3 General Question of Higher Dimensional Spaces

In this section, we will generalize the three classical questions into general n-dimensional space. Second, we discuss the properties of maximizing regions during point, line, and space. And then, we will generalize these properties for n-dimensional space with a system of equations. At last, we solve this general question by recurrence relation to get its formula.

3.1 Generalizing These Three Classical Questions

In light of the results of these questions, we are really interested in the rules of their formulas -each increasing combination "C" for each increasing dimension:

Maximum number of regions that k points partition a line is $\underline{C_0^k + C_1^k}$

Maximum number of regions that k lines partition a plane is $\underline{C_0^k + C_1^k + C_2^k}$

Maximum number of regions that k plane partition a space is $\underline{C_0^k + C_1^k + C_2^k + C_3^k}$

Hence, we generalize the questions into the following one:

If there are k different $(n-1)$-dimensional spaces in an n-dimensional space, then what is the largest number of the different regions which are formed by these k different $(n-1)$-dimensional spaces?

Here, the partitioner in an n-dimensional space is a set of $(n-1)$-dimensional space. It is just like the 3-dimensional space that the partitioner is a set of 2-dimensional space (*plane*).

Now, we would like to see if the formula of this solution is of the following form:

$$C_0^k + C_1^k + C_2^k + \cdots + C_{n-1}^k + C_n^k$$

For this purpose, we have to generalize the properties of maximizing the number of regions by k partitioner in lower dimensional space -line, plane, and 3D space which are described in the Section 3.2.

3.2 The Properties of Point, Line, and 3-D Space

Return to Question 2: **If there are k different 1-dimensional lines in an 2-dimensional plane, then what is the largest number of the different regions which are formed by these k different lines?**

The sufficient and necessary condition to maximize the number of regions made up by the k lines is:

$\langle 1 \rangle$ any 2 of the k lines intersect in a common point.

$\langle 2 \rangle$ any 3 of three of the k lines do not intersect in a common point.

Now is Question 3: **If there are k different 2-dimensional planes in an 3-dimensional space, then what is the largest number of the different regions which are formed by these k different planes?**

The sufficient and necessary condition to maximize the number of regions made up by the k planes is:

$\langle 1 \rangle$ any 2 of the k planes intersect in a common line.

$\langle 2 \rangle$ any 3 of the k planes intersect in a common point.

$\langle 3 \rangle$ any 4 of the k planes do not intersect in a common point.

3.3 The Properties of General Question and Standard Partition System of n-Dimensional Space

Having the conclusion in the former section, we will easily generalize the properties of maximizing the number of regions made up by k different $(n-1)$-dimensional space in an n-dimensional space in the following. For convenience, we have the following definitions that an 0-dimensional space as a point, an 1-dimensional space as a line, an 2-dimensional space as a plane, and an 3-dimensional space as our familiar living space. These are just what we know in *Euclidean Geometry.* Then we state the properties as follows:

$\langle 1 \rangle$ any 2 of the k different $(n-1)$-dimensional spaces intersect in a common $(n-2)$-dimensional space.

$\langle 2 \rangle$ any 3 of the k different $(n-1)$-dimensional spaces intersect in a common $(n-3)$-dimensional space.

$\langle 3 \rangle$ any 4 of the k different $(n-1)$-dimensional spaces intersect in a common $(n-4)$-dimensional space.

$\vdots$

$\langle n-2 \rangle$ any $n-1$ of the k different $(n-1)$-dimensional spaces intersect in a common 1-dimensional space.

$\langle n-1 \rangle$ any n of the k different $(n-1)$-dimensional spaces intersect in a common 0-dimensional space.

$\langle n \rangle$ any $n+1$ of the k different $(n-1)$-dimensional spaces do not intersect in a common 0-dimensional space.

We build up a system called ***Standard Partition System of n-Dimensional Space*** in the following. The system not only ensures that the properties of the generalization are realized but also shows how the partitions are presented in the lower dimensional spaces as line, plane, and space:

$$\widetilde{E}_m : x_n = \frac{1}{m}x_1 + \frac{1}{m+1}x_2 + \cdots + \frac{1}{m+(n-2)}x_{n-1} + (m-1),\ k \in \mathbb{N} \tag{3.1}$$

which is equivalent to

$$\widetilde{E}_m : \frac{1}{m}x_1 + \frac{1}{m+1}x_2 + \cdots + \frac{1}{m+(n-2)}x_{n-1} - x_n = 1-m,\ k \in \mathbb{N} \tag{3.2}$$

hence we have the following system of $n+1$ different equations:

$$\begin{cases}
\frac{1}{m_1}x_1 & + & \frac{1}{m_1+1}x_2 & + & \cdots & + & \frac{1}{m_1+(n-2)}x_{n-1} & - & x_n & = & 1-m_1 \\
\frac{1}{m_2}x_1 & + & \frac{1}{m_2+1}x_2 & + & \cdots & + & \frac{1}{m_2+(n-2)}x_{n-1} & - & x_n & = & 1-m_2 \\
 & \vdots & & \vdots & & & \vdots & & & & \vdots \\
\frac{1}{m_{n-1}}x_1 & + & \frac{1}{m_{n-1}+1}x_2 & + & \cdots & + & \frac{1}{m_{n-1}+(n-2)}x_{n-1} & - & x_n & = & 1-m_{n-1} \\
\frac{1}{m_n}x_1 & + & \frac{1}{m_n+1}x_2 & + & \cdots & + & \frac{1}{m_n+(n-2)}x_{n-1} & - & x_n & = & 1-m_n \\
\frac{1}{m_{n+1}}x_1 & + & \frac{1}{m_{n+1}+1}x_2 & + & \cdots & + & \frac{1}{m_{n+1}+(n-2)}x_{n-1} & - & x_n & = & 1-m_{n+1}
\end{cases}$$

3.4 Proof of the Properties

First, we shows property $\langle\, n-1 \,\rangle$ **any n of the k different $(n-1)$-dimensional spaces intersecting in a common 0-dimensional space.** That is to say, these n different equations will have a common solution.

We check the determinant of the coefficient matrix:

$$\det\begin{pmatrix}
\frac{1}{m_1} & \frac{1}{m_1+1} & \frac{1}{m_1+2} & \cdots & \frac{1}{m_1+(n-3)} & \frac{1}{m_1+(n-2)} & -1 \\
\frac{1}{m_2} & \frac{1}{m_2+1} & \frac{1}{m_2+2} & \cdots & \frac{1}{m_2+(n-3)} & \frac{1}{m_2+(n-2)} & -1 \\
\frac{1}{m_3} & \frac{1}{m_3+1} & \frac{1}{m_3+2} & \cdots & \frac{1}{m_3+(n-3)} & \frac{1}{m_3+(n-2)} & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{1}{m_{n-1}} & \frac{1}{m_{n-1}+1} & \frac{1}{m_{n-1}+2} & \cdots & \frac{1}{m_{n-1}+(n-3)} & \frac{1}{m_{n-1}+(n-2)} & -1 \\
\frac{1}{m_n} & \frac{1}{m_n+1} & \frac{1}{m_n+2} & \cdots & \frac{1}{m_n+(n-3)} & \frac{1}{m_n+(n-2)} & -1
\end{pmatrix}$$

As we take all denominators of each row out of this determinant to be multipliers, we will obtain the following form

$$\prod_{i=1}^{n}\prod_{j=0}^{n-2}\frac{1}{m_i+j}\cdot\det\begin{pmatrix} f_0(m_1) & f_1(m_1) & \cdots & f_{n-2}(m_1) & f_{n-1}(m_1) \\ f_0(m_2) & f_1(m_2) & \cdots & f_{n-2}(m_2) & f_{n-1}(m_2) \\ f_0(m_3) & f_1(m_3) & \cdots & f_{n-2}(m_3) & f_{n-1}(m_3) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_0(m_{n-1}) & f_1(m_{n-1}) & \cdots & f_{n-2}(m_{n-1}) & f_{n-1}(m_{n-1}) \\ f_0(m_n) & f_1(m_n) & \cdots & f_{n-2}(m_n) & f_{n-1}(m_n) \end{pmatrix}$$

Where $f_t(m_i) = m_i(m_i+1)(m_i+2)\cdots[m_i+(t-1)][m_i+(t+1)]\cdots[m_i+(n-2)]$,for $t = 0, 1, 2, \ldots, n-1$ and $i = 1, 2, 3, \ldots, n$.

Hence $deg(f_t(m_i)) = n-2$ for $t = 0, 1, 2, \ldots, n-2$, and $deg(f_t(m_i)) = n-1$ for $t = n-1$

We can easily check that $\beta = \{f_0(m_i), f_1(m_i), \ldots, f_{n-2}(m_i)\}$ forms a basis of $P_{n-2}(\mathbb{R})$[6]. Since each row is of the same form, we could simplify first n elements of each row in this determinant into a standard basis of $P_{n-2}(\mathbb{R})$ by column operation in the following

$$\prod_{i=1}^{n}\prod_{j=0}^{n-2}\frac{1}{m_i+j}\cdot\det\begin{pmatrix} 1 & m_1 & {m_1}^2 & \cdots & {m_1}^{n-1} & f_{n-1}(m_1) \\ 1 & m_2 & {m_2}^2 & \cdots & {m_2}^{n-1} & f_{n-1}(m_2) \\ 1 & m_3 & {m_3}^2 & \cdots & {m_3}^{n-1} & f_{n-1}(m_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & m_{n-1} & {m_{n-1}}^2 & \cdots & {m_{n-1}}^{n-1} & f_{n-1}(m_{n-1}) \\ 1 & m_n & {m_n}^2 & \cdots & {m_n}^{n-1} & f_{n-1}(m_n) \end{pmatrix}$$

Hence, we use the first n columns to reduce the last column by column operations to get the following form:

$$\prod_{i=1}^{n}\prod_{j=0}^{n-2}\frac{1}{m_i+j}\cdot\det\begin{pmatrix} 1 & m_1 & {m_1}^2 & \cdots & {m_1}^{n-1} & {m_1}^n \\ 1 & m_2 & {m_2}^2 & \cdots & {m_2}^{n-1} & {m_2}^n \\ 1 & m_3 & {m_3}^2 & \cdots & {m_3}^{n-1} & {m_3}^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & m_{n-1} & {m_{n-1}}^2 & \cdots & {m_{n-1}}^{n-1} & {m_{n-1}}^n \\ 1 & m_n & {m_n}^2 & \cdots & {m_n}^{n-1} & {m_n}^n \end{pmatrix}$$

$$= \prod_{i=1}^{n}\prod_{j=0}^{n-2}\frac{1}{m_i+j}\cdot\prod_{0\le i<j\le n}(m_i-m_j)\neq 0$$

Then these n different equations have a common solution since the determinant of the coefficient matrix is not equal to zero. i.e. any n of k different $(n-1)$-dimensional space will intersect in a common 0-dimensional space.

Second, we will prove property $\langle\, 1\,\rangle \sim \langle\, n-2\,\rangle$. For property (p), we choose p different equations randomly. Here we use first p equations for convenience since each of m_1, m_2, ..., m_p is different.

Now we check the matrix of the system of the p equations.

$$\begin{bmatrix} \frac{1}{m_1} & \frac{1}{m_1+1} & \frac{1}{m_1+2} & \cdots & \frac{1}{m_1+(n-3)} & \frac{1}{m_1+(n-2)} & -1 \\ \frac{1}{m_2} & \frac{1}{m_2+1} & \frac{1}{m_2+2} & \cdots & \frac{1}{m_2+(n-3)} & \frac{1}{m_2+(n-2)} & -1 \\ \frac{1}{m_3} & \frac{1}{m_3+1} & \frac{1}{m_3+2} & \cdots & \frac{1}{m_3+(n-3)} & \frac{1}{m_3+(n-2)} & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{m_{p-1}} & \frac{1}{m_{p-1}+1} & \frac{1}{m_{p-1}+2} & \cdots & \frac{1}{m_{p-1}+(n-3)} & \frac{1}{m_{p-1}+(n-2)} & -1 \\ \frac{1}{m_p} & \frac{1}{m_p+1} & \frac{1}{m_p+2} & \cdots & \frac{1}{m_p+(n-3)} & \frac{1}{m_p+(p-2)} & -1 \end{bmatrix}$$

We use the same way as above to get the form:

$$\prod_{i=1}^{p}\prod_{j=0}^{n-2}\frac{1}{m_i+j}\cdot\begin{pmatrix} 1 & m_1 & {m_1}^2 & \cdots & {m_1}^{n-1} & {m_1}^n \\ 1 & m_2 & {m_2}^2 & \cdots & {m_2}^{n-1} & {m_2}^n \\ 1 & m_3 & {m_3}^2 & \cdots & {m_3}^{n-1} & {m_3}^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & m_{p-1} & {m_{p-1}}^2 & \cdots & {m_{p-1}}^{n-1} & {m_{p-1}}^n \\ 1 & m_p & {m_p}^2 & \cdots & {m_p}^{n-1} & {m_p}^n \end{pmatrix}$$

Then we can say that the matrix is full rank, and the rank is $min\{row\,rank, column\,rnak\} = p$. Hence the $nullity = n - p$. This implies that the solution of these p equations have $n - p$ free variables and p basic variables. Therefore, any p of the k different equation will intersect with a $(n - p)$-dimensional space. For $p = 1 \sim (n - 2)$ we proof the property $\langle\, 1 \,\rangle \sim \langle\, n - 2 \,\rangle$.

At last, we show property $\langle\, n \,\rangle$ **any $n + 1$ of the k different $(n - 1)$-dimensional spaces do not intersect in a common 0-dimensional space.**

Here we use the augmented matrix of the system composed of these $n + 1$ equations, and obtain that its determinant is not equal to zero, from which we conclude that this matrix is full rank.

$$\det\begin{pmatrix} \frac{1}{m_1} & \frac{1}{m_1+1} & \frac{1}{m_1+2} & \cdots & \frac{1}{m_1+(n-3)} & \frac{1}{m_1+(n-2)} & -1 & 1-m_1 \\ \frac{1}{m_2} & \frac{1}{m_2+1} & \frac{1}{m_2+2} & \cdots & \frac{1}{m_2+(n-3)} & \frac{1}{m_2+(n-2)} & -1 & 1-m_2 \\ \frac{1}{m_3} & \frac{1}{m_3+1} & \frac{1}{m_3+2} & \cdots & \frac{1}{m_3+(n-3)} & \frac{1}{m_3+(n-2)} & -1 & 1-m_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m_{n-1}} & \frac{1}{m_{n-1}+1} & \frac{1}{m_{n-1}+2} & \cdots & \frac{1}{m_{n-1}+(n-3)} & \frac{1}{m_{n-1}+(n-2)} & -1 & 1-m_{n-1} \\ \frac{1}{m_n} & \frac{1}{m_n+1} & \frac{1}{m_n+2} & \cdots & \frac{1}{m_n+(n-3)} & \frac{1}{m_n+(n-2)} & -1 & 1-m_n \\ \frac{1}{m_{n+1}} & \frac{1}{m_{n+1}+1} & \frac{1}{m_{n+1}+2} & \cdots & \frac{1}{m_{n+1}+(n-3)} & \frac{1}{m_{n+1}+(n-2)} & -1 & 1-m_{n+1} \end{pmatrix}$$

We use the same way as proving property $\langle\, n - 1 \,\rangle$ to derive that the determinant is not equal to zero.

Hence the *Reduced Row Echelon Form* of the matrix is:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

This tell us the system has no solution since the last row is a contradiction.

Finally, we complete this proof by combining all the three parts.

3.5 Solution by Recurrence Relation for General Question

With the properties in the former section, we could solve the general question stated again in the following:

If there are k different $(n-1)$-dimensional spaces in an n-dimensional space, then what is the largest number of the different regions which are formed by these k different $(n-1)$-dimensional spaces?

In the general question, we can use the similar form of recurrence relation and the useful equation 2.1 to count the maximum number of regions made up by k different partitioner in a n-dimensional space that we conjectured in Section 3.1 as $C_0^k + C_1^k + C_2^k + \cdots + C_{n-1}^k + C_n^k$.

For the goal of obtaining the similar form of recurrence relation, we change the notations of a_k, b_k, and c_k into the following two numbers $P_{n,k}$ and $P_{n-1,k}$ for this question.

Let $P_{n,k}$ be the largest number of the different regions which are formed by k different $(n-1)$-dimensional spaces in an n-dimensional space.

Hence $P_{n-1,k}$ is the largest number of the different regions which are formed by k different $(n-2)$-dimensional spaces in an $(n-1)$-dimensional space.

In the following Figure 4, we can see that the total number of regions is counted as original regions and increasing regions. The number of original regions is $P_{n,k-1}$, made up by the former $k-1$ different $(n-1)$-dimensional spaces, won't be eliminated by the k^{th} $(n-1)$-dimensional space. And, each of the increasing regions made by the k^{th} $(n-1)$-dimensional space is associated with each regions made up by $k-1$

different $(k-2)$-dimensional space in the k^{th} $(n-1)$-dimensional space. Therefore, the number of increasing regions is just $P_{n-1,\,k-1}$.

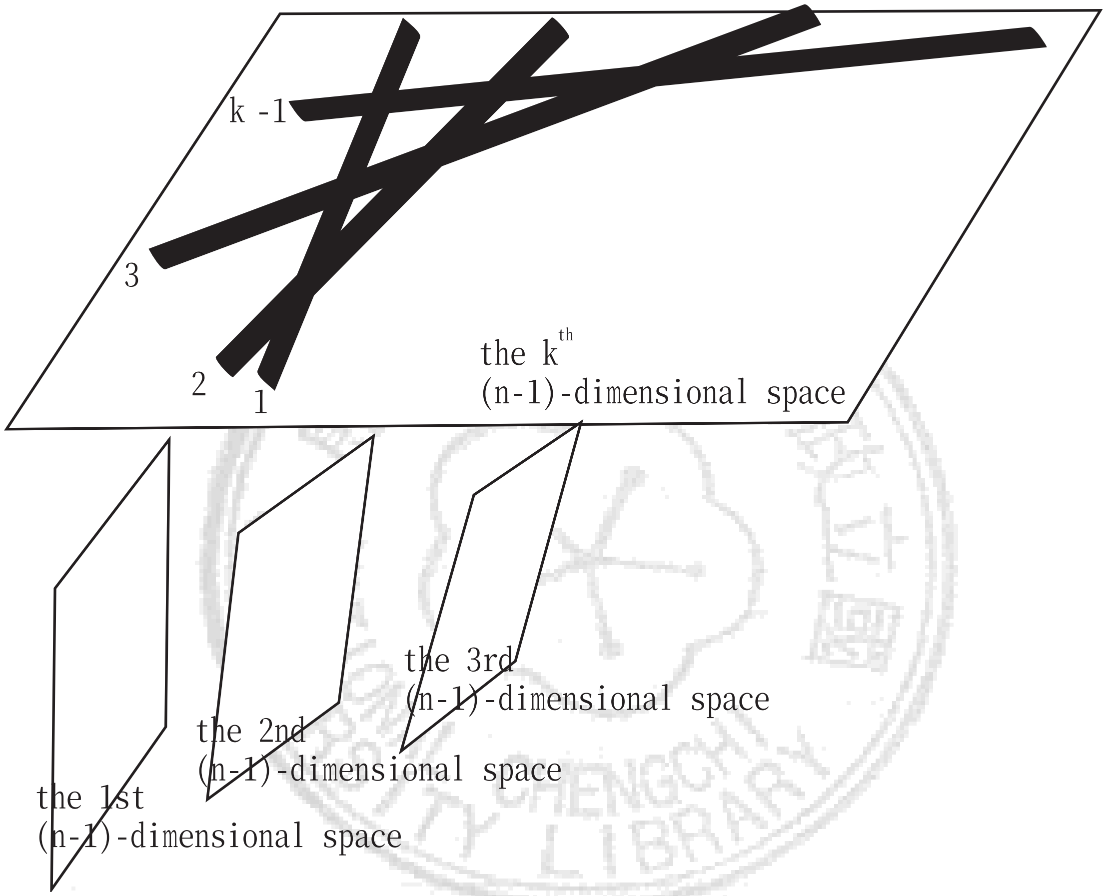

Figure 4: Translation of Recurrence Relation for Higher Dimensional Space

Hence, we have the similar form of recurrence relation

$$P_{n,\,k} = P_{n,\,k-1} + P_{n-1,\,k-1}$$

Again, use the Equation 2.1

$$C_r^r + C_r^{r+1} + C_r^{r+2} + \cdots + C_r^n = C_{r+1}^{n+1}$$

We can easily obtain that

$$P_{n,\,k} = C_0^k + C_1^k + C_2^k + \cdots + C_n^k$$