Chapter 3

On Craigen-de Launey's Constructions of Hadamard Matrices

In 1867, Sylvester [77] noted that the Kronecker product of two Hadamard matrices is again an Hadamard matrix. In 1893, Hadamard [37] himself showed that if H and K are Hadamard matrices of orders $4h$ and $4k$, then $H \otimes K$ is an Hadamard matrix of order 2^4hk. In 1992, for the above given Hadamard matrices H, K, R. Craigen [20] gave a simpler proof of a result due to Agayan-Sarukhanyan which asserts the existence of an Hadamard matrix of order 2^3hk (see our Theorem 3.0.2). By use of the above-mentioned Agayan-Sarukhanyan's result, for any four Hadamard matrices of orders $4h, 4k, 4m, 4n$, there is an Hadamard matrix of order 2^5hkmn. In the same year 1992, Craigen, Seberry and Zhang [21] used orthogonal pairs and weighing matrices to strengthen the result to obtain an Hadamard matrix of order 2^4hkmn (see our Theorem 3.0.4). Repeating Craigen-Seberry-Zhang's method, there exists an Hadamard matrix of order $2^{10}m_1m_2\cdots m_{12}$ for any 12 Hadamard matrices of orders $4m_1, 4m_2, ..., 4m_{12}$. In 1993, de Launey [22] further improved Craigen-Seberry-Zhang's method to yield the following result:

Theorem 3.0.1 (de Launey) *If there are twelve Hadamard matrices of orders $4m_1, 4m_2, ..., 4m_{12}$, then there exists an Hadamard matrix of order $2^9m_1m_2...m_{12}$.*

A natural problem arises: What happens to any t Hadamard matrices of orders $4m_1, 4m_2, ..., 4m_t$? In Section 3.1, we follow de Launey's idea to yield two results, Theorem 3.1.1 and Theorem 3.1.2 which allow us to construct, for any given t Hadamard matrices of orders $4m_1, 4m_2, ..., 4m_t$, an Hadamard matrix of order $2^k m_1 m_2 \cdots m_t$ with $k \leq t$ as small as possible. In Section 3.2, we introduce the *minimum exponent* E_t, for any t, such that there exists, for any given t Hadamard matrices of orders $4m_1, 4m_2, ..., 4m_t$, an Hadamard matrix of order $2^{E_t} m_1 m_2 \cdots m_t$ (for precise definition, see Section 3.2). Moreover, we explore some particular properties of the minimum exponent E_t which turns out to be a monotonic increasing step function with step jump 1 (Lemma 3.2.1). To obtain a calculable upper bound for E_t, we bring in another number ε_t which will be defined recursively using an algorithm derived from our Theorem 3.1.1 and Theorem 3.1.2. Finally, for illustrating the results, we give a list of ε_t for $1 \leq t \leq 20$.

For the sake of proving Theorem 3.1.1 and Theorem 3.1.2 and for fixing our notation, we recall the definitions of orthogonal pairs and disjoint weighing matrices and some well-known relevant results.

A pair (S, P), where $S, P \in \mathbb{M}_{4h \times 4h}(\{\pm 1\})$, is an *orthogonal pair*, notation: (S, P) is an $OP(4h)$, if it satisfies

$$SS^T + PP^T = 8hI_{4h} \text{ and } SP^T = PS^T = O_{4h}.$$

Following Craigen [20], Theorem 3, for any two Hadamard matrices $H = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{pmatrix}$ and $K = \begin{pmatrix} K_1 & K_2 & K_3 & K_4 \end{pmatrix}$, where $H_i \in \mathbb{M}_{h \times 4h}(\{\pm 1\})$ and $K_i \in \mathbb{M}_{4k \times k}(\{\pm 1\})$ for $i = 1, 2, 3, 4$, put

$$S = \frac{1}{2}\{(H_1 + H_2) \otimes K_1 + (H_1 - H_2) \otimes K_2\} \text{ and}$$

$$P = \frac{1}{2}\{(H_3 + H_4) \otimes K_3 + (H_3 - H_4) \otimes K_4\},$$

then (S, P) is an $OP(4hk)$. Combining this with Craigen's Lemma 2,3 of [20], we have:

Theorem 3.0.2 (Craigen) *If there are Hadamard matrices of orders $4h$ and $4k$, then there is an $OP(4hk)$ (S,P). Moreover,* $\begin{pmatrix} S & P \\ P & S \end{pmatrix}$ *is an Hadamard matrix of order $8hk$.*

A matrix $M \in \mathbb{M}_{4m\times 4m}(\{0, \pm 1\})$ is a *weighing matrix* of order $4m$ with weight $2m$ if $MM^T = 2mI_{4m}$. Two weighing matrices of order $4m$, namely $W = (w_{ij})$ and $U = (u_{ij})$, are *disjoint* if $w_{ij}u_{ij} = 0$. For convenience, we say that (W, U) is a pair of $DW(4m)$ if W and U are two *disjoint weighing matrices* of order $4m$ with weight $2m$. For the same H, K, S, and P as above, if we set $W = \frac{1}{2}(S + P)$ and $U = \frac{1}{2}(S-P)$, it is easy to check that (W, U) is a pair of $DW(4hk)$ (see e.g. Craigen [20], Theorem 7). We formulate this important result as follows (see [22], Theorem B):

Lemma 3.0.3 (Seberry and Zhang) *If there are two Hadamard matrices of orders $4m$ and $4n$, there exists a pair of $DW(4mn)$.*

Combining Theorem 3.0.2 and Lemma 3.0.3, for four Hadamard matrices of orders $4h, 4k, 4m, 4n$, then we gain an $OP(4hk)$ (S, P) and a pair of $DW(4mn)$ (X, Y), respectively. [21], Theorem 1 asserts that $\hat{H} = X \otimes S + Y \otimes P$ is an Hadamard matrix of order 2^4hkmn.

Theorem 3.0.4 (Craigen, Seberry and Zhang) *If there are four Hadamard matrices of orders $4h, 4k, 4m, 4n$, then there is an Hadamard matrix of order 2^4hkmn.*

3.1 Generalizations of Craigen's Theorem and of Craigen-Seberry-Zhang's Theorem

To begin with, the following theorem is a generalization of Theorem 3.0.2 in which the construction's idea comes from de Launey [22]. In fact, de Launey's Theorem deals with twelve Hadamard matrices grouped into three groups consisting each of four Hadamard matrices out of twelve Hadamard matrices of orders $4m_i, i = 1, 2, ..., 12$. The first group and the second group produce, by Theorem 3.0.4, two

Hadamard matrices of orders $2^4m_1m_2m_3m_4$ and $2^4m_5m_6m_7m_8$, respectively. The third group produces two different pairs of $DW(4m_9m_{10})$ and $DW(4m_{11}m_{12})$, by Lemma 3.0.3. The following result generalizes de Launey's construction for $l = 4$ to arbitrary l.

Theorem 3.1.1 *If there are two Hadamard matrices H and K of orders 2^lm and 2^ln, respectively, and there are $l-2$ different pairs of $DW(4p_i)$ for $i = 1, 2, ..., l-2$, then there is an $OP(2^{3l-4}mnp_1p_2...p_{l-2})$ and hence an Hadamard matrix of order $2^{3l-3}mnp_1p_2...p_{l-2}$.*

Proof. Following de Launey's proof of Theorem ([6], p.126), set $H = \begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_{2^l} \end{pmatrix}$ and $K = \begin{pmatrix} K_1 & K_2 & \cdots & K_{2^l} \end{pmatrix}$, where $H_i \in \mathbb{M}_{m\times 2^l m}(\{\pm1\})$ and $K_i \in \mathbb{M}_{2^l n\times n}(\{\pm1\})$ for $i = 1, 2, \ldots, 2^l$. Then

$$H_iH_j^T = \begin{cases} 2^lmI_m & \text{, if } i = j, \\ O_m & \text{, otherwise,} \end{cases} \tag{3.1.1}$$

and

$$KK^T = K_1K_1^T + K_2K_2^T + \cdots + K_{2^l}K_{2^l}^T = 2^lnI_{2^ln}. \tag{3.1.2}$$

Let (X_i, Y_i) be $l-2$ different pairs of $DW(4p_i)$ for $i = 1, 2, ..., l-2$, and set $\mathbb{F} = \{Z_1 \otimes Z_2 \otimes \cdots \otimes Z_{l-2} \mid Z_i = X_i \text{ or } Y_i \text{ for } i = 1, 2, ..., l-2\}$. Clearly, the cardinal number of $\mathbb{F}$ is 2^{l-2}, and we have for $F_i \in \mathbb{F}$ for $i = 1, 2, ..., 2^{l-2}$:

$$\begin{aligned} F_iF_i^T &= (Z_1 \otimes Z_2 \otimes \cdots \otimes Z_{l-2})(Z_1 \otimes Z_2 \otimes \cdots \otimes Z_{l-2})^T \\ &= Z_1Z_1^T \otimes Z_2Z_2^T \otimes \cdots \otimes Z_{l-2}Z_{l-2}^T \\ &= 2p_1I_{4p_1} \otimes 2p_2I_{4p_2} \otimes \cdots \otimes 2p_{l-2}I_{4p_{l-2}}, \\ &= 2^{l-2}p_1p_2\cdots p_{l-2}I_{4^{l-2}p_1p_2\cdots p_{l-2}}, \end{aligned}$$

and generalizing de Launey's proof, we define

$$2S = \sum_{i=1}^{2^{l-2}} F_i \otimes \{(H_{2i-1} + H_{2i}) \otimes K_{2i-1} + (H_{2i-1} - H_{2i}) \otimes K_{2i}\}$$

$$2P = \sum_{i=1}^{2^{l-2}} F_i \otimes \{(H_{2^{l-1}+2i-1} + H_{2^{l-1}+2i}) \otimes K_{2^{l-1}+2i-1} + (H_{2^{l-1}+2i-1} - H_{2^{l-1}+2i}) \otimes K_{2^{l-1}+2i}\}.$$

The following algebraic calculation shows that (S, P) is an $OP(2^{3l-4}mnp_1p_2 \cdots p_{l-2})$. In fact, first we calculate SS^T using Equation (3.1.1):

SS^T

$= \frac{1}{4}\sum_{i=1}^{2^{l-2}} F_iF_i^T \otimes \{(H_{2i-1}H_{2i-1}^T + H_{2i}H_{2i}^T) \otimes K_{2i-1}K_{2i-1}^T + (H_{2i-1}H_{2i-1}^T + H_{2i}H_{2i}^T) \otimes K_{2i}K_{2i}^T\} + \frac{1}{4}\sum_{i \neq j} F_iF_j^T \otimes \{(H_{2i-1} + H_{2i})(H_{2j-1} + H_{2j})^T \otimes K_{2i-1}K_{2j-1}^T + \cdots\}$

$= \frac{1}{4}\sum_{i=1}^{2^{l-2}} F_iF_i^T \otimes \{(H_{2i-1}H_{2i-1}^T + H_{2i}H_{2i}^T) \otimes K_{2i-1}K_{2i-1}^T + (H_{2i-1}H_{2i-1}^T + H_{2i}H_{2i}^T) \otimes K_{2i}K_{2i}^T\}$, since the mixed summation with $i \neq j$ in the big parentheses is zero, by Equation (3.1.1),

$= \frac{1}{4}\sum_{i=1}^{2^{l-2}} 2^{l-2}p_1p_2 \cdots p_{l-2}I_{4^{l-2}p_1p_2 \cdots p_{l-2}} \otimes 2 \cdot 2^l mI_m \otimes \{K_{2i-1}K_{2i-1}^T + K_{2i}K_{2i}^T\}.$

Analogously,

PP^T

$= \frac{1}{4}\sum_{i=1}^{2^{l-2}} 2^{l-2}p_1p_2 \cdots p_{l-2}I_{4^{l-2}p_1p_2 \cdots p_{l-2}} \otimes 2 \cdot 2^l mI_m \otimes \{K_{2^{l-1}+2i-1}K_{2^{l-1}+2i-1}^T + K_{2^{l-1}+2i}K_{2^{l-1}+2i}^T\}.$

Using Equation (3.1.2), we get:

$SS^T + PP^T$

$= \frac{1}{4}\sum_{i=1}^{2^l} 2^{l-2}p_1p_2 \cdots p_{l-2}I_{4^{l-2}p_1p_2 \cdots p_{l-2}} \otimes 2 \cdot 2^l mI_m \otimes K_iK_i^T$

$= 2^{3l-3}mnp_1p_2 \cdots p_{l-2}I_{2^{3l-4}mnp_1p_2 \cdots p_{l-2}}.$

Finally, a direct calculation, using Equation (3.1.1), proves that $SP^T = PS^T = O_{2^{3l-4}mnp_1p_2 \cdots p_{l-2}}$. This shows (S, P) is an $OP(2^{3l-4}mnp_1p_2 ... p_{l-2})$ and this orthogonal pair (S, P) produces an Hadamard matrix $\begin{pmatrix} S & P \\ P & S \end{pmatrix}$ of order $2^{3l-3}mnp_1p_2 \cdots p_{l-2}$, by Craigen's Theorem 3.0.2. ❑

By putting $l = 2$ in Theorem 3.1.1, we obtain the above Theorem 3.0.2. To illustrate how Theorem 3.1.1 yields a better result, we choose a special example of 12 Hadamard matrices of orders $2^7m_1, 2^7m_2, 4m_3, 4m_4, ..., 4m_{12}$: de Launey's Theorem 3.0.1 yields the existence of an Hadamard matrix of order $2^{19}m_1m_2 \cdots m_{12}$. However, we can improve the exponent 19 to 18, by Theorem 3.1.1.

Our next following result seems to be unnoticed which is a generalization of Theorem 3.0.4: If we are given $l-1$ different pairs of $DW(4p_i)$ for $i = 1, 2, ..., l-1$ (instead of $l-2$ different pairs), surprisingly we might construct, similar as in Theorem 3.1.1, an Hadamard matrix of even smaller exponent using a combination of Theorem 3.0.2 and Lemma 3.0.3. Pertinent examples will be given in the sequel.

Theorem 3.1.2 *If there are two Hadamard matrices of orders $2^l m$ and $2^l n$, and there are $l-1$ different pairs of $DW(4p_i)$ for $i = 1, 2, ..., l-1$, then there is an Hadamard matrix of order $2^{3l-2} mnp_1p_2...p_{l-1}$.*

***Proof*.** As in the proof of Theorem 3.1.1, we obtain an $OP(2^{3l-4}mnp_1p_2 \cdots p_{l-2})$, say (S, P), from the two given Hadamard matrices of orders $2^l m$ and $2^l n$, and $l-2$ different pairs of $DW(4p_i)$ for $i = 1, 2, ..., l-2.$, with (X, Y) being the $l-1^{th}$ pair of $DW(4p_{l-1})$.
Now put $\hat{H} = X \otimes S + Y \otimes P$. Then

$$\begin{aligned} \hat{H}\hat{H}^T &= XX^T \otimes SS^T + YY^T \otimes PP^T \\ &= 2p_{l-1} I_{4p_{l-1}} \otimes (SS^T + PP^T) \\ &= 2^{3l-2} mnp_1p_2...p_{l-1} I_{2^{3l-2}mnp_1p_2...p_{l-1}}. \end{aligned}$$

Thus $\hat{H}$ is the desired Hadamard matrix of order $2^{3l-2}mnp_1p_2...p_{l-1}$. ❑

Combining Lemma 3.0.3 and Theorem 3.1.2 for $l = 2$, we gain easily Theorem 3.0.4. Using Lemma 3.0.3, Theorem 3.0.4 and Theorem 3.1.1 for $l = 4$, we obtain Theorem 3.0.1. Similar to Theorem 3.0.2 visa-à-vis Theorem 3.0.1, our Theorem 3.1.2 also yields a better bound than Theorem 3.0.1: For specially chosen 12 Hadamard matrices of orders $2^6 m_1, 2^6 m_2, 4m_3, 4m_4, ..., 4m_{12}$, Theorem 3.0.1 yields an Hadamard matrix of order $2^{17} m_1 m_2 \cdots m_{12}$, whereas, Theorem 3.1.2 allows us to produce a better exponent 16.

3.2 Minimum Exponent of Hadamard Matrices Resulting from t Hadamard Matrices

Given any t Hadamard matrices of orders $4m_1, 4m_2, ..., 4m_t, t \geq 4$, using Theorem 3.0.2 and Theorem 3.0.4 repeatedly, one gets a new Hadamard matrix of order

$2^k m_1 m_2 \cdots m_t$ with $k \le t$. An interesting problem is how to minimize k. At the end of this section, we will utilize Theorem 3.1.1 and Theorem 3.1.2 to find the exponent k as small as possible.

To this end, we define the minimum exponent as follows:
$E(4m_1, 4m_2, ..., 4m_t) = \min\{k \mid$ Given any t Hadamard matrices of orders $4m_1, 4m_2, ..., 4m_t$, there is an Hadamard matrix of order $2^k m_1 m_2 \cdots m_t\}$

and

$E_t = \max\{E(4m_1, 4m_2, ..., 4m_t) \mid$ There are t Hadamard matrices of orders $4m_1, 4m_2, ..., 4m_t$ $\}$.

Note that by Well-Ordering Principle, $E(4m_1, 4m_2, ..., 4m_t)$ and E_t are well defined. Clearly, $E_1 = 2, E_2 \le 3, E_3 \le 4$ by Agayan-Sarukhanyan's result (our Theorem 3.0.2), and $E_t \le t$ for $t \ge 4$ as a consequence of Craigen's result (Theorem 3.0.2 and Theorem 3.0.4). De Launey's construction [22] leads to $E_{12} \le 9$. An important property of E_t is that E_t is a monotonic increasing step function of t with step jump 1.

Lemma 3.2.1 *1. $E_{t+1} \ge E_t$ for $t \in \mathbb{N}$.*

2. $E_{t+1} = E_t$ or $E_{t+1} = E_t + 1$, i.e. E_t is a step function.

***Proof*.**

1. By definition, there exists t Hadamard matrices of orders $4m_1, 4m_2, ..., 4m_t$ such that $E_t = E(4m_1, 4m_2, ..., 4m_t)$. Obviously, this implies the existence of $t+1$ Hadamard matrices of orders $4m_1, 4m_2, ..., 4m_t, 4$, hence the existence of an Hadamard matrix of order $2^{E(4m_1, 4m_2, ..., 4m_t, 4)} m_1 m_2 \cdots m_t$. Now, on the one hand, $E(4m_1, 4m_2, ..., 4m_t) \le E(4m_1, 4m_2, ..., 4m_t, 4)$. On the other hand, by definition, $E(4m_1, 4m_2, ..., 4m_t, 4) \le E_{t+1}$. This yields $E_t \le E_{t+1}$.

2. Partition the $t+1$ Hadamard matrices into two parts consisting of 1 and t Hadamard matrices, respectively, which produce two Hadamard matrices of orders $2^2 m_1$ and $2^{E_t} m_2 m_3 \cdots m_{t+1}$. This implies the existence of an Hadamard

matrix of order $2^{E_t+1}m_1m_2\cdots m_{t+1}$, by Theorem 3.0.2, which yields $E(4m_1, 4m_2, ..., 4m_{t+1}) \le E_t + 1$, for any $t+1$ Hadamard matrices of orders $4m_1, 4m_2, ..., 4m_{t+1}$. Thus, $E_{t+1} \le E_t + 1$, and hence $E_t \le E_{t+1} \le E_t + 1$.

❑

Our next goal is to find E_t which is difficult. First we give two examples to illustrate how to find a bound of E_t by use of Theorem 3.1.2 and Theorem 3.1.1, respectively, and find out that Theorem 3.1.2 yields a better bound than Theorem 3.1.1 in Example 3.2.2, and the other way around in Example 3.2.3. However, in most cases Theorem 3.1.2 yields a better bound than Theorem 3.1.1.

Example 3.2.2 $E_{10} \le 8$.

Proof. It suffices to show that there is an Hadamard matrix of order $2^8m_1m_2\cdots m_{10}$. From the first six Hadamard matrices, we obtain two Hadamard matrices of orders $2^3m_1m_2$ and $2^4m_3m_4m_5m_6$, respectively. The rest four Hadamard matrices yield two pairs of $DW(4m_7m_8)$ and $DW(4m_9m_{10})$. By Theorem 3.1.2, there is an Hadamard matrix of order $2^8m_1m_2\cdots m_{10}$. ❑

Note that if we partition the 10 Hadamard matrices into three parts which contain $3, 3$ and 4 Hadamard matrices, then we gain two Hadamard matrices of orders $2^4m_1m_2m_3$ and $2^4m_4m_5m_6$, and two pairs of $DW(4m_7m_8)$ and $DW(4m_9m_{10})$. Thus by Theorem 3.1.1, there exists an Hadamard matrix of order $2^9m_1m_2\cdots m_{10}$ but not $2^8m_1m_2\cdots m_{10}$.

The next example is the well-known de Launey's result (Theorem 3.0.1).

Example 3.2.3 $E_{12} \le 9$.

Here if we partition the 12 Hadamard matrices into three parts which contain $3, 3$ and 6 Hadamard matrices, then we get two Hadamard matrices of orders $2^4m_1m_2m_3$ and $2^4m_4m_5m_6$, and three pairs of $DW(4m_7m_8), DW(4m_9m_{10})$ and $DW(4m_{11}m_{12})$. Thus by Theorem 3.1.2, there exists an Hadamard matrix of order $2^{10}m_1m_2\cdots m_{12}$ instead of $2^9m_1m_2\cdots m_{12}$.

Next we attempt to derive some upper bounds of E_t. The first step is to prove some recursive inequalities. It is easily shown that $E_{t+3} \le E_t + 2$ and $E_t \le E_k + E_{t-k} - 1$ for $1 \le k \le t-1$. In order to make use of Theorem 3.1.1 and Theorem 3.1.2, we have to partition the given t Hadamard matrices into suitable three parts. The following result illustrates how to do it. For $1 \le k \le t-1$, in partitioning the t Hadamard matrices into three parts, the first part consists of k Hadamard matrices which yields the existence of an Hadamard matrices of order $2^{E(4m_1,4m_2,...4m_k)}m_1m_2...m_k$ which, by Sylvester's construction, can be enlarged to an Hadamard matrix of order $2^{E_k}m_1m_2\cdots m_k$. To utilize Theorem 3.1.1, we put $2(E_k-2)$ Hadamard matrices into a group which yields E_k-2 pairs of DWs, and the number of the rest of Hadamard matrices is $t-k-2(E_k-2)$ which is supposed to be equal or greater than k. Analogously, to use Theorem 3.1.2, we proceed in a similar way by partitioning $2(E_k-1)$ Hadamard matrices to yield E_k-1 DWs instead of E_k-2.

Theorem 3.2.4 *For $t, k \in \mathbb{N}$, we get the following results.*

1. (a) $E_t \le 2E_k - 3 + E_{t-k-2(E_k-2)}$, *where* $1 \le k \le t-1$ *and* $t \ge 2(k+E_k-2)$.

   (b) $E_{2(t+E_t-2)} \le 3E_t - 3$.

2. (a) $E_t \le 2E_k - 2 + E_{t-k-2(E_k-1)}$, *where* $1 \le k \le t-1$ *and* $t \ge 2(k+E_k-1)$.

   (b) $E_{2(t+E_t-1)} \le 3E_t - 2$.

Proof. Suppose there are t Hadamard matrices of orders $4m_1, 4m_2, ..., 4m_t$.

1. Partition the t Hadamard matrices into three parts which contains $k, 2(E_k-2)$, and $t-k-2(E_k-2)$ Hadamard matrices, respectively. Since $t \ge 2(k+E_k-2)$, we get $t-k-2(E_k-2) \ge k$. From the first part, there exists an Hadamard matrix of order $2^{E_k}m_1m_2\cdots m_k$. The second part yields E_k-2 different pairs of DWs, by Lemma 3.0.3. From the third part, it yields an Hadamard matrix of order $2^{E_{t-k-2(E_k-2)}}m_{k+1}m_{k+2}\cdots m_{E_{t-2(E_k-2)}}$. Since $t-k-2(E_k-2) \ge k$, we obtain, by Theorem 3.1.1, an Hadamard matrix of order $2^{3E_k-3}(2^{E_{t-k-2(E_k-2)}-E_k}m_1m_2\cdots m_t)$, i.e. there is an Hadamard matrix of order $2^{2E_k-3+E_{t-k-2(E_k-2)}}m_1m_2\cdots m_t$. Then we finish the proof of part (a).

If we partition $2(t+E_t-2)$ Hadamard matrices into three parts consisting of t, t, and $2(E_t-2)$ Hadamard matrices, then we get (b) using Theorem 3.1.1.

2. To use Theorem 3.1.2, we partition t Hadamard matrices into three parts consisting of $k, 2(E_t-1))$ and $t-k-2(E_t-1)$ Hadamard matrices, respectively. Since $t \geq 2(k+E_k-1)$, hence $t-k-2(E_k-1) \geq k$. This partition yields the inequality 2,(a). To prove 2,(b), analogous as in 1,(b), we partition $2(t+E_t-1)$ Hadamard matrices into three parts consisting of t, t, and $2(E_t-1)$ Hadamard matrices, then we obtain (b) using Theorem 3.1.2.

❑

The above Theorem gives some inequalities of E_t which might be of no use in calculating upper bounds of E_t. For practical purposes, using the same strategy as before, we develop the following methods for calculating explicit upper bounds of E_t whose proofs are similar to those of Theorem 3.2.4.

Corollary 3.2.5 *For $t, k, k_1, k_2 \in \mathbb{N}$, we obtain the following results.*

1. (a) *If $E_k \leq k_1, E_{t-k-2(k_1-2)} \leq k_2$, $k_1 \leq k_2$ and $t \geq 2(k+k_1-2)$, then $E_t \leq 2k_1-3+k_2$.*

   (b) *If $E_t \leq k$, then $E_{2(t+k-2)} \leq 3k-3$.*

2. (a) *If $E_k \leq k_1, E_{t-k-2(k_1-1)} \leq k_2$, $k_1 \leq k_2$ and $t \geq 2(k+k_1-1)$, then $E_t \leq 2k_1-2+k_2$.*

   (b) *If $E_t \leq k$, then $E_{2(t+k-1)} \leq 3k-2$.*

From Corollary 3.2.5,1,(b) with $t=k=4$ and $E_4 \leq 4$, we immediately obtain de Launey's result: $E_{12} \leq 9$. As mentioned above, t is an upper bound of E_t. To get a better upper bound for E_t, we use Corollary 3.2.5 to define recursively ε_t as follows:

1. Set $\varepsilon_1 = 2$.

2. Using Corollary 3.2.5,1,(a), we replace k_1 and k_2 with ε_k and $\varepsilon_{t-k-2(\varepsilon_k-2)}$, respectively. Put $\alpha_t = \min\limits_{1 \leq k \leq t-1} \{2\varepsilon_k - 3 + \varepsilon_{t-k-2(\varepsilon_k-2)} \mid \varepsilon_{t-k-2(\varepsilon_k-2)} \geq \varepsilon_k\}$.

3. Using Corollary 3.2.5,2,(a), we replace k_1 and k_2 with ε_k and $\varepsilon_{t-k-2(\varepsilon_k-1)}$, respectively. Put $\beta_t = \min_{1\leq k\leq t-1}\{2\varepsilon_k - 2 + \varepsilon_{t-k-2(\varepsilon_k-1)} \mid \varepsilon_{t-k-2(\varepsilon_k-1)} \geq \varepsilon_k\}$.

4. $\varepsilon_t = \min\{\alpha_t, \beta_t\}$.

Note that $E_1 = \varepsilon_1 = 2, E_2 \leq \varepsilon_2 = 3, E_3 \leq \varepsilon_3 = 4$ and $E_t \leq \varepsilon_t \leq t$ for $t \geq 4$. In fact, by definition, $E(4m_1, 4m_2, ..., 4m_t) \leq \varepsilon_t$, hence, $E_t \leq \varepsilon_t$. On the other hand, again by definition, we have: $\varepsilon_t \leq \alpha_t \leq 2\varepsilon_1 - 3 + \varepsilon_{t-1} = 1 + \varepsilon_{t-1} \leq 1 + (t-1)$, and by inductive hypotheses for $t - 1 \geq 4$.
To illustrate the above calculation, we give a list below of ε_t for $1 \leq t \leq 20$.

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
ε_t	2	3	4	4	5	6	6	7	8	8	9	9	10	10	11	12	12	13	14	14

Finally as by-products of Corollary 3.2.5,1,(b) and Corollary 3.2.5,2,(b), by starting e.g. with $t = 4$ and $k = 4$, a series of various t for which upper bounds ε_t of E_t can be calculated immediately.

t	4	12	14	38	40	120	122	374	376	1152	1154
ε_t	4	9	10	24	25	69	70	204	205	609	610

Remark. Referring to the above definitions of α_t and β_t, we can prove the following result: Put $\hat{\alpha}_t = \{2\varepsilon_k - 3 + \varepsilon_{t-k-2(\varepsilon_k-2)} \mid k$ is the maximum value satisfying $t - k - 2(\varepsilon_k - 2) \geq k\}$ and $\hat{\beta}_t = \{2\varepsilon_k - 2 + \varepsilon_{t-k-2(\varepsilon_k-1)} \mid k$ is the maximum value satisfying $t - k - 2(\varepsilon_k - 1) \geq k\}$, then $\varepsilon_t = \min\{\hat{\alpha}_t, \hat{\beta}_t\}$. Moreover, ε_t is also a monotonic increasing step function of t. Proofs of these assertions are tedious case by case verifications, therefore, we omit it.