customers in the closed system, we only need to solve $\boldsymbol{\pi}_0$, $\boldsymbol{\pi}_N$ and coefficients c_i. Hence the computational complexity is greatly reduced.

5. Illustrative Examples

Example 1. This example illustrates a fundamental model that consists of a $-/M/1 \rightarrow /M/1$ closed system having $\mu_2 \leq \mu_1$. The LST of service times at each node are given by

$$S_1^*(s_1) = (\frac{\mu_1}{s_1+\mu_1}), \quad S_2^*(s_2) = (\frac{\mu_2}{s_2+\mu_2}).$$

We first solve equation (2.3). The solutions are:

$$(h, s_1, s_2) = (\frac{\mu_2}{\mu_1}, \mu_1 - \mu_2, \mu_2 - \mu_1).$$

In particular with one dimension, we have $\mathbf{u}_1 = \mathbf{u}_2 = 1$.

According to $\mathbf{w}_m = \eta^m(\mathbf{u}_1 \otimes \mathbf{u}_2)$ and let $\rho_k = \lambda/\mu_k$ where $\lambda = \mu_2$, we obtain and

$$\mathbf{w}_m = (\frac{\mu_2}{\mu_1})^m = (\frac{\lambda}{\mu_1})^m(\frac{\lambda}{\mu_2})^{N-m} = \rho_1^m \rho_2^{N-m}. \tag{5.1}$$

Thus, let $\pi_m = c\rho_1^m\rho_2^{N-m}$, for $1 \leq m \leq N-1$. After solving π_0 and π_N, we have $\pi_0 = c\rho_1^0\rho_2^N$ and $\pi_N = c\rho_1^N\rho_2^0$, Hence, $\pi_m = c\rho_1^m\rho_2^{N-m}$, for $0 \leq m \leq N$, where

$$c = (\sum_{n_1+n_2=N} \rho_1^{n_1}\rho_2^{n_2})^{-1}$$

is a normalization constant. It agrees with the general results in a closed Jackson network [4].

Example 2. The example presents a model that consists of a $-/E_2/1 \rightarrow /E_2/1$ closed system having $\mu_2 \geq \mu_1$. The system has the following features:

$$N = 6, \quad \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = (1, 0), \quad \mu_1 = 0.5, \quad \mu_2 = 0.6.$$

With these data, we are able to write down the matrix $\mathbf{Q}$. In addition, we have

$$S_1^*(s_1) = (\frac{1}{s_1+1})^2, \quad S_2^*(s_2) = (\frac{1.2}{s_2+1.2})^2.$$

We first solve the system of equation (2.3). We obtain three roots for (h, s_1, s_2), i.e.,

$$\{(0.204682, -1.45242, 1.45242), (1.44, 0.2, -0.2), (7.03532, 1.65242, -1.65242)\}.$$

Denote them by indices i, $i = 1, 2, 3$. Based on the solutions, we have

$$\mathbf{w}_m(1) = (0.204682)^m(-0.568851, -0.257358, 1.25736, 0.568851);$$

$$\mathbf{w}_m(2) = (1.44)^m(0.247934, 0.297521, 0.206612, 0.247934);$$

$$\mathbf{w}_m(3) = (7.03532)^m(-0.439483, 1.16569, -0.165691, 0.439483).$$

Let

$$\boldsymbol{\pi}_m = \sum_{i=1}^{3} c_i \mathbf{w}_m(i)$$
$$\boldsymbol{\pi}_0 = (\pi_{0,1}, \pi_{0,2})$$
$$\boldsymbol{\pi}_6 = (\pi_{6,1}, \pi_{6,2}).$$

Inserting above equations into equation (3.12), (3.13), (3.15), (3.16) and (3.11). Solve the linear nonhomogeneous system of equations. We finally obtain

$$\begin{aligned}
\boldsymbol{\pi}_0 &= (0.00753841, 0.0199861),\\
\boldsymbol{\pi}_1 &= (0.0199014, 0.0220067, 0.00904609, 0.0149373),\\
\boldsymbol{\pi}_2 &= (0.0256346, 0.0302131, 0.0197997, 0.0245331),\\
\boldsymbol{\pi}_3 &= (0.0365342, 0.0425703, 0.0299881, 0.0357073),\\
\boldsymbol{\pi}_4 &= (0.0542142, 0.0566464, 0.0441195, 0.0498136),\\
\boldsymbol{\pi}_5 &= (0.0902349, 0.049219, 0.0681869, 0.0595651),\\
\boldsymbol{\pi}_6 &= (0.0590628, 0.130541).
\end{aligned}$$

Let $p_i(n_i)$ denote the marginal probability of number of customers n_i at node i, $i = 1, 2$. Thus, we have

$$p_1(n_1) = p_2(6 - n_1) = \boldsymbol{\pi}_{n_1}\mathbf{e}'.$$

Conventionally, it may be computed by a formula given in [1] which gives

$$\begin{aligned}
&p_1(0) = p_2(6) = 0.0275,\ p_1(1) = p_2(5) = 0.0659,\\
&p_1(2) = p_2(4) = 0.1002,\ p_1(3) = p_2(3) = 0.1448,\\
&p_1(4) = p_2(2) = 0.2048,\ p_1(5) = p_2(1) = 0.2672,\\
&p_1(6) = p_2(0) = 0.1896.
\end{aligned}$$

In conclusion, it reconfirms our derivation in this paper.

6. Conclusions

In this study, we have conjectured that the Laplace-Stieltjes transforms of the service times distributions may satisfy (2.3). For a closed system, we have shown the stationary probability of unboundary states can be written as a linear combination of Kronecker product-forms. Furthermore, we found that each component of these products can be expressed in terms of roots of the system which involve the Laplace-Stieltjes transforms of the service time distributions. This results are useful because solving a closed network with a large number of system states may be complicated due to a great number of customers in system. We provided an efficient computational algorithm for solving stationary probabilities. Hence the computational complexity can be reduced significantly. In specific, we give examples to reconfirm our results and show how the method works.

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