Concluding Remarks

To conclude our thesis, we summarize the open questions left and add some new ones: In Part I, Chapter 1, p. 15, we raised the following open questions:

1. Marrero's Proposition mentioned that every Hadamard matrix belongs to a J_2-class CJ_2. For a given Hadamard matrix, does it belong to a J_m-class CJ_m for some $m \geq 4$?

2. Examples 1.2.1 and 1.2.2 seemed to provide counterexamples to Question 1, if the following is true: A J_8-Hadamard matrix belongs to CJ_4? This question has been answered positively by Theorem 2.2.3.

In Chapter 2, p. 25, Corollary 2.1.3 showed that a J_m-Hadamard matrix H is a J_l-Hadamard matrix for some $l \mid m$, where l depends on m and H. The question whether l depends only on m is extremely difficult. However, since $CJ_8 \subsetneqq CJ_4 \subsetneqq CJ_2$, it seems likely that $CJ_{2^n} \subseteq CJ_{2^m}$ for some $1 \neq m < n$. In Chapter 3, using an algorithm, we devoted ourself to concretely finding the upper bound of the minimal exponent E_t for given t Hadamard matrices. We wonder whether there are more useful methods to modify the upper bound of the minimal exponent.

In Part II, Chapter 4, we enumerated plane forests on various parameters and applied them to Dyck paths with flaws. We believe that there are more plane forests on other parameters worthy of investigating. We shall hope to apply the results of plane forests to other families, e.g., noncrossing partitions. In Chapter 5, we found a new Catalan identity, provided three families related to Chung-Feller Theorem, and studied Chung-Feller Theorem for Motzkin number and Riordan number. Hence,

we shall attempt to discover new Catalan identities and seek new families related to Chung-Feller Theorem for Catalan-like numbers. In Chapter 6, we presented graceful labellings of some $n-$caterpillars. In the future, we wish to study new families of graceful trees.