Chapter 6

Graceful Labellings of Some n-Caterpillars

Let $G = (V, E)$ be a graph with vertex set V and edge set E. We say that G has a *graceful labelling* f if

1. f is an injective mapping from V to $\{0, 1, 2, ..., |E|\}$, and

2. the mapping g from E to $\{1, 2, ..., |E|\}$, defined by $g(uv) = |f(u) - f(v)|$, is bijective.

In 1964, Ringel [55] conjectured that if T is a fixed tree with n edges, then the complete graph on $2n + 1$ vertices can be decomposed into $2n + 1$ copies of T (see [84], 2.2.15. Conjecture.). To prove Ringel's conjecture, one is led to focus on a stronger conjecture, called *Graceful Tree Conjecture* claiming that all trees have graceful labellings; following [84], this is also called Ringel-Kotzing conjecture. In fact, Rosa [60] introduced the notion of graceful labelling originally called $\beta-$ valuation and was renamed as such by Golomb in [36]. Rosa also showed that *caterpillars* (trees with a path incident to every edge) and paths both have a graceful labelling. As an application of graceful labellings of trees, Rosa proved if a tree T with n edges has a graceful labelling, then the complete graph on $2n + 1$ edges has a decomposition into $2n + 1$ copies of T (see [84], 2.2.17. Theorem.) The

Graceful Tree Conjecture is still unsolved. A good reference for graceful labelling is a survey paper by J. A. Gallian [34]. From given graceful trees to obtain new graceful trees was described on [12, 38, 42, 76]. For convenience's sake, we list the following references on other families of graceful trees appeared in the literiture: [2, 3, 5, 8, 9, 10, 11, 39, 40, 43, 50, 54, 61, 62, 80, 86].

In 1979, Bermond [6] conjectured that *lobsters* are graceful which is still open where lobsters are trees with a path from which each vertex has distance at most 2. Recently, Mishra and Pangrahi proved that some classes of lobsters, satisfying the conditions in [47], p. 368, have graceful labellings. For other families of graceful lobsters, we refer to see [15, 48, 53, 82]. In this Chapter, our first purpose is to present a graceful labelling of a special class of lobsters, named *2-caterpillars* (Definition 6.1.1). These results supports the validity of Bermond's conjecture and we are motivated to study graceful labellings of some families of lobsters.

The strongest motivation for studying graceful labellings of 2-caterpillars is the fact that caterpillars have *up/down* labelling which are the generalizations of those of paths and stars (see [84], p. 70). We are not aware of any existence literatures, including [47], that discuss 2-caterpillars, and it seem that this part is left unstudied. In Section 6.1, we use algorithm A_1 to yield graceful labellings of 2-caterpillars by partitioning them into a union of 2-stars after suitably removing some legs (Theorem 6.1.2). The algorithm is essentially simple to understand and the technique of the proof is by induction.

Naturally, it comes to our mind whether *n-caterpillars* (Definition 6.1.1) have graceful labellings analogous to those of 2-caterpillars for $n \geq 3$. Unfortunately, similar method to algorithm A_1 doesn't work for some 3-caterpillars. Therefore, we turn to study graceful labellings of *regular $n-$caterpillars* (Definition 6.1.1). In Section 6.2, we devote ourself to studying graceful labellings of $(r, m, n)-caterpillars$ (Definition 6.1.1). The Algorithm A_2 uses two graceful labellings of small regular $n-$caterpillars to yield a graceful labelling of a large regular $n-$caterpillar. On the one hand, we combine two graceful labellings of a $(2k, m, n)-$caterpillar and a $(1, m, n)-$caterpillar, respectively, to yield a graceful labelling of a $(2k + 1, m, n)-$caterpillar. On the other hand, we combine two graceful labellings of a $(2k, m, n)-$caterpillar and a $(2, m, n)$ caterpillar, respectively, to yield a graceful la-

belling of $(2k+2, m, n)-$caterpillar. The technique of the proof is also by induction. In fact, algorithm A_2 can be simplified as algorithm A'_2.

Let $\mathbb{T}_n$ be the set of $n-$caterpillars with the single path of length divisible by n. In Section 6.3, our main purpose is to generalize the result of Lemma 6.1.3 to $n-$caterpillars in $\mathbb{T}_n$. We partition $[kn]$ into k $n-$sets $X_i, i = 1, 2, ..., k$ and partition any $n-$caterpillar $T \in \mathbb{T}_n$ into a union of $n-$stars. Using algorithm A_4 to label T will yield a graceful labelling of T (Proposition 6.3.11).

At step 3 of algorithm A_1, as we restore removed legs and replace two vertex labels $2n + 1 - y$ and y with y and $2n + 1 - y$, then the reconstructed $2-$caterpillar still has a graceful labelling. If we fill these four vertex labels in a square of order 2, then it looks like a Latin square. This motivates us to study an application of Latin squares to graceful labellings of 2^n-caterpillars. The case of 4-caterpillars has been settled recently in a master thesis by Wu [85]. In Section 6.4, we first construct a special class of Latin squares of orders 2^n, named *graceful Latin squares* (Definition 6.4.1). Next, using Latin square, we provide algorithm A_4, analogous but more complicated than algorithm A_1, to yield graceful labellings of 2^n-caterpillars.

6.1   Graceful Labellings of 2-Caterpillars

For extending graceful labellings of paths and 2-stars and towards Bermond's conjecture, in this Section we present graceful labellings of 2-caterpillars, special class of lobsters, defined as follows:

Definition 6.1.1 *$n-$Caterpillars are trees with a single path with vertices either a single vertex or the root of an $n-$star. Those paths attached to the single path are called their legs. Caterpillars are 1-caterpillars. Regular $n-$caterpillars are $n-$caterpillars where each vertex of the single path has the same number of legs. $(r, m, n)-$Caterpillars are regular $n-$caterpillars where the single path has r vertices and each vertex of the single path has exactly m legs.*

In this thesis we allow n-caterpillars to grow from the leftmost vertex of the single path by either adding an horizontal edge or a leg. Clearly, for any n-caterpillar,

we can iterate such construction to obtain it from a single vertex and the construction is unique. Therefore we can say that an $n-$caterpillar has a unique single path with vertices either a single vertex or the root of a $n-$star.

We herein wish to complete the following result.

Theorem 6.1.2 *2-Caterpillars have graceful labellings.*

For this purpose, we first offer an algorithm to yield graceful labellings of $2-$caterpillars with the single path of even length and the first vertex label of the single path is 0 where the first vertex of the single path is the leftmost vertex of the single path. Combing this with $up/down$ labelling yields graceful labellings of $2-$caterpillars with the single path of odd length.

Recall that $\mathbb{T}_n$ be the set of $n-$caterpillars with the single path of length divisible by n.

Algorithm A_1: Labelling of $2-$caterpillars.
Let T be a 2-caterpillar.

1. Assume that $T \in \mathbb{T}_2$ with $2n + 1$ vertices and we put $[2n] = \{1, 2, ..., 2n\}$.

   (a) Remove each leg incident to the $2i^{th}$ vertex of the single path to be incident to the $2i + 1^{st}$ vertex of the single path for $i \geq 1$.

   (b) Partition T into a union of $2-$stars ($2-$caterpillars whose single path is a single vertex) such that each odd vertex (except the first one) of the single path is the last leaf of a $2-$star and the root of next $2-$star.

   (c) Label the first vertex of the single path with 0.

   (d) Label each leaf of the first $2-$star with the smallest unused number odd x in $[2n]$ number and label its adjacent vertex with $2n + 1 - x$.

   (e) If each leaf of a $2-$star is labelled with odd (even) number, then we label each leaf of next $2-$star with the smallest unused even (odd) number and label its adjacent vertex with $2n + 1 - x$.

2. Assume that $T \notin \mathbb{T}_2$.

(a) Remove each leg incident to the $2i - 1^{st}$ vertex of the single path to be incident to the $2i^{th}$ vertex of the single path for $i \geq 1$.

(b) Label the first vertex u of the single path with the number of edges of T.

(c) Apply step 1 to label vertices of the 2-caterpillar $T - u$.

3. Restore each removed leg and interchange the vertex labels between leaf and its adjacent vertex.

Figure 6.1 and Figure 6.2 are examples of graceful labellings of 2-caterpillars with the single paths of even and odd length, respectively.
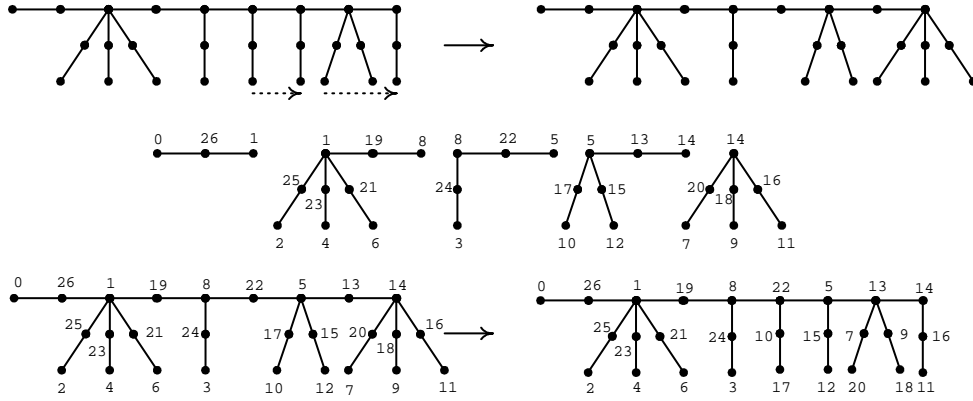

Figure 6.1: A graceful labelling of a 2−caterpillar with the single path of even length.
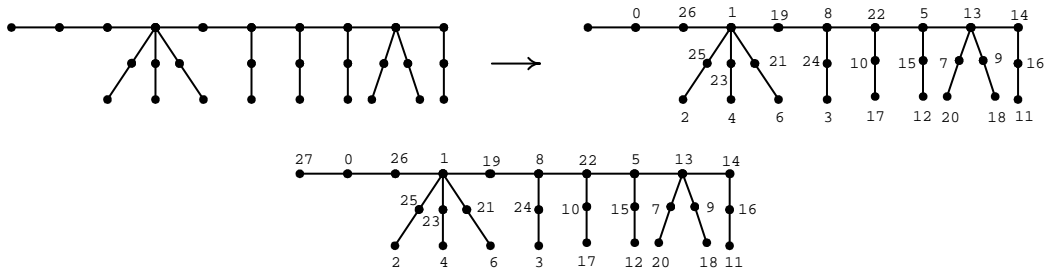

Figure 6.2: A graceful labelling of a 2−caterpillar with the single path of odd length.

Lemma 6.1.3 *If each $2i^{th}$ vertex of the single path of any 2-caterpillar in $\mathbb{T}_2$ has no legs, then algorithm A_1 yields a graceful labelling and the first vertex label of the single path is 0.*

Proof. Assume that the $2i^{th}$ vertex for $i \geq 1$ of the single path of any 2-caterpillar, with $2n + 1$ vertices, has no legs. We use induction on n. For $n = 1$, it is a P_3 which has a graceful labelling 0,2, and 1 on its vertices in order. Suppose that for any 2-caterpillar with $2k + 1$ vertices, where each $2i^t h$ vertex of the single path has no legs, always has a graceful labelling by algorithm A_1.

Consider a $T \in \mathbb{T}_2$ with $2k + 3$ vertices, where each $2i^{th}$ vertex of the single path has no legs.

Case 1: The first vertex of the single path of T has no legs (see Figure 6.3-(a)). Use algorithm A_1 to label vertices of T and partition T into two parts P_3 and T_1, where P_3 is a labelled path with 3 vertices and T_1 is a labelled $2-$caterpillar with $2k + 1$ vertices. If we replace each vertex label s with $s - 1$ in T_1, then there yields a new labelled $2-$caterpillar T_2, where the first vertex label of the single path is 0 and the other vertices are labelled by algorithm A_1. By induction hypothesis, T_2 has a graceful labelling which derives that T has a graceful labelling where the first vertex label of the single path is 0.

Case 2: The first vertex of the single path of T has at least one leg (see Figure 6.3-(b)). Use algorithm A_1 to label vertices of T and partition T into two parts P_3 and T_1, where P_3 is a labelled path with 3 vertices and T_1 is a labelled $2-$caterpillar with $2k + 1$ vertices. If we replace each odd number s with $s - 2$ in T_1, then there yields a new labelled $2-$caterpillar T_2, where the first vertex of the single path is labelled 0 and the other vertices are labelled by algorithm A_1. By induction hypothesis, T_2 has a graceful labelling and the set of edge labels in T_2 is $[2k]$ which equals to

$$\{2, 4, ..., 2k\} \cup \{|x - y| : x + y = 2k + 1, x \text{ is odd, and } y \text{ is even, for } 1 \leq x, y \leq 2k\}.$$

It is easy check that

$$\{|x - y| : \ x + y = 2k + 1, x \text{ is odd, and } y \text{ is even, for } 1 \leq x, y \leq 2k\}$$

$$= \{|x - y| : \ x + y = 2k + 3, x \text{ is odd, and } y \text{ is even, for } 2 \leq x, y \leq 2k + 1\}.$$

This implies $[2k]$ equals to

$$\{2, 4, ..., 2k\} \cup \{|x - y| : x + y = 2k + 3, x \text{ is odd, and } y \text{ is even, for } 2 \leq x, y \leq 2k + 1\},$$

which is the set of edge labels in T_1. Combining this with the edge labels $2k + 1, 2k + 2$ in P_3, we obtain that T has a graceful labelling, where the first vertex label of the single path is 0. $\qquad\square$
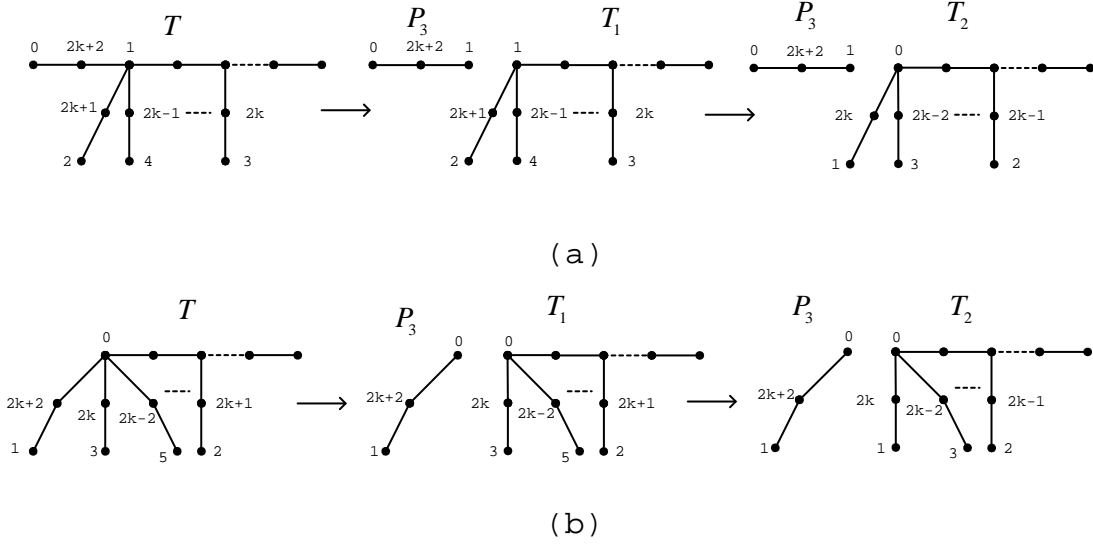
Figure 6.3: An illustration for the proof of Lemma 6.1.3.

Proof of Theorem 6.1.2. Let T be a 2-caterpillar. Two cases are discussed.

Case 1: $T \in \mathbb{T}_2$ has $2n + 1$ vertices. Step 1a of algorithm A_1 turns T into a new T' such that each $2i^{th}$ vertex of the single path has no legs. By Lemma 6.1.3, T' has a graceful labelling. The remaining is to prove after restoring each removed leg, T still has a graceful labelling. In Figure 6.4, before restoring the removed leg, two edge labels of the leg are $|x - y|$ and $|2n + 1 - 2y|$. After restoring the removed leg and exchanging the vertex labels between the leaf and its adjacent vertex, two edge labels of the leg are still $|x - y|$ and $|2n + 1 - 2y|$, i.e. T still has a graceful labelling, where the first vertex label of the single path is 0.

Case 2: $T \notin \mathbb{T}_2$ has $2n + 2$ vertices. Let u be the first vertex of the single path of T. Step 2 of algorithm A_1 turns $T - u$ into a new $T' \in \mathbb{T}_e$. By case 1, T' has a graceful labelling, where the first vertex label of the single path is 0. Labelling vertex u with $2n + 1$ yields a graceful labelling of T. $\qquad\square$
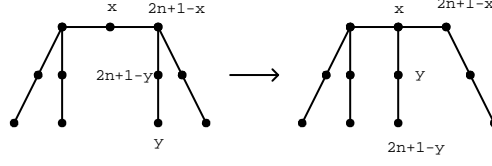
Figure 6.4: An illustration for the proof of Theorem 6.1.2.

6.2 Graceful Labellings of $(r, m, n)-$Caterpillars

Another question arises: Are $n-$caterpillars graceful? Unfortunately, algorithm A_1 doesn't work for some 3-caterpillar. Hence, in this Section we focus on the existence of graceful labellings of regular $n-$caterpillars.

Algorithm A_2: Labelling of $(r, m, n)-$caterpillars

For $m, n \in \mathbb{N}$, let C_r be the $(r, m, n)-$caterpillar for $r \geq 1$.

1. For $r = 1, 2$, let $x = r(mn + 1) - 1$.

   (a) Label the ith vertex of the single path with x_i for $i = 1, 2, ..., r$, where

   $$x_i = \begin{cases} \frac{i-1}{2}(mn + 1), & \text{if } i \text{ is odd,} \\ (r - \frac{i}{2})(mn + 1), & \text{if } i \text{ is even.} \end{cases}$$

   (b) For the first vertex of the single path, label the kth vertex of the jth leg with $a_{1,j,k}$ for $j = 1, 2, ..., m$, and $k = 1, 2, ..., n$, where

   $$a_{1,j,k} = \begin{cases} x - (j - 1)n - \frac{k-1}{2}, & \text{if } k \text{ is odd,} \\ (j - 1)n + \frac{k}{2}, & \text{if } k \text{ is even.} \end{cases}$$

   (c) For the second vertex of the single path, label the kth vertex of the jth leg with $a_{2,j,k}$ for $j = 1, 2, ..., m$, and $k = 1, 2, ..., n$, where

   $$a_{2,j,k} = \begin{cases} -x_2 + a_{1,j,k}, & \text{if } k \text{ is odd,} \\ x_2 + a_{1,j,k}, & \text{if } k \text{ is even.} \end{cases}$$

2. Assume that the $(2t, m, n)-$caterpillar C_{2t} and the $(1, m, n)-$caterpillar C_1 are labelled. We obtain a labelling of the $(2t + 1, m, n)-$caterpillar C_{2t+1} by the following construction.

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(a) For C_{2t}, add $mn + 1$ to each vertex label whose corresponding vertex has odd distance to the vertex labelled 0.

(b) For C_1, add $t(mn + 1)$ to each vertex label.

(c) Add an edge between the last vertex of the single path in C_{2t} and the root in C_1.

3. Assume that the $(2t, m, n)-$caterpillar C_{2t} and the $(2, m, n)-$caterpillar C_2 are labelled. We obtain a labelling of the $(2t + 2, m, n)-$caterpillar $C_{(}2t + 2)$ by the following construction.

(a) For C_{2t}, add $2(mn + 1)$ to each vertex label whose corresponding vertex has odd distance to the vertex labelled 0.

(b) For C_2, add $t(mn + 1)$ to each vertex label.

(c) Add an edge between the last vertex of the single path in C_{2t} and the root in C_2.

Figure 6.5 and Figure 6.6 are examples of graceful labellings of $(3, 3, 3)-$caterpillar and $(4, 3, 3)-$caterpillar by algorithm A_2, respectively.

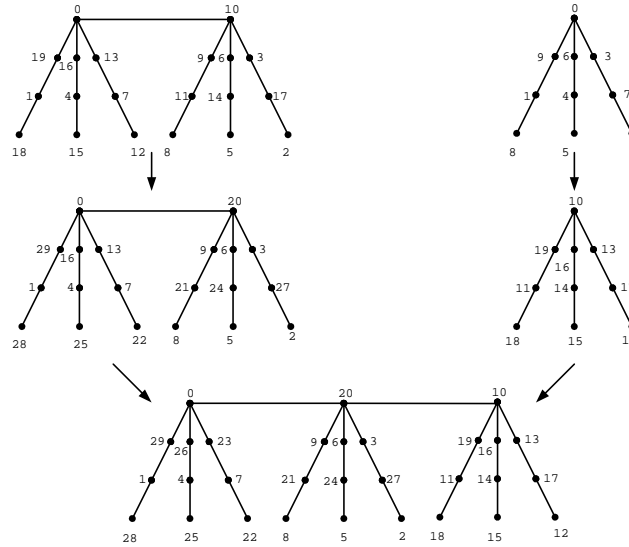

Figure 6.5: A graceful labelling of the $(3, 3, 3)-$caterpillar.

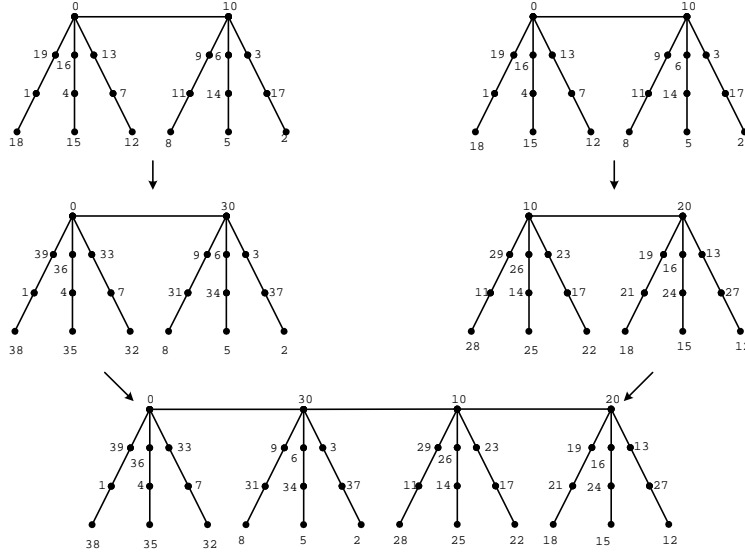**Lemma 6.2.1** *Algorithm A_2 yields graceful labellings of $(1, m, n)-$caterpillars.*

Figure 6.6: A graceful labelling of the $(4, 3, 3)-$caterpillar.

Proof. It suffices to prove that $a_{1,j,k}$'s are all different and all edge labels are different and less than $mn + 1$ for $j = 1, 2, ..., m$ and $k = 1, 2, ..., n$.

By step 1a of algorithm A_2, we label the first vertex of the single path with 0. By step 1b of algorithm A_2, we obtain

$$a_{1,j,k} = \begin{cases} x - (j-1)n - \frac{k-1}{2}, & \text{if } k \text{ is odd,} \\ (j-1)n + \frac{k}{2}, & \text{if } k \text{ is even,} \end{cases}$$

where $x = mn$. We first claim that all $a_{1,j,k}$ are different. Clearly, $a_{1,j,k}$ doesn't equal to 0. If $x - (j-1)n - \frac{k-1}{2} = x - (j'-1)n - \frac{k'-1}{2}$, then $(j-j')n = \frac{k'-k}{2}$. Since the left-hand side of the equation is a multiple of n and the right-hand side of the equation is between $\frac{1-n}{2}$ and $\frac{n-1}{2}$, then the equality holds only when $j = j'$ and $k = k'$. With similar argument, the forth equality holds only when $j = j'$ and $k = k'$. The second and the third equalities are impossible.

Secondly, we claim that all edge labels are different and less than $mn + 1$. Let $e_{1,j,k}$ be the edge label of the kth edge in j^{th} leg for $j = 1, 2, ..., m$ and $k = 1, 2, ..., n$. Then

$$e_{1,j,1} = a_{1,j,k} = x - (j-1)n$$

and

$$e_{1,j,k} = |a_{1,j,k} - a_{1,j,(k-1)}| = |x - 2(j-1)n - (k-1)|, \text{ if } k \neq 1.$$

Clearly, $e_{1,j,1} \neq e_{1,j,k}$ for $k \neq 1$. For $k, k' \neq 1$, if $e_{1,j,k} = e_{1,j',k'}$, i.e.

$$|x - 2(j-1)n - (k-1)| = |x - 2(j'-1)n - (k'-1)|,$$

then it is easy to check that the equality holds only when $j = j'$ and $k = k'$. This means $e_{1,j,k}$'s are all different.

Finally, a routine work finds $e_{1,j,k} \leq mn$ for $j = 1, 2, ..., m$ and $k = 1, 2, ..., n$. Hence, we finish this proof. ❏

Lemma 6.2.2 *Algorithm A_2 yields graceful labellings of $(2, m, n)$−caterpillars. Moreover, the label of each vertex with even distance to the first vertex of the single path is less than $mn + 1$.*

Proof. It suffices to prove that $a_{i,j,k} \neq 0, mn + 1$, $a_{i,j,k}$'s are all different, and all edge labels are different and less than $2(mn + 1)$ for $i = 1, 2$, $j = 1, 2, ..., m$, and $k = 1, 2, ..., n$.

By step 1a of algorithm A_2, we assign $0, mn + 1$ to the first vertex and the second vertex of the single path, respectively. Step 1b implies

$$a_{1,j,k} = \begin{cases} x - (j-1)n - \frac{k-1}{2}, & \text{if } k \text{ is odd,} \\ (j-1)n + \frac{k}{2}, & \text{if } k \text{ is even,} \end{cases}$$

and

$$a_{2,j,k} = \begin{cases} -(mn+1) + x - (j-1)n - \frac{k-1}{2}, & \text{if } k \text{ is odd,} \\ (mn+1) + (j-1)n + \frac{k}{2}, & \text{if } k \text{ is even,} \end{cases}$$

where $x = 2mn + 1$. We first claim that $a_{i,j,k} \neq 0, mn + 1$ and $a_{i,j,k}$'s are all different for $1 \leq i, i' \leq 2$, $1 \leq j, j' \leq m$, and $1 \leq k, k' \leq n$. Clearly, $a_{i,j,k} \neq 0, mn + 1$.

With the same argument as the proof of Lemma 6.2.1, it is easy to check that for $i = 1, 2$, $a_{i,j,k} = a_{i,j',k'}$ holds only when $j = j'$ and $k = k'$.

If $a_{1,j,k} = a_{2,j',k'}$, then

$$x - (j-1)n - \frac{k-1}{2} = -(mn+1) + x - (j'-1)n - \frac{k'-1}{2},$$

$$x - (j-1)n - \frac{k-1}{2} = mn + 1 + (j'-1)n + \frac{k'}{2},$$

$$(j-1)n - \frac{k}{2} = -(mn+1) + x - (j'-1)n - \frac{k'-1}{2},$$

or

$$(j-1)n - \frac{k}{2} = mn + 1 + (j'-1)n + \frac{k'}{2}.$$

If $x - (j-1)n - \frac{k-1}{2} = -(mn+1) + x - (j'-1)n - \frac{k'-1}{2}$, then $mn + (j'-j)n = \frac{k-k'-2}{2}$. Since the left-hand side of the equation is a multiple of n and the right-hand side of the equation is between $\frac{-1-n}{2}$ and $\frac{n-3}{2}$, equality holds only when $j = j'$ and $k = k'$. With similar argument, the forth equality holds only when $j = j'$ and $k = k'$. The second and the third equalities are impossible.

Secondly, we claim that all edge labels are different and less than $2(mn+1)$. Note that the edge label of the single path is $mn + 1$. For the ith vertex of the single path, let $e_{i,j,k}$ be the label of the kth edge in the jth leg, where $i = 1, 2$, $j = 1, 2, ..., m$, and $k = 1, 2, ..., n$. Then

$$e_{1,j,1} = a_{1,j,1} = 2mn + 1 - (j-1)n,$$

$$e_{1,j,k} = |a_{1,j,k} - a_{1,j,(k-1)}| = x - 2(j-1)n - (k-1), \text{ if } k \neq 1,$$

$$e_{2,j,1} = mn + 1 - a_{2j1} = 1 + (j-1)n,$$

and

$$e_{2,j,k} = |a_{2,j,k} - a_{2,j,(k-1)}| = k + 2(j-1)n, \text{ if } k \neq 1.$$

Clearly, all $e_{i,j,k}$'s are less than $2mn + 1$. The remaining is to check that all $e_{i,j,k}$'s are different and not equal to $mn + 1$.

If $e_{i,j,k} = e_{i',j',k'}$, then a routine work finds $i = i', j = j'$ and $k = k'$.

If some $e_{i,j,k} = mn + 1$, then

$$2mn + 1 - (j-1)n = mn + 1,$$

$$x - 2(j-1)n - k - 1 = mn + 1,$$

$$1 + (j-1)n = mn + 1,$$

or

$$k + 2(j-1)n = mn + 1.$$

$2mn + 1 - (j-1)n = mn + 1$ implies $j = m + 1$ contradicting to the fact $j \leq m$. Neither of the other three equalities hold.

Finally, the label of any vertex with even distance to the first vertex of the single path is either $(j-1)n + \frac{k}{2}$ for even k, or $\frac{x-1}{2} - (j-1)n - \frac{k-1}{2}$ for odd k. In any case, it is less than $mn + 1$. Hence, we complete this proof. $\qquad\square$

Using Lemma 6.2.1 and Lemma 6.2.2, we shall prove that regular $n-$caterpillars have graceful labellings.

Theorem 6.2.3 (r, m, n)-*Caterpillars have graceful labellings. In particular, if r is even, then the label of each vertex with even distance to the first vertex of the single path is less than $\frac{r}{2}(mn + 1)$.*

Proof. For any $(r, m, n)-$caterpillar, we use induction on r to verify the result. For $r = 1, 2$, by Lemma 6.2.1 and Lemma 6.2.2, $(1, m, n)-$caterpillars and $(2, m, n)-$caterpillars have a graceful labelling. Assume that any $(i, m, n)-$caterpillar, $1 \leq i \leq r$, always has a graceful labelling and the label of any vertex with even distance to the first vertex u of the single path is less than $\frac{i}{2}(mn + 1)$ for even i.

Consider the $(r + 1, m, n)-$caterpillar.
Case 1: $r + 1$ is odd and let it be $2t + 1$. By step 2 of algorithm A_2, it is formed by a labelled $(2t, m, n)-$caterpillar C_1 and a labelled $(1, m, n)-$caterpillar C_2. We first claim that step 2 of algorithm A_2 forms a set $\{0, 1, 2, ..., (2t + 1)(mn + 1) - 1\}$ of vertex labels for a $(2t + 1, m, n)-$caterpillar.

For a labelled $(2t, m, n)-$caterpillar, the set of vertex labels is $\{0, 1, ..., 2t(mn + 1) - 1\}$ and the label of any vertex with even distance to u is less than $t(mn + 1)$. After adding $mn + 1$ to the label of any vertex with odd distance to u, the set of new vertex labels is $\{0, 1, ..., t(mn + 1) - 1\} \cup \{(t + 1)(mn + 1), (t + 1)(mn + 1) + 1, ..., (2t + 1)(mn + 1) - 1\}$.

For a labelled $(1, m, n)-$caterpillar, the set of vertex labels is $\{0, 1, ..., mn\}$. After adding $t(mn + 1)$ to all vertex labels, the set of new vertex labels is $\{t(mn + 1), t(mn + 1) + 1, ..., (t + 1)(mn + 1) - 1\}$. Combing these two sets of vertex labels yields a set $\{0, 1, ..., (2t + 1)(mn + 1) - 1\}$ which is the set of vertex labels for a given $(2t + 1, m, n)-$caterpillar.

Secondly, we claim that the set of edge labels is $[(2t + 1)(mn + 1) - 1]$. Since C_1 has a graceful labelling whose label of vertex with even distance to u is less than

$t(mn + 1)$, adding $mn + 1$ to the label of any vertex with odd distance to u yields a set of edge labels $\{(mn + 1) + 1, (mn + 1) + 2, ..., (mn + 1) + 2t(mn + 1) - 1\}$, denoted by $\mathbb{S}_1$. Since C_2 has a graceful labelling, adding $t(mn + 1)$ to each vertex label never change the set of edge labels $\{1, 2, ..., mn\}$, denoted by $\mathbb{S}_2$. Note that the new label of the last vertex of the single path in C_1 is $(t + 1)(mn + 1)$ and the new label of the only vertex of the single path in C_2 is $t(mn + 1)$. Hence, the label of edge adjacent to the last vertex of the single path in C_1 and the first vertex of the single path in C_2 is $mn + 1$. That is to say the set of edge labels in the given $(2t + 1, m, n)-$caterpillar is $\mathbb{S}_1 \cup \mathbb{S}_2 \cup \{mn + 1\}$ which is $[(2t + 1)(mn + 1) - 1]$.

Case 2: $r + 1$ is even and let it be $2t$. By similar argument as case 1, a $(2t + 2, m, n)-$caterpillar has a graceful labelling. The remaining is to prove that the label of any vertex with even distance to the vertex labelled 0 is less than $(t+1)(mn+1)$. Assume that a $(2t, m, n)-$caterpillar C_1 and a $(2, m, n)-$caterpillar C_2 can form a $(2t + 2, m, n)-$caterpillar. In C_1, by inductive hypothesis, the label of any vertex with even distance to the vertex labelled 0 is less than $t(mn + 1)$, so does its new label. In C_2, the label of any vertex is less than $mn + 1$. After adding $t(mn + 1)$, the new label of vertex is less than $(t + 1)(mn + 1)$. Hence, for a $(2t + 2, m, n)-$caterpillar, the label of any vertex with even distance to the vertex labelled 0 is less than $(t + 1)(mn + 1)$ and the proof follows. ❑

In fact, algorithm A_2 can be simplified as follows:

Algorithm A_2' : Labelling of $(r, m, n)-$caterpillars.

Let T be the $(r, m, n)-$caterpillar.

1. Assign x_i to the ith vertex of the single path, where

$$x_i = \begin{cases} \frac{i-1}{2}(mn + 1), & \text{if } i \text{ is odd,} \\ (r - \frac{i}{2})(mn + 1), & \text{if } i \text{ is even.} \end{cases}$$

2. Let $x = r(mn + 1) - 1$. For the first vertex of the single path, orderly assign $a_{1,j,k}$ to the kth vertex of the jth leg for $j = 1, 2, ..., m$, and $k = 1, 2, ..., n$, where

$$a_{1,j,k} = \begin{cases} x - (j - 1)n - \frac{k-1}{2}, & \text{if } k \text{ is odd,} \\ (j - 1)n + \frac{k}{2}, & \text{if } k \text{ is even.} \end{cases}$$

For the ith vertex of the single path, orderly assign $a_{i,j,k}$ to the kth vertex of

the jth leg for $i = 2, 3, ..., r$, $j = 1, 2, ..., m$, and $k = 1, 2, ..., n$, where

$$a_{i,j,k} = \begin{cases} -x_i + a_{1,j,k}, & \text{if } k \text{ is odd,} \\ x_i + a_{1,j,k}, & \text{if } k \text{ is even.} \end{cases}$$

Figure 6.7 is an an example of a graceful labelling of the $(4, 3, 3)$−caterpillar by algorithm A_2'.
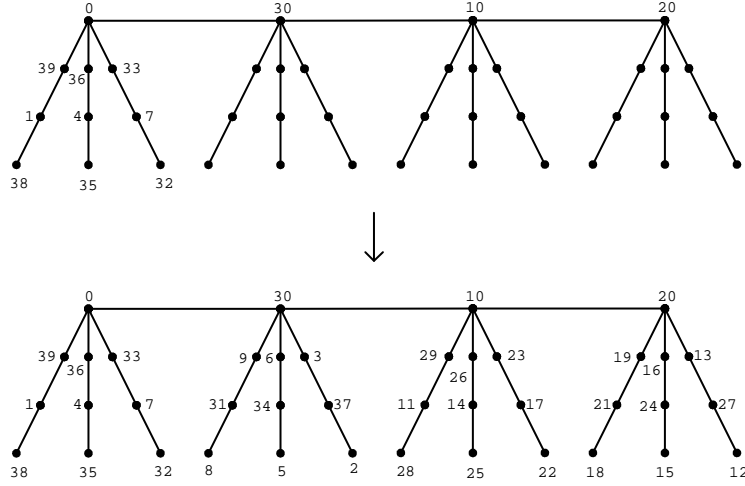

Figure 6.7: A graceful labelling of the $(4, 3, 3)$−caterpillar by algorithm A_2'.

6.3 n−Partitions with Parameter k

In this section we wish to generalize the result of Lemma 6.1.3 to n−caterpillars. For this purpose, we first study some results in partitioning $[kn]$.

Definition 6.3.1 $\{X_i\}_{i=1}^k$ *is an* n−*partition with parameter* k *if*

$$x_{ij} = \begin{cases} (k + 1 - i)n - \frac{j-1}{2}, & \text{if } j \text{ is odd,} \\ (i - 1)n + \frac{j}{2}, & \text{if } j \text{ is even,} \end{cases}$$

where $X_i = \{x_{ij}\}_{j=1}^n$ *for* $i = 1, 2, ..., k$.

Example 6.3.2 *Set* $X_1 = \{15, 1, 14, 2, 13\}, X_2 = \{10, 6, 9, 7, 8\}$, *and* $X_3 = \{5, 11, 4, 12, 3\}$. *Then* $\{X_1, X_2, X_3\}$ *is a 5-star partition with parameter 3.*

The following are three $n-$star partitions with parameter k obtained from three $n-$star partitions with parameter $k + 1$, respectively.

Let $\{X_i\}_{i=1}^{k+1}$ be an $n-$star partition with parameter $k + 1$.

1. For $i = 1, 2, ...k$, set $y_{ij} = \begin{cases} x_{(i+1)j}, & \text{if } j \text{ is odd} \\ x_{(i+1)j} - n, & \text{if } j \text{ is even.} \end{cases}$

2. Assume that n is even. For $i = 1, 2, ..., k$ and $j = 1, 2, ..., n$, we set

$$x'_{ij} = x_{(i+1)j} - \frac{n}{2}, \text{ and}$$

$$z_{ij} = x'_{f_i g_j}, \text{ where } f_i = k + 1 - i \text{ and } g_j = n + 1 - j.$$

3. Assume that n is odd. For $i = 1, 2, ..., k$ and $j = 1, 2, ..., n$, we set

$$x'_{ij} = x_{(i+1)j} - \frac{n+1}{2},$$

$$x''_{ij} = (kn + 1) - x'_{ij}, \text{ and}$$

$$w_{ij} = x''_{f_i g_j}, \text{ where } f_i = k + 1 - i \text{ and } g_j = n + 1 - j.$$

By algebraic calculation, we easily obtain the following result.

Lemma 6.3.3 *For $i = 1, 2, ..., k$, let $Y_i = \{y_{ij}\}_{j=1}^n$, $Z_i = \{z_{ij}\}_{j=1}^n$, and $W_i = \{w_{ij}\}_{j=1}^n$.*

1. *For $n \in \mathbb{N}$,*
$$y_{ij} = \begin{cases} (k + 1 - i)n - \frac{j-1}{2}, & \text{if } j \text{ is odd,} \\ (i - 1)n + \frac{j}{2}, & \text{if } j \text{ is even,} \end{cases}$$
i.e., $\{Y_i\}_{i=1}^k$ is an $n-$star partition with parameter k.

2. *For even n,*
$$z_{ij} = \begin{cases} (k + 1 - i)n - \frac{j-1}{2}, & \text{if } j \text{ is odd,} \\ (i - 1)n + \frac{j}{2}, & \text{if } j \text{ is even,} \end{cases}$$
i.e., $\{Z_i\}_{i=1}^k$ is an $n-$star partition with parameter k.

3. For odd n,

$$w_{ij} = \begin{cases} (k + 1 - i)n - \frac{j-1}{2}, & \text{if } j \text{ is odd,} \\ (i - 1)n + \frac{j}{2}, & \text{if } j \text{ is even,} \end{cases}$$

i.e., $\{W_i\}_{i=1}^k$ is an $n-$star partition with parameter k.

Example 6.3.4 *Using Lemma 6.3.3-1, we shall yield a 6-star partition with parameter 4 obtained from a 6-star partition with parameter 5 as following.*

X_1	30	1	29	2	28	3							
X_2	24	7	23	8	22	9		Y_1	24	1	23	2	22	3
X_3	18	13	17	14	16	15	$\Rightarrow$	Y_2	18	7	17	8	16	9
X_4	12	19	11	20	10	21		Y_3	12	13	11	14	10	15
X_5	6	25	5	26	4	27		Y_4	6	19	5	20	4	21

Example 6.3.5 *Using Lemma 6.3.3-2, we shall yield a 4-star partition with parameter 4 obtained from a 4-star partition with parameter 5 as following.*

X_1	20	1	19	2												
X_2	16	5	15	6		14	3	13	4		Z_1	16	1	15	2	
X_3	12	9	11	10	$\Rightarrow$	10	7	9	8	$\Rightarrow$	Z_2	12	5	11	6	
X_4	8	13	7	14		6	11	5	12		Z_3	8	9	7	10	
X_5	4	17	3	18		2	15	1	16		Z_4	4	13	3	14	

Example 6.3.6 *Using Lemma 6.3.3-3, we shall yield a 3-star partition with parameter 4 obtained from a 3-star partition with parameter 5 as following.*

X_1	15	1	14													
X_2	12	4	11		10	2	9		2	10	3		W_1	12	1	11
X_3	9	7	8	$\Rightarrow$	7	5	6	$\Rightarrow$	5	7	6	$\Rightarrow$	W_2	9	4	8
X_4	6	10	5		4	8	3		8	4	9		W_3	6	7	5
X_5	3	13	2		1	11	0		11	1	12		W_4	3	10	2

To prove the Proposition 6.3.11, we need the following results.

Lemma 6.3.7 *Let*

$$F_1' = \{|x - y| : x, y > m, x + y = 2(k + 1)m + 1\},$$

$$F_2' = \{|x - y| : x, y > m, x + y = 2(k+1)m, \ x, y \equiv i \ (mod \ 2m), \ i \neq 0, m\},$$

$$F_1'' = \{|x - y| : x, y > 0, \ x + y = 2km + 1\}, \ and$$

$$F_2'' = \{|x - y| : x, y > 0, \ x + y = 2km, \ x, y \equiv i \ (mod \ 2m), \ i \neq 0, m\}.$$

Then $F_1' = F_1''$ and $F_2' = F_2''$.

Proof. For $i = 1, 2$, if we correspond x', y' satisfying conditions in F_i' to x'', y'' satisfying conditions in F_i'', respectively, such that $x'' = x' - m$ and $y'' = y' - m$, then we complete the proof. $\quad\square$

Lemma 6.3.8 *Let*

$$F_1' = \{|x-y| : x, y > m-1, x+y = (k+1)(2m-1)+1, \ x, y \equiv i \ (mod \ 2m-1), \ i \neq m\},$$

$$F_2' = \{|x - y| : x, y > m-1, x+y = (k+1)(2m-1), \ x, y \equiv i \ (mod \ 2m-1), \ i \neq 0\},$$

$$F_1'' = \{|x - y| : x, y > 0, x + y = k(2m - 1) + 1\}, \ x, y \equiv i \ (mod \ 2m - 1), \ i \neq m\},$$

and

$$F_2'' = \{|x - y| : x, y > 0, x + y = k(2m - 1), \ x, y \equiv i \ (mod \ 2m - 1), \ i \neq 0\}.$$

Then $F_1' = F_2''$ and $F_2' = F_1''$.

Proof. If we correspond x', y' satisfying conditions in F_1' to x'', y'' satisfying conditions in F_2'' such that $x'' = x' - m$ and $y'' = y' - m$, then we have $F_1' = F_2''$. If we correspond x', y' satisfying conditions in F_2' to x'', y'' satisfying conditions in F_1'', such that $x'' = x' - m + 1$ and $y'' = y' - m + 1$, then we have $F_2' = F_1''$. $\quad\square$

Example 6.3.9 *For $m = 3$ and $k = 2$,*

$$
\begin{aligned}
F_1' &= \{|x - y| : x, y > 3, x + y = 19\} \\
&= \{|4 - 15|, |5 - 14|, |6 - 13|, |7 - 12|, |8 - 11|, |9 - 10|\} \\
&= \{11, 9, 7, 5, 3, 1\}; \\
F_2' &= \{|x - y| : x, y > 3, \ x + y = 18, \ x, y \equiv 1, 2, 4, 5 \ (mod \ 6)\} \\
&= \{|4 - 14|, |5 - 13|, |7 - 11|, |8 - 10|\} \\
&= \{10, 8, 4, 2\};
\end{aligned}
$$

$$\begin{aligned}
F_1'' &= \{|x-y| : x, y > 0, \ x+y = 13\} \\
&= \{|1-12|, |2-11|, |3-10|, |4-9|, |5-8|, |6-7|\} \\
&= \{11, 9, 7, 5, 3, 1\}; \\
F_2'' &= \{|x-y| : x, y > 0, \ x+y = 12, \ x, y \equiv 1, 2, 4, 5 \ (mod \ 6)\} \\
&= \{|1-11|, |2-10|, |4-8|, |5-7|\} \\
&= \{10, 8, 4, 2\}.
\end{aligned}$$

Example 6.3.10 *For $m = 3$ and $k = 2$,*

$$\begin{aligned}
F_1' &= \{|x-y| : x, y > 2, \ x+y = 16, \ x, y \equiv 0, 1, 2, 4 \ (mod \ 5)\} \\
&= \{|4-12|, |5-11|, |6-10|, |7-9|\} \\
&= \{8, 6, 4, 2\}; \\
F_2' &= \{|x-y| : x, y > 2, \ x+y = 15, \ x, y \equiv 1, 2, 3, 4 \ (mod \ 5)\} \\
&= \{|3-12|, |4-11|, |6-9|, |7-8|\} \\
&= \{9, 7, 3, 1\}; \\
F_1'' &= \{|x-y| : x, y > 0, \ x+y = 11, \ x, y \equiv 0, 1, 2, 4 \ (mod \ 5)\} \\
&= \{|1-10|, |2-9|, |4-7|, |5-6|\} \\
&= \{9, 7, 3, 1\}; \\
F_2'' &= \{|x-y| : x, y > 0, \ x+y = 10, \ x, y \equiv 1, 2, 3, 4 \ (mod \ 5)\} \\
&= \{|1-9|, |2-8|, |3-7|, |4-6|\} \\
&= \{8, 6, 4, 2\}.
\end{aligned}$$

Now we shall show our main result Proposition 6.3.11 below which is a generalization of Lemma 6.1.3. To this end, we need an algorithm to yield graceful labellings of $n-$caterpillars in $\mathbb{T}_n$. Recall that $\mathbb{T}_n$ is the set of $n-$caterpillars which have a single path with length divisible by n.

Algorithm A_3: Labellings of $n-$caterpillars in $\mathbb{T}_n$.

Assume that $T \in \mathbb{T}_n$ has $kn + 1$ vertices. Let $\{X_i\}_{i=1}^k$ be an $n-$star partition with parameter k and $X_i = \{x_{ij}\}_{j=1}^n$, i.e.,

$$
\begin{array}{cccccc}
X_1 & kn & 1 & (kn-1) & 2 & \cdots \\
X_2 & (k-1)n & (1+n) & (k-1)n-1 & 2+n & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
X_k & n & 1+(k-1)n & n-1 & 2+(k-1)n & \cdots
\end{array}
$$

1. Partition T into a union of $n-$stars such that each $(ni+1)^{st}$ vertex (except the first one) of the single path is the last leaf of an $n-$star and the root of next $n-$star.

2. Assign 0 to the first vertex of the single path.

3. In $(2j+1)^{st}$ n-star for $j \geq 0$, choose a unlabelled leg from the left to the right and label vertices of the leg with the numbers $x_{i1}, x_{i2}, ..., x_{in}$ in X_i where i is the unused minimum in $[k]$.

4. In $(2j)^{th}$ n-star for $j \geq 1$, choose a unlabelled leg from the left to the right and label vertices of the leg with the numbers $x_{in}, x_{i(n-1)}, ..., x_{i1}$ in X_i where i is the unused maximum in $[k]$.

Figure 6.8 is an illustration of algorithm A_3.

Now we prove our main result as follows:

Proposition 6.3.11 *Let $T \in \mathbb{T}_n$. If the single path of T has no legs except in the $ni + 1^{st}$ vertex, then algorithm A_3 yields a graceful labelling of T and the first vertex label of the single path is 0.*

Proof. There are two cases to be discussed.

Case 1: $n = 2m$. Assume that T has $2mk + 1$ vertices and each vertex except the $2mi + 1^{st}$ vertex of the single path of T has no legs. Algorithm A_3 clearly shows that all vertex labels of T are different and the set of vertex labels is $\{0, 1, 2, ..., 2mk\}$. It suffices to prove that the set of edge labels on T is $[2km]$.

We use induction on k. For $k = 1$, no matter T has no leg or only one leg , it is a P_{2m+1} which has a graceful labelling $0, 2m, 1, ..., m+1, m$ on its vertices in order.

$$\mathbf{X}_1 \quad 40 \quad 1 \quad 39 \quad 2 \quad 38 \quad 3 \quad 37 \quad 4$$
$$\mathbf{X}_2 \quad 32 \quad 9 \quad 31 \quad 10 \quad 30 \quad 11 \quad 29 \quad 12$$
$$\mathbf{X}_3 \quad 24 \quad 17 \quad 23 \quad 18 \quad 22 \quad 19 \quad 21 \quad 20$$
$$\mathbf{X}_4 \quad 16 \quad 25 \quad 15 \quad 26 \quad 14 \quad 27 \quad 13 \quad 28$$
$$\mathbf{X}_5 \quad 8 \quad 33 \quad 7 \quad 34 \quad 6 \quad 35 \quad 5 \quad 36$$
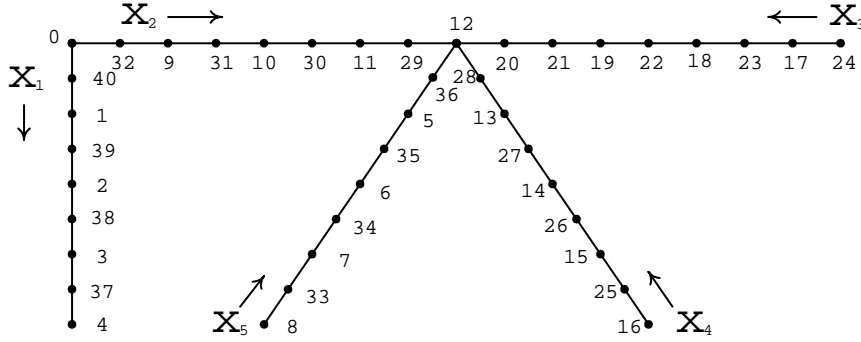

Figure 6.8: An illustration for algorithm A_3 .

Suppose that for any $T \in \mathbb{T}_{2m}$, if T has $2mk+1$ vertices and each vertex except the $2mi + 1^{st}$ vertex of the single path has no legs, then T has a graceful labelling by algorithm A_3.

Consider a $T \in \mathbb{T}_{2m}$ with $2m(k+1)+1$ vertices, where each vertex except the $2mi + 1^{st}$ vertex of the single path has no legs.

Subcase 1.1: The first vertex of the single path in T has no legs. Use algorithm A_3 to label vertices of T and partition T into two parts P_{2m+1} and T_1, where P_{2m+1} is a labelled path with vertex labels $\{0, 2(k+1)m, 1, 2(k+1)m - 1, ..., 2(k+1)m - m + 1, m\}$ and T_1 is a labelled $2m-$caterpillar with $2mk + 1$ vertices. If we replace each vertex label s with $s - m$ in T_1 and rename it as T_2, then the set of vertex labels in T_2 corresponds to $\{Z_i\}_{i=1}^{k}$ the $2m-$star partition with parameter k. By Lemma 6.3.3-2, the vertices of T_2 are labelled by algorithm A_3. Since T_2 has $2km+1$ vertices and is labelled by algorithm A_3, by inductive hypothesis, T_2 has a graceful labelling with the set of edge labels $[2mk]$, i.e., the set of edge labels in T_1 is $[2mk]$. Since the set of edge labels in P_{2m+1} is $\{2m(k+1), 2m(k+1) - 1, ..., 2mk + 1\}$, we conclude that the set of edge labels in T is $[2m(k+1)]$. Note that the first vertex

label of the single path is 0. Figure 6.9-(a) is an example for $m = 4$ and $k = 4$.

Subcase 1.2: The first vertex of the single path in T has at least one leg. Use algorithm A_3 to label vertices of T and partition T into two parts P_{2m+1} and T_1, where P_{2m+1} is a labelled path with vertex labels $\{0, 2(k + 1)m, 1, 2(k + 1)m - 1, ..., 2(k + 1)m - m + 1, m\}$ and T_1 is a labelled $2m-$caterpillar with $2km + 1$ vertices. Figure 6.9-(b) is an example for $m = 2$ and $k = 2$.

It suffices to prove that the set of edge labels in T_1 is $[2mk]$. We first replace the vertex label s of the vertex u in T_1 with $s - 2m$, where u is an even vertex of legs in $2i + 1^{th}$ $2m$-star or an odd vertex of legs in $2i + 2^{nd}$ $2m$-star. Then we obtain a new $2m-$caterpillar T_2 in $\mathbb{T}_{2m}$ whose set of vertex labels corresponds to $\{Y_i\}_{i=1}^k$. By Lemma 6.3.3-1, T_2 is labelled by algorithm A_3. Since T_2 has $2km + 1$ vertices and is labelled by algorithm A_3, by inductive hypothesis, T_2 has a graceful labelling, i.e., the set of edge labels in T_2 is $[2km]$. The remaining is to prove that the set of edge labels in T_1 is equal to that in T_2.

In T_2, let

$$S_i'' = \left\{s : s \text{ is the } i^{th} \text{ edge label of some leg in some } 2m - \text{star}\right\}$$

for $i = 1, 2, ..., 2m$, then the set of edge labels in T_2 is $[2km] = \bigcup_{i=1}^{2m} S_i''$. In fact,

$$\bigcup_{i=1}^{m} S_{2i}'' = \{|x - y| : x, y > 0, x + y = 2km + 1\}, \text{ and}$$

$$\bigcup_{i=1}^{m} S_{2i-1}'' = S_1'' \cup \{|x - y| : x, y > 0, \ x + y = 2km, \ x, y \equiv i \ (mod \ 2m), \ i \neq 0, m\}.$$

In T_1, let

$$S_i' = \left\{s : s \text{ is the } i^{th} \text{ edge label of some leg in some } 2m - \text{star}\right\}$$

for $i = 1, 2, ..., 2m$, then the set of edge labels in T_1 is $\bigcup_{i=1}^{2m} S_i'$. In fact,

$$\bigcup_{i=1}^{m} S_{2i}' = \{|x - y| : x, y > m, x + y = 2(k + 1)m + 1\}, \text{ and}$$

$$\bigcup_{i=1}^{m} S_{2i-1}' = S_1' \cup \{|x - y| : x, y > m, x + y = 2(k+1)m, x, y \equiv i (mod \ 2m), i \neq 0, m\}.$$

Note that $S_1' = S_1''$.

By Lemma 6.3.7,

$$\bigcup_{i=1}^{m} S_{2i}' = F_1' = F_1'' = \bigcup_{i=1}^{m} S_{2i}'', \text{ and}$$

$$\bigcup_{i=2}^{m} S_{2i-1}' = F_2' = F_2'' = \bigcup_{i=2}^{m} S_{2i-1}''.$$

Since $S_1' = S_1''$, we obtain $\bigcup_{i=1}^{2m} S_i' = \bigcup_{i=1}^{2m} S_i'' = [2km]$.

Case 2: $n = 2m - 1$. Assume that T has $(2m - 1)k + 1$ vertices and each vertex except the $(2m - 1)i + 1^{st}$ vertex of the single path of T has no legs. Algorithm A_3 clearly shows that all vertex labels of T are different and the set of vertex labels is $\{0, 1, 2, ..., (2m - 1)k\}$. It suffices to prove that the set of edge labels on T is $[(2m - 1)k]$.
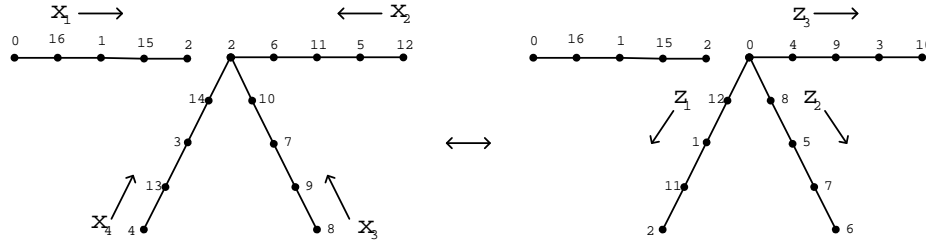
We use induction on k. For $k = 1$, no matter T has no leg or only one leg , it is a P_{2m} which has a graceful labelling $0, 2m - 1, 1, ..., m - 1, m$ on its vertices in order.

Suppose that for any $T \in \mathbb{T}_{2m-1}$, if T has $(2m - 1)k + 1$ vertices and each vertex except the $(2m - 1)i + 1^{st}$ vertex of the single path has no legs, then T has a graceful labelling by algorithm A_3.
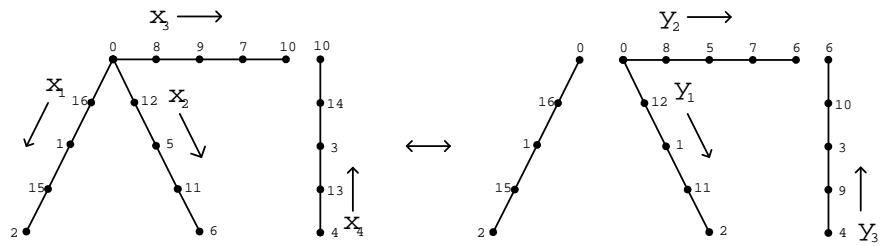
Consider a $T \in \mathbb{T}_{2m-1}$ with $(2m - 1)(k + 1) + 1$ vertices, where each vertex except the $(2m - 1)i + 1^{st}$ vertex of the single path has no legs.
Subcase 2.1: The first vertex of the single path in T has no legs. The proof is analogous to that in Subcase 1.1. In this subcase we use Lemma 6.3.3-3 instead of Lemma 6.3.3-2. Figure 6.9-(c) is an example for $m = 2$ and $k = 3$.
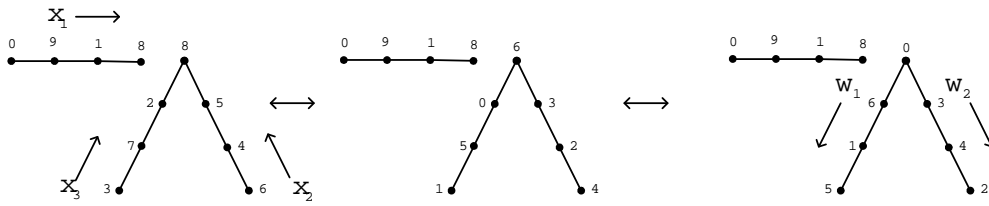**Subcase 2.2**: The first vertex of the single path in T has at least one leg. The proof is is analogous to that in Subcase 1.2. In this subcase we use Lemma 6.3.8 instead of Lemma 6.3.7. Figure 6.9-(d) is an example for $m = 2$ and $k = 3$. Hence we complete the proof. $\qquad\qquad\Box$
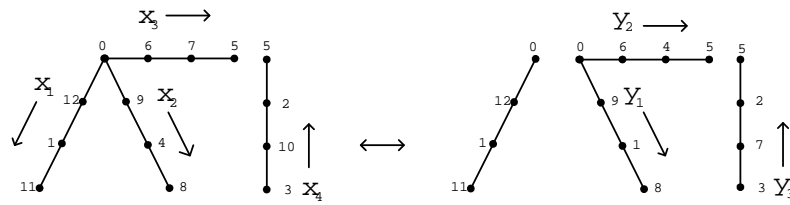
Figure 6.9: An illustration for the proof of Proposition 6.3.11 .

6.4 Latin Squares and Graceful Labellings of 2^n-Caterpillars

In this section we wish to study graceful labellings of 2^n-caterpillars using Latin squares. First, we offer some results derived from Latin squares.

Recall that a Latin square is an $n \times n$ array such that each element in $[n]$ appears exactly one time in each row and each column.

Definition 6.4.1 *A graceful Latin square of order n is a Latin square such that the set of absolute value between the number and its neighbor in each column is $[n-1]$. A Latin square of order n on $\{y_1, y_2, ..., y_n\}$ is a square such that each element in $\{y_1, y_2, ..., y_n\}$ appears exactly one time in each column and each row. A graceful Latin square of order n on $\{y_1, y_2, ..., y_n\}$ is a Latin square of order n on $\{y_1, y_2, ..., y_n\}$ such that the set of absolute value between the number and its neighbor in each column is $\{v, v-1, ..., v-n+2\}$, where $v = \max\{|y_i - y_j| : i \neq j\}$.*

Example 6.4.2 *Let*

$$A = \begin{array}{|c|c|c|c|} \hline 4 & 1 & 3 & 2 \\ \hline 1 & 4 & 2 & 3 \\ \hline 3 & 2 & 4 & 1 \\ \hline 2 & 3 & 1 & 4 \\ \hline \end{array} \quad and$$

$$B = \begin{array}{|c|c|c|c|} \hline 12 & 3 & 11 & 4 \\ \hline 3 & 12 & 4 & 11 \\ \hline 11 & 4 & 12 & 3 \\ \hline 4 & 11 & 3 & 12 \\ \hline \end{array}.$$

Then A is a graceful Latin square of order 4 and B is a graceful square of order 4 on $\{3, 4, 11, 12\}$.

We mention an interesting result which will be used to prove Theorem 6.4.4. The proof is obvious.

Lemma 6.4.3 *1. Let $\{x_i\}_{i=1}^{2^n}$ and $\{y_i\}_{i=1}^{2^n}$ be two sequences where*

$$x_i = \begin{cases} 2^n - (i-1)/2, & \text{if } i \text{ is odd,} \\ i/2, & \text{otherwise,} \end{cases}$$

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and

$$y_i = x_{2^n+1-i} \quad for \ \ i = 1, 2, ..., 2^n.$$

Then $|x_i - y_i| = 2^{n-1}$.

2. *Let* $\{u_i\}_{i=1}^{2^n}$ *and* $\{v_i\}_{i=1}^{2^n}$ *be two sequences where*

$$u_i = \begin{cases} x_i + 2^n, & \text{if } i \text{ is odd,} \\ x_i, & \text{otherwise,} \end{cases}$$

and

$$v_i = y_i + 2^{n-1} \quad for \ \ i = 1, 2, ..., 2^n.$$

Then $|u_i - v_i| = 2^n$ *for* $i = 1, 2, ..., 2^n$.

Theorem 6.4.4 *There exists a symmetric graceful Latin square of order 2^n for $n \in \mathbb{N}$.*

Proof. We use induction to construct this symmetric graceful Latin square of order 2^n for $n \in \mathbb{N}$. Let $A_2 = \left(a_{ij}^{(2)} \right)$ be a Latin square of order 2 with $a_{11}^{(2)} = a_{22}^{(2)} = 2$ and $a_{12}^{(2)} = a_{21}^{(2)} = 1$. It has the following three properties:

1. The sequence $\{2, 1\}$ simultaneously appears in the first row and the first column.

2. $a_{ij}^{(2)} > a_{lm}^{(2)}$, if $i + j$ is even and $l + m$ is odd.

3. It is a symmetric graceful Latin square.

Assume that $A_{2^n} = \left(a_{ij}^{(2^n)} \right)$ is a Latin square of order 2^n satisfying the following three properties:

P1. The sequence $\{x_i\}_{i=1}^{2^n}$ defined in Lemma 6.4.3 simultaneously appears in the first row and the first column.

P2. $a_{ij}^{(2^n)} > a_{lm}^{(2^n)}$, if $i + j$ is even and $l + m$ is odd.

P3. A_{2^n} is a symmetric graceful Latin square.

Now we construct a graceful Latin square of order 2^{n+1} as follows:

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Let $B_{2^n} = \left(b_{ij}^{(2^n)} \right)$, where $b_{ij}^{(2^n)} = \begin{cases} a_{ij}^{(2^n)}, & \text{if } i+j \text{ is odd,} \\ a_{ij}^{(2^n)} + 2^n, & \text{otherwise,} \end{cases}$ for $i, j = 1, 2, ..., 2^n$.

Let $C_{2^n} = \left(c_{ij}^{(2^n)} \right)$, where $c_{ij}^{(2^n)} = a_{ij}^{(2^n)} + 2^{n-1}$ for $i, j = 1, 2, ..., 2^n$.

Put $A_{2^{n+1}} = \left(a_{ij}^{(2^{n+1})} \right) = \begin{pmatrix} B_{2^n} & C_{2^n} \\ C_{2^n} & B_{2^n} \end{pmatrix}$.

Under such construction, we obtain the following results:

1. The squares B_{2^n} and C_{2^n} are symmetric.

2. $b_{ij}^{(2^n)} > b_{lm}^{(2^n)}$ and $c_{ij}^{(2^n)} > c_{lm}^{(2^n)}$, if $i+j$ is even and $l+m$ is odd.

3. The sequence $\{2^{n+1}, 1, 2^{n+1} - 1, 2, ..., 2^{n-1} + 1 + 2^n, 2^{n-1}\}$ simultaneously appears in the first row and the first column of B_{2^n}. The sequence $\{x_i + 2^{n-1}\}_{i=1}^{2^n}$ simultaneously appears in the first row and the first column of C_{2^n}.

4. The set of numbers in each row and each column of B_{2^n} is

$$\{1, 2, ..., 2^{n-1}, (2^{n-1} + 1) + 2^n, (2^{n-1} + 2) + 2^n, ..., 2^n + 2^n\},$$

and the set of numbers in each row and each column of C_{2^n} is

$$\{1 + 2^{n-1}, 2 + 2^{n-1}, ..., 2^n + 2^{n-1}\}.$$

5. The set of absolute value between the number and its neighbor in each row and each column of the square B_{2^n} is

$$\{1 + 2^n, 2 + 2^n, ..., (2^n - 1) + 2^n\},$$

and the set of absolute value between the number and its neighbor in each row and each column of the square C_{2^n} is

$$\{1, 2, ..., 2^n - 1\}.$$

6. $\left| b_{i2^n}^{(2^n)} - c_{i1}^{(2^n)} \right| = \left| b_{2^n i}^{(2^n)} - c_{1i}^{(2^n)} \right| = \left| u_{2^n - i + 1} - v_{2^n - i + 1} \right| = 2^n$ for $i = 1, 2, ..., 2^n$ by Lemma 6.4.3-2.

   Similarly, $\left| b_{i1}^{(2^n)} - c_{i2^n}^{(2^n)} \right| = \left| b_{1i}^{(2^n)} - c_{2^n i}^{(2^n)} \right| = \left| u_i - v_i \right| = 2^n$ for $i = 1, 2, ..., 2^n$ by Lemma 6.4.3-2 .

It is a routine work to check that $A_{2^{n+1}}$ satisfies three properties similar to **P1**, **P2**, and **P3**. Firstly, result 3 implies that the sequence $\{x_i\}_{i=1}^{2^{n+1}}$ simultaneously appears in the first row and the first column of square $A_{2^{n+1}}$ which satisfies property **P1**. Secondly, result 2 implies that $a_{ij}^{(2^{n+1})} > a_{lm}^{(2^{n+1})}$, if $i+j$ is even and $l+m$ is odd which satisfies property **P2**. Thirdly, we claim that $A_{2^{n+1}}$ satisfies property **P3**. Result 1 implies that $A_{2^{n+1}}$ is symmetric. Result 4 implies that the set of numbers in each row and each column of the square $A_{2^{n+1}}$ is $[2^{n+1}]$, i.e., $A_{2^{n+1}}$ is a Latin square. Results 5 and 6 imply that the set of absolute value between the number and its neighbor in each row and each column of the square $A_{2^{n+1}}$ is $[2^{n+1}-1]$. These results verify that $A_{2^{n+1}}$ is a symmetric graceful Latin square. Hence we complete the proof. $\qquad\square$

Example 6.4.5 *The following is a construction of symmetric graceful Latin square of order 8 from a symmetric graceful Latin square of order 4. Let*

$$A_4 = \begin{array}{|c|c|c|c|} \hline 4 & 1 & 3 & 2 \\ \hline 1 & 4 & 2 & 3 \\ \hline 3 & 2 & 4 & 1 \\ \hline 2 & 3 & 1 & 4 \\ \hline \end{array}.$$

As the construction in the proof of Theorem 6.4.4, we obtain

$$B_4 = \begin{array}{|c|c|c|c|} \hline 8 & 1 & 7 & 2 \\ \hline 1 & 8 & 2 & 7 \\ \hline 7 & 2 & 8 & 1 \\ \hline 2 & 7 & 1 & 8 \\ \hline \end{array} \text{ and } C_4 = \begin{array}{|c|c|c|c|} \hline 6 & 3 & 5 & 4 \\ \hline 3 & 6 & 4 & 5 \\ \hline 5 & 4 & 6 & 3 \\ \hline 4 & 5 & 3 & 6 \\ \hline \end{array}.$$

Hence

$$A_8 = \begin{pmatrix} B_4 & C_4 \\ C_4 & B_4 \end{pmatrix} = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 8 & 1 & 7 & 2 & 6 & 3 & 5 & 4 \\ \hline 1 & 8 & 2 & 7 & 3 & 6 & 4 & 5 \\ \hline 7 & 2 & 8 & 1 & 5 & 4 & 6 & 3 \\ \hline 2 & 7 & 1 & 8 & 4 & 5 & 3 & 6 \\ \hline 6 & 3 & 5 & 4 & 8 & 1 & 7 & 2 \\ \hline 3 & 6 & 4 & 5 & 1 & 8 & 2 & 7 \\ \hline 5 & 4 & 6 & 3 & 7 & 2 & 8 & 1 \\ \hline 4 & 5 & 3 & 6 & 2 & 7 & 1 & 8 \\ \hline \end{array}.$$

Clearly, A_8 is a symmetric graceful Latin square.

Using Theorem 6.4.4, we obtain the following important result which will be used to prove Theorem 6.4.9.

Corollary 6.4.6 *Let $\{X_i\}_{i=1}^k$ be the 2^n-star partition with parameter k. Then there exists a symmetric graceful Latin square of order 2^n on X_i for $i = 1, 2, ..., k$.*

Proof. Let $X_i = \{x_i\}_{i=1}^{2^n}$ for $i = 1, 2, ..., k$. Then $\{x_{i1}, x_{i3}, ..., x_{i(2^n-1)}\}$ is a arithmetic sequence with the common difference -1 and $\{x_{i2}, x_{i4}, ..., x_{i(2^n)}\}$ is an arithmetic sequence with the common difference 1. We first claim that either $x_{i1} < x_{i2}$ or $x_{i(2^n-1)} > x_{i(2^n)}$, i.e., either $x_{ij} < x_{it}$ for odd j and even t, or $x_{ij} > x_{it}$ for odd j and even t. As before, $x_{i1} = (k - i + 1)2^n$ and $x_{i2} = (i - 1)2^n + 1$.

Case 1: $x_{i1} < x_{i2}$. It is nothing to show.

Case 2: $x_{i1} > x_{i2}$. We obtain $(k - 2i + 2)2^n > 1$. This implies $k - 2i + 2 \geq 1$, i.e., $k - 2i + 1 \geq 0$, Then

$$
\begin{aligned}
x_{i(2^n-1)} - x_{i(2^n)} &= \left(x_{i1} - 2^{n-1} + 1\right) - \left(x_{i2} + 2^{n-1} - 1\right) \\
&= \left((k - i + 1)2^n - 2^{n-1} + 1\right) - \left((i - 1)2^n + 1 + 2^{n-1} - 1\right) \\
&= (k - 2i + 1)2^n + 1 \geq 1 > 0.
\end{aligned}
$$

Therefore $x_{i(2^n-1)} > x_{i(2^n)}$.

By Theorem 6.4.4, there is a symmetric graceful Latin square of order 2^n with the property: $a_{ij}^{(2^n)} > a_{lm}^{(2^n)}$, if $i+j$ is even and $l+m$ is odd. In the case $x_{i1} < x_{i2}$, if we replace $\{1, 2, ..., 2^{n-1}\}$ with $\{x_{i1}, x_{i3}, ...x_{i(2^n-1)}\}$ and replace $\{2^{n-1}+1, 2^{n-1}+2, ..., 2^n\}$ with $\{x_{i2}, x_{i4}, ...x_{i(2^n)}\}$, then we obtain a symmetric graceful Latin square of order 2^n on X_i for $i = 1, 2, ..., k$. Similarly, when $x_{i1} > x_{i2}$, there exists a symmetric graceful Latin square of order 2^n on X_i for $i = 1, 2, ..., k$. $\qquad \square$

By suitable permutations of rows and columns, we shall obtain the following result.

Corollary 6.4.7 *Let $z \in X_i$ for $i = 1, 2, ..., k$. Then there exists a symmetric graceful Latin square of order 2^n on X_i such that z appears in the first row and the first column.*

Example 6.4.8 *Let A_8 be the symmetric Latin square of order 8 obtained from Example 6.4.5. Let $\{X_i\}_{i=1}^5$ be the 8-star partition with parameter 5. If we correspond 8,1,7,2,6,3,5, 4 to 36,5,35,6,34,7,33, 8, respectively, then there exists a symmetric graceful Latin square of order 8 on X_5 as follows:*

36	5	35	6	34	7	33	8
5	36	6	35	6	34	8	33
35	6	36	5	33	8	34	7
6	35	5	36	8	33	7	34
34	7	33	8	36	5	35	6
7	34	8	33	5	36	6	35
33	8	34	7	35	6	36	5
8	33	7	34	6	35	5	36

Now we are in position to prove that 2^n-caterpillars have graceful labellings (Theorem 6.4.9 below).

For this purpose, we first provide an algorithm to yield graceful labellings of 2^n−caterpillars. The main technique is to deal with 2^n-caterpillars which have a single path with length divisible by 2^n.

Algorithm A_4: Labellings of 2^n−caterpillars.
Let T be a 2^n-caterpillar.

1. Assume that $T \in \mathbb{T}_{2^n}$ has $k2^n + 1$ vertices.

   (a) Remove all legs of the $(i2^n + j)^{th}$ vertices of the single path to be incident to the $((i + 1)2^n + 1)^{st}$ vertex of the single path for $i \geq 0, 2 \leq j \leq 2^n$.

   (b) Use algorithm A_3 to label the new 2^n−caterpillar.

2. Assume that $T \notin \mathbb{T}_{2^n}$ has $k2^n + j$ vertices for $j \in \{2, 3, ..., 2^n\}$.

   (a) Remove all legs incident to the first $j - 1$ vertices of the single path to the jth vertex.

   (b) Let the first $j - 1$ vertices be $u_1, u_2, ..., u_{j-1}$. Label $u_{j-1}, u_{j-2}, ..., u_1$ with the numbers $a_1, a_2, ..., a_{j-1}$,

$$\text{where } a_i = \begin{cases} k2^n + (i+1)/2, & \text{if } i \text{ is odd} \\ -i/2, & \text{otherwise.} \end{cases}$$

(c) Apply step 1 to label the remaining 2^n−caterpillar.

(d) Replace each vertex label s with

$$s - min\{x | x \text{ is a label of some } u_i \text{ and} x < 0\}.$$

3. Restore each removed leg and relabel its vertices according to the following rules: Let $x_1^{(i)}, x_2^{(i)}, ..., x_{2^n}^{(i)}$ be the vertex labels of the leg before restoring for some $i \in [k]$.

(a) Using the construction of Theorem 6.4.4 and Corollary 6.4.6, we shall obtain a graceful Latin square L on $\{x_{i1}, x_{i2}, ..., x_{i2^n}\}$ such that $x_1^{(i)}, x_2^{(i)}, ..., x_{2^n}^{(i)}$ simultaneously appears in the first column and the first row.

(b) If a leg is removed m places for $m = 1, 2, ..., 2^n - 1$, then we orderly relabel vertices of this leg with the numbers in m^{th} column of L after restoring.

Figure 6.10 is an example of labelling an 8-caterpillar in $\mathbb{T}_8$. Figure 6.11 is an example of labelling an 8-caterpillar not in $\mathbb{T}_8$.

Theorem 6.4.9 *Every 2^n-caterpillar has a graceful labelling.*

Proof. Let T be a 2^n−caterpillar with $k2^n + 1$ vertices. There are two cases to be considered.

Case 1: Assume that T is in $\mathbb{T}_{2^n}$. Step $1 - (a)$ of algorithm A_4 makes T be a new T' such that each vertex of the single path has no legs except in the $2^n i + 1$ vertex for $i \geq 0$. By Lemma 6.3.11, T' has a graceful labelling and the first vertex label of the single path is 0. The remaining is to prove that after restoring each removed leg, T still has a graceful labelling. There are two subcases to be considered:

Subcase 1.1: The vertex labels of the leg before being removed are $x_{i1}, x_{i2}, ..., x_{i(2^n)}$ and the vertex labels of the last leg in the previous 2^n−star are $x_{j(2^n)}, x_{j(2^n - 1)}, ..., x_{j1}$ for some $i, j \in [k]$ and $j > i$. By our construction of algorithm A_4, we have

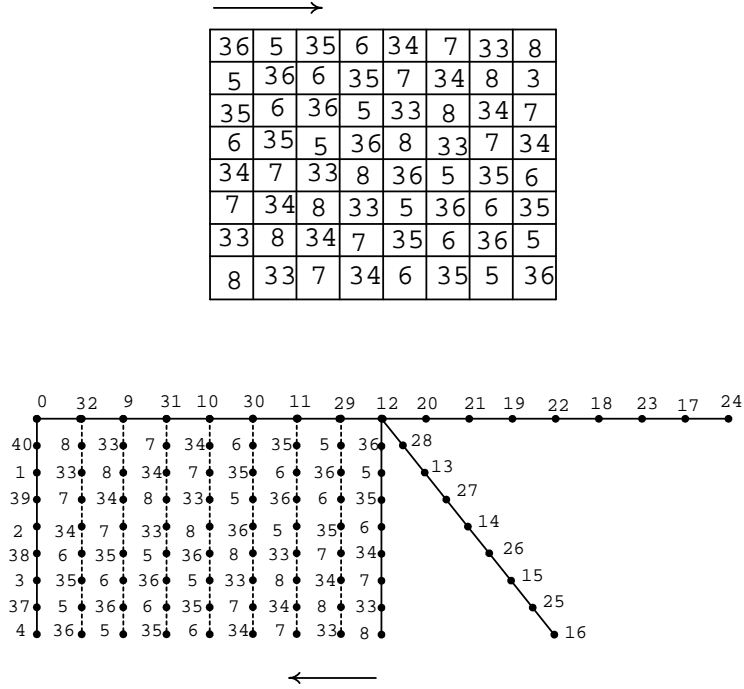$$|x_{i1} - x_{j1}| = |x_{im} - x_{jm}| \text{ for } m \in [2^n].$$

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36	5	35	6	34	7	33	8
5	36	6	35	7	34	8	3
35	6	36	5	33	8	34	7
6	35	5	36	8	33	7	34
34	7	33	8	36	5	35	6
7	34	8	33	5	36	6	35
33	8	34	7	35	6	36	5
8	33	7	34	6	35	5	36

Figure 6.10: A graceful labelling of an $8-$caterpillar in $\mathbb{T}_8$.

Moreover, in step 3 of algorithm A_4, the set $\{e_2, e_3, ...e_{2^n}\}$ of edge labels of a given leg P_{2^n} before restoring is equal to the set $\{e_{i_2}, e_{i_3}, ..., e_{i_{2^n}}\}$ of edge labels of the leg after restoring, since it is constructed by Theorem 6.4.4 and Corollary 6.4.6. Note that $|x_{i1} - x_{j1}|$ is the edge label between the vertex in the single path and the first vertex of the leg before restoring; $|x_{im} - x_{jm}|$ is the edge label between the vertex in the single path and the first vertex of the leg after restoring m places. By the above facts, we obtain that the set of edge labels never changes after restoring each removed leg. Figure 6.12 is an illustration for this case.

Subcase 1.2: The vertex labels of the leg before being removed are $x_{i(2^n)}, x_{i(2^n-1)}, ...,$ x_{i1} and the vertex labels of the last leg in the previous 2^n-star are $x_{j1}, x_{j2}, ..., x_{j(2^n)}$ for some $i, j \in [k]$ and $i > j$. By our construction of algorithm A_4, we have $|x_{i(2^n)} - x_{j(2^n)}| = |x_{im} - x_{jm}|$ for $m \in [2^n]$. Moreover, in step 3 of algorithm A_4, the set $\{e_2, e_3, ..., e_{2^n}\}$ of edge labels of a given leg P_{2^n} before restoring is equal to the set $\{e_{i_2}, e_{i_3}, ..., e_{i_{2^n}}\}$ of edge labels of the leg after restoring, since it is constructed by Theorem 6.4.4 and Corollary 6.4.6. Note that $|x_{i(2^n)} - x_{j(2^n)}|$ is the edge label between the vertex in the the single path and the first vertex of the leg before restoring; $|x_{im} - x_{jm}|$ is the edge label between the vertex in the the single path and
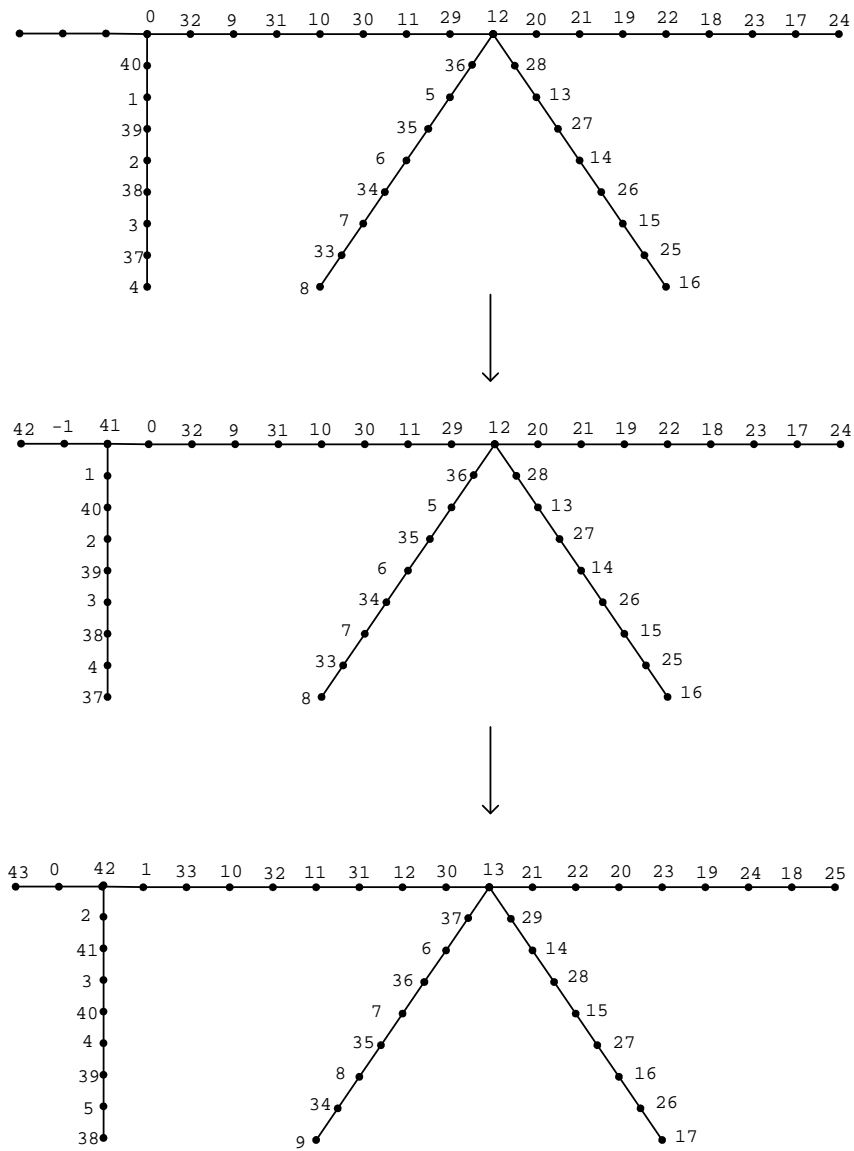
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Figure 6.11: A graceful labelling of an 8−caterpillar not in $\mathbb{T}_8$.

the first vertex of the leg after restoring m places. By the above facts, we obtain that the set of edge labels never changes after restoring.

Case 2: Assume that $T \notin \mathbb{T}_{2^n}$. The step 2 of algorithm A_4 makes $T - \{u_1, u_2, ..., u_{j-1}\}$ to be a new $T' \in \mathbb{T}_{2^n}$. By case 1, T' has a graceful labelling and the first vertex label of the single path is 0. The step $2 - (b)$ of algorithm A_4 makes the set of the other edge labels to be $\{k2^n + 1, k2^n + 2, ..., k2^n + j - 1\}$; hence the set of edge labels of T is $[k2^n + j - 1]$. The step $2 - (d)$ of algorithm A_4 ensures that the set of vertex labels is $\{0, 1, 2, ..., k2^n + j - 1\}$ and never changes the set of edge labels. That is to say that the step 2 of algorithm A_4 yields a graceful labelling of T. Hence we complete the proof. $\qquad\qquad\square$
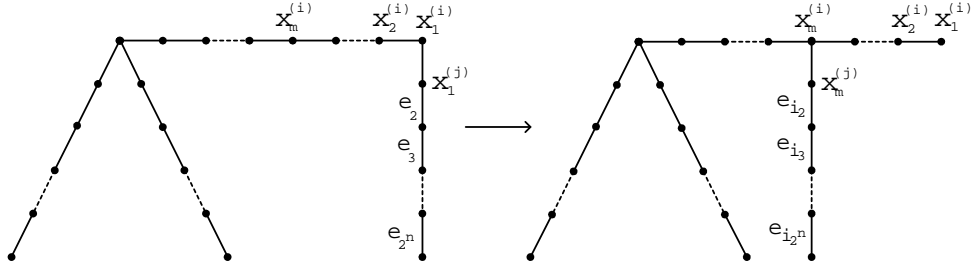

Figure 6.12: An illustration for the proof of Theorem 6.4.9.

Remark. Algorithm A_4 seems to suggest that decomposing any $n-$caterpillar to a union of $n-$stars may yield a graceful labelling. Unfortunately, similar method to algorithm A_4 doesn't work for some $3-$caterpillars (see Figure 6.13). Hence, the question we raise here is to modify algorithm A_4 so that it works for any $n-$caterpillar.
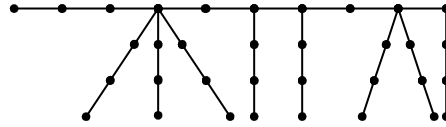

Figure 6.13: An example of remark.