2 Existence and Uniqueness of Solution

2.1 Existence of solutions

To gain a roughly estimate of the life-span of the solution for the initial value problem (2.1) below, in this subsection we reconsider the existence of the solutions of the following initial value problem for the nonlinear equation:

$$\begin{cases} u''(t) = u'(t)^q (c_1 + c_2 u(t)^p), \quad p,\ q \geq 1,\ c_1^2 + c_2^2 \neq 0, \\ u(0) = u_0, u'(0) = u_1. \end{cases} \tag{2.1}$$

For $p \in \mathbb{Q}$, we say that p is odd (even, respectively) if $p = r/s$, $r \in \mathbb{N}$, $s \in 2\mathbb{N}+1, (r, s) = 1$(common factor) and r is odd(even, respectively).

Define

$$T = \min \left\{ \begin{array}{l} \frac{1}{\mid u_1 \mid}, \frac{1}{\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p}, \\ \frac{-\mid u_1 \mid + \sqrt{u_1^2 + 2(\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p)}}{\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p}, \\ -1 + \sqrt{1 + \frac{1}{\alpha_1 + \alpha_2 + \alpha_3}} \end{array} \right\},$$

where

$$N = \mid u_0 \mid +1,\ M = \mid u_1 \mid +1,$$
$$\alpha_1 = \mid c_1 \mid M^q p N^{p-1},\ \alpha_2 = \mid c_1 \mid q M^{q-1},\ \alpha_3 = \mid c_2 \mid q N^p M^{q-1},$$

and

$$\mathbb{X}_T = \{u \in C^2[0, T) : \parallel u \parallel_\infty \leq N \text{ and } \parallel u' \parallel_\infty \leq M\}.$$

We have the following result:

Theorem 2.1 *For any initial values u_0 and u_1, there exists a constant T given as above such that the problem* (2.1) *possesses a solution u in $\mathbb{X}_T$.*

Proof. Thanks to $u \in C^1$, (2.1) can be transformed as follows

$$u(t) = u_0 + u_1 t + \int_0^t \int_0^r u'(s)^q (c_1 + c_2 u(s)^p)\ dsdr. \tag{2.2}$$

We take the sequence

$$\begin{cases} u_0(t) = u_0, \\ u_1(t) = u_0 + u_1 t, \\ \vdots \\ u_n(t) = u_0 + u_1 t + \int_0^t \int_0^r u'_{n-1}(s)^q (c_1 + c_2 u_{n-1}(s)^p)\ dsdr \quad , n \geq 2. \end{cases} \tag{2.3}$$

Now we claim that $\{u_n\}$ converges in

$$\mathbb{X}_1 = \{u \in C^1[0,T) : \| u \|_\infty \leq N \text{ and } \| u' \|_\infty \leq M\}.$$

At first, to check $\{u_n\} \subseteq \mathbb{X}_1$ (see Proposition 5.2 in Appendices). $\mathbb{X}_1$ is a closed bounded set in $C^1[0,T)$, we prove that $\{u_n\}$ is a Cauchy sequence in $\mathbb{X}_1$. We have

$$\begin{aligned} & \mid u_{n+1}(t) - u_n(t) \mid \\ = & \mid \int_0^t \int_0^r c_1(u'_n(s)^q - u'_{n-1}(s)^q) + c_2(u'_n(s)^q u_n(s)^p - u'_{n-1}(s)^q u_{n-1}(s)^p)\ dsdr \mid \\ \leq & \int_0^t \int_0^r \mid c_1(u'_n(s)^q - u'_{n-1}(s)^q) \mid + \mid c_2(u'_n(s)^q u_n(s)^p - u'_{n-1}(s)^q u_{n-1}(s)^p) \mid \ dsdr. \end{aligned}$$

Estimate the first and second integrand respectively.

$$\begin{aligned} & \int_0^t \int_0^r \mid c_1(u'_n(s)^q - u'_{n-1}(s)^q) \mid \ dsdr \\ \leq & \ qM^{q-1} \mid c_1 \mid \int_0^t \int_0^r \mid u'_n(s) - u'_{n-1}(s) \mid \ dsdr \\ \leq & \ qM^{q-1} \mid c_1 \mid \| u'_n - u'_{n-1} \|_\infty T^2/2 \quad \forall t \in [0,T) \end{aligned}$$

and also

$$\begin{aligned} & \int_0^t \int_0^r \mid c_2(u'_n(s)^q u_n(s)^p - u'_{n-1}(s)^q u_{n-1}(s)^p) \mid \ dsdr \\ = & \int_0^t \int_0^r \mid c_2(u'_n(s)^q u_n(s)^p - u'_n(s)^q u_{n-1}(s)^p + u'_n(s)^q u_{n-1}(s)^p - u'_{n-1}(s)^q u_{n-1}(s)^p) \mid \ dsdr \\ \leq & \mid c_2 \mid \int_0^t \int_0^r M^q \mid u_n(s)^p - u_{n-1}(s)^p \mid \ dsdr \\ & + \mid c_2 \mid \int_0^t \int_0^r N^p \mid u'_n(s)^q - u'_{n-1}(s)^q \mid \ dsdr \quad \forall t \in [0,T). \end{aligned}$$

By Proposition 5.3 (see Appendices), we get

$$| u_n(t)^p - u_{n-1}(t)^p | \leq pN^{p-1} | u_n(t) - u_{n-1}(t) |$$

and

$$| u_n'(t)^p - u_{n-1}'(t)^p | \leq qM^{q-1} | u_n'(t) - u_{n-1}'(t) | .$$

Therefore,

$$\begin{aligned}
& \int_0^t \int_0^r | c_2(u_n'(s)^q u_n(s)^p - u_{n-1}'(s)^q u_{n-1}(s)^p) | \ dsdr \\
\leq \ & | c_2 | M^q pN^{p-1} \int_0^t \int_0^r | u_n(s) - u_{n-1}(s) | \ dsdr \\
& + | c_2 | N^p qM^{q-1} \int_0^t \int_0^r | u_n'(s) - u_{n-1}'(s) | \ dsdr \\
\leq \ & \frac{1}{2} | c_2 | M^q pN^{p-1} T^2 \| u_n - u_{n-1} \|_\infty \\
& + \frac{1}{2} | c_2 | N^p qM^{q-1} T^2 \| u_n' - u_{n-1}' \|_\infty .
\end{aligned}$$

We obtain

$$\begin{aligned}
\| u_{n+1} - u_n \|_\infty \ \leq \ & \frac{1}{2} | c_2 | M^q pN^{p-1} T^2 \| u_n - u_{n-1} \|_\infty \\
& + \frac{1}{2} qM^{q-1}(| c_1 | + | c_2 | N^p) T^2 \| u_n' - u_{n-1}' \|_\infty .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\| u_{n+1}' - u_n' \|_\infty \ \leq \ & | c_2 | M^q pN^{p-1} T \| u_n - u_{n-1} \|_\infty \\
& + qM^{q-1}(| c_1 | + | c_2 | N^p) T \| u_n' - u_{n-1}' \|_\infty .
\end{aligned}$$

We conclude that

$$\begin{aligned}
& \| u_{n+1} - u_n \|_{c_1} \\
\leq \ & | c_2 | M^q pN^{p-1}(T + \frac{1}{2}T^2) \| u_n - u_{n-1} \|_\infty \\
& + qM^{q-1}(| c_1 | + | c_2 | N^p)(T + \frac{1}{2}T^2) \| u_n' - u_{n-1}' \|_\infty \\
\leq \ & (\alpha_1 + \alpha_2 + \alpha_3)(T + \frac{T^2}{2}) \| u_n - u_{n-1} \|_{c_1} \\
\leq \ & [(\alpha_1 + \alpha_2 + \alpha_3)(T + \frac{T^2}{2})]^2 \| u_{n-1} - u_{n-2} \|_{c_1} \\
& \vdots \\
\leq \ & [(\alpha_1 + \alpha_2 + \alpha_3)(T + \frac{T^2}{2})]^n \| u_1 - u_0 \|_{c_1} .
\end{aligned}$$

Let $K = (\alpha_1 + \alpha_2 + \alpha_3)(T + \frac{T^2}{2})$, then $K < 1$. For $n > m > 0$, we obtain that

$$\begin{aligned} & \| u_n - u_m \|_{c_1} \\ = \; & \| u_n - u_{n-1} + u_{n-1} - u_{n-2} + \cdots + u_{m+1} - u_m \|_{c_1} \\ \leq \; & (K^n + K^{n-1} + \cdots + K^{m+1}) \mid u_1 - u_0 \mid . \end{aligned}$$

Since that $K < 1$, therefore, $\{u_n\}$ is a Cauchy sequence in $C^1[0, T)$. Hence there is a solution of problem (2.2) in $\mathbb{X}_1$. Thus, we have a solution of problem (2.1) in $\mathbb{X}_T = \{u \in C^2[0, T) : \| u \|_\infty \leq N \text{ and } \| u' \|_\infty \leq M\}$. ❑

2.2 Uniqueness of solution

Theorem 2.2 *Suppose that u is a classical solution of problem* (2.1) *in $\mathbb{X}_T$. Then u is the unique solution of problem* (2.1) *in $\mathbb{X}_T$.*

Proof. Suppose that u and v are two solutions of (2.1) in $\mathbb{X}$. We have

$$u(t) = u_0 + u_1 t + \int_0^t \int_0^r u'(s)^q (c_1 + c_2 u(s)^p) \, dsdr$$

and

$$v(t) = u_0 + u_1 t + \int_0^t \int_0^r v'(s)^q (c_1 + c_2 v(s)^p) \, dsdr.$$

We recall the proof of existence of solution in the last section. It is easy to get following results. For $t \in [0, T)$

$$\begin{aligned} & \mid u(t) - v(t) \mid \\ = \; & \int_0^t \int_0^r c_1 [u'(t)^q - v'(t)^q] + c_2 [u'(t)^q u(t)^p v'(t)^q v(t)^p] \, dsdr \\ \leq \; & q M^{q-1} \mid c_1 \mid \| u' - v' \| \frac{T^2}{2} \\ & + \frac{1}{2} \mid c_2 \mid M^q p N^{p-1} T^2 \| u - v \|_\infty + \frac{1}{2} \mid c_2 \mid N^p q M^{q-1} T^2 \| u' - v' \|_\infty . \end{aligned}$$

Hence

$$\begin{aligned} & \| u - v \|_\infty \\ \leq \; & \frac{1}{2} \mid c_2 \mid M^q p N^{p-1} T^2 \| u - v \|_\infty + \frac{1}{2} q M^{q-1} (\mid c_1 \mid + \mid c_2 \mid N^p) T^2 \| u' - v' \|_\infty \end{aligned}$$

and

$$\begin{aligned} &\| u' - v' \|_\infty \\ \leq\ & | c_2 | M^q p N^{p-1} T \| u - v \|_\infty + q M^{q-1} (| c_1 | + | c_2 | N^p) T \| u' - v' \|_\infty . \end{aligned}$$

We conclude that

$$\begin{aligned} &\| u - v \|_{c_1} \\ \leq\ & | c_2 | M^q p N^{p-1} (T + \frac{1}{2} T^2) \| u - v \|_\infty + \\ & q M^{q-1} (| c_1 | + | c_2 | N^p)(T + \frac{1}{2} T^2) \| u' - v' \|_\infty \\ \leq\ & (\alpha_1 + \alpha_2 + \alpha_3)(T + \frac{T^2}{2}) \| u - v \|_{c_1} \end{aligned}$$

and then

$$[1 - (\alpha_1 + \alpha_2 + \alpha_3)(T + \frac{T^2}{2})] \| u - v \|_{c_1} \leq 0,$$

seeing that $(\alpha_1 + \alpha_2 + \alpha_3)(T + \frac{T^2}{2}) < 1$; thus we have

$$0 \leq \| u - v \|_{c_1} \leq 0,$$

that means that $u = v$ in $\mathbb{X}_T$. ❑