1 Introduction

A mapping $f = (f_1, \ldots, f_n) : \Omega_1 \to \Omega_2$, where Ω_1 and Ω_2 are domains in $\mathbb{C}^n$, is said to be a biholomorphism if f is a bijective holomorphic mapping with holomorphic inverse. In this case, we say that Ω_1 and Ω_2 are biholomorphic. If $\Omega_1 = \Omega_2 = \Omega$, then f is called an automorphism of Ω. Denote $\mathrm{Aut}(\Omega)$ the set of all automorphisms of Ω which is obviously a group.

To classify domains in $\mathbb{C}^n$ up to biholomorphic and to find $\mathrm{Aut}(\Omega)$ are two important research areas in complex geometric function theory. For $n = 1$, every proper simply connected domain in $\mathbb{C}$ is biholomorphic to the unit disk D which is the remarkable Riemann mapping theorem. Also, $\mathrm{Aut}(\mathbb{C})$ is the group of all nonsingular affine linear mappings. However, every thing becomes complicated and difficult if $n \geq 2$. For example, there is no Riemann mapping theorem in $\mathbb{C}^n$, $n \geq 2$, since it is well-known that the unit polydisc and the unit ball in $\mathbb{C}^2$ are not biholomorphic [6]. Also, $\mathrm{Aut}(\mathbb{C}^n)$ is no long the set of all nonsingular affine linear transformations if $n \geq 2$. Furthermore, there is a class of domains in $\mathbb{C}^n$, $n \geq 2$, called Fatou-Bieberbach domains which are biholomorphic to $\mathbb{C}^n$.

The purpose of this thesis is to review the automorphism group $\mathrm{Aut}(\mathbb{C})$ of $\mathbb{C}$, to present some classes of automorphisms of $\mathbb{C}^n$ and to use the complex dynamic method to obtain a large class of Fatou-Bieberbach domains in $\mathbb{C}^n$.

The thesis contains four sections. In section 1, we give an introduction. In section 2, we present a well-known proof of the group $\mathrm{Aut}(\mathbb{C})$. In section 3, we collect some well-known and new examples of automorphisms of $\mathbb{C}^n$ and derive their inverse mappings, Jacobian matrices and Jacobians. In section 4, we use the complex dynamic method to obtain a large class of Fatou-Bieberbach domains in $\mathbb{C}^n$, $n \geq 2$, which is our main work in this thesis.

2 The Automorphism Group of $\mathbb{C}$

Given a domain Ω in $\mathbb{C}$, denote $\text{Aut}(\Omega)$ the set of all automorphisms of Ω, that is, all biholomorphisms from Ω onto Ω. Clearly, $\text{Aut}(\Omega)$ is a group under the composition of mappings.

The group $\text{Aut}(\mathbb{C})$ is well-known in elementary complex analysis. For completeness, we present a proof in [4] that every automorphism of $\mathbb{C}$ is a nonsingular affine linear mapping.

First, we prove some lemmas.

Lemma 2.1 *Let $f : \mathbb{C} \to \mathbb{C}$ be an automorphism, then f satisfies*

$$\lim_{|z| \to \infty} |f(z)| = \infty,$$

that is, given $M > 0$, there exists $R > 0$, such that if $|z| > R$, then $|f(z)| > M$.

Proof. Given $M > 0$. $\overline{D}(0; M) \subseteq \mathbb{C}$ is compact. Since f^{-1} is continuous. $f^{-1}\left(\overline{D}(0; M)\right)$ is also compact. So there exists $R > 0$, such that

$$f^{-1}\left(\overline{D}(0; M)\right) \subseteq \overline{D}(0; R),$$

that is, if $z \notin \overline{D}(0; R)$, then

$$z \notin f^{-1}\left(\overline{D}(0; M)\right),$$

which is equivalent to

$$|f(z)| > M.$$

So we have claimed that, given $M > 0$, there exists $R > 0$ such that, for all $|z| > R$ we have $|f(z)| > M$. ❏

Roughly speaking, we may think f in Lemma 2.1 as a function that continuous at ∞ with value ∞.

Lemma 2.2 *Let* $f : \mathbb{C} \to \mathbb{C}$ *be an automorphism, then there exists* $M_1, M_2 > 0$ *such that*

$$|\, f(z)\,| \leq M_1 \,|\, z\,|$$

for all $|\, z\,| > M_2$.

Proof. Let M and R be as in Lemma 2.1, and let g be defined on $D^*(0; \frac{1}{R}) = \{z \in \mathbb{C} \,|\, 0 < |\, z\,| < \frac{1}{R}\}$ by

$$g(z) = \frac{1}{f(1/z)}.$$

By Lemma 2.1,

$$|\, g(z)\,| = \frac{1}{|\, f(1/z)\,|} < \frac{1}{M}$$

for all $z \in D^*(0; \frac{1}{R})$. By the Riemann removable singularities theorem, g can be holomorphically extended to $D(0; \frac{1}{R})$ so that $g(0) = 0$. Now, since f is one-to-one, it follows that g is also one-to-one on $D(0; \frac{1}{R})$. Hence $g'(0) \neq 0$.

$$0 \neq |\, g'(0)\,| = \lim_{z \to 0} \left| \frac{g(z) - g(0)}{z - 0} \right| = \lim_{|\, z\,| \to 0^+} \frac{|\, g(z)\,|}{|\, z\,|}.$$

For $\varepsilon = \frac{1}{2}\,|\, g'(0)\,| > 0$, there exists $\delta > 0$ such that, if $0 < |\, z\,| < \delta$, then we have

$$\left| \frac{|\, g(z)\,|}{|\, z\,|} - |\, g'(0)\,| \right| < \frac{1}{2}\,|\, g'(0)\,|.$$

By a simple calculation, we have

$$|\, f(z)\,| \leq \frac{2}{g'(0)}\,|\, z\,|$$

for all $|\, z\,| > \frac{1}{\delta}$. Set $M_1 = \frac{2}{|\, g'(0)\,|}$ and $M_2 = \frac{1}{\delta}$. Then $|\, f(z)\,| \leq M_1\,|\, z\,|$ for all $|\, z\,| > M_2$. ❏

Theorem 2.3 $f : \mathbb{C} \to \mathbb{C}$ *is an automorphism if, and only if*

$$f(z) = az + b$$

for some $a, b \in \mathbb{C}$, $a \neq 0$, *that is,* $\mathrm{Aut}(\mathbb{C}) = \{\, az + b \,|\, a \neq 0,\ a, b \in \mathbb{C}\,\}$.

Proof. If $f(z) = az + b$ for some $a, b \in \mathbb{C}$, $a \neq 0$. Then, clearly, f is an automorphism of $\mathbb{C}$. Conversely, if $f : \mathbb{C} \to \mathbb{C}$ is an automorphism. By Lemma 2.2 we know that $|\,f(z)\,| \leq M_1\,|\,z\,|$ for all $|\,z\,| > M_2$. Using the Cauchy estimates, it is easy to see that f is a polynomial of degree at most 1, i.e., $f(z) = az + b$ for some $a, b \in \mathbb{C}$. Since f is an automorphism of $\mathbb{C}$, we must have $a \neq 0$. Therefore, we have proved that every automorphism of $\mathbb{C}$ is a nonsingular linear mapping.

❑

3 Some Examples of Automorphisms of $\mathbb{C}^n$

In Section 2, we have seen that the automorphism group $\mathrm{Aut}(\mathbb{C})$ of $\mathbb{C}$ is nothing but all nonsingular affine linear mappings, so the group is quite easy to describe. Therefore one may ask the same question for $\mathbb{C}^n$, namely, is every automorphism of $\mathbb{C}^n$, $n \geq 2$, a nonsingular affine linear mapping? More precisely, what is $\mathrm{Aut}(\mathbb{C}^n)$? Unfortunately, the answer is no. In fact, the automorphism group $\mathrm{Aut}(\mathbb{C}^n)$, $n \geq 2$, is quite large and there is no explicit description of the group $\mathrm{Aut}(\mathbb{C}^n)$. In this section, we exhibit some examples of automorphism of $\mathbb{C}^n$. Of course, some of them are well-known in the literatures [3, 7]. Moreover, we can also identify their inverse mappings, their Jacobian matrices and Jacobians.

In the following, $F'(z)$ and $J_F(z)$ denote the Jacobian matrix and the Jacobian of a holomorphic mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ at z respectively.

In order to compute the Jacobian of holomorphic mapping, we need a simple lemma in linear algebra.

Lemma 3.1 *Let $v = (v_1, \ldots, v_n)$, $u = (u_1, \ldots, u_n)$ be two vectors in $\mathbb{C}^n$ and I_n be the $n \times n$ identity matrix. Then*

$$\det\left(I_n + v^T u\right) = 1 + uv^T.$$

Proof. If $v = 0$, then it is obvious. Now, we assume that $v \neq 0$. Write $T = I_n + v^T u$. Then T can be regarded as a linear transformation of $\mathbb{C}^n$. Choose a basis $\beta = \{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$ with $e_1 = v$. We have

$$\begin{aligned} T(e_j) &= (I_n + v^T u)(e_j) \\ &= e_j + v^T(ue_j^T) \\ &= (ue_j^T)e_1 + e_j \end{aligned}$$

for $j = 1, 2, \ldots, n$. So the matrix representation of T with respect to β is

$$[T]_\beta = \begin{pmatrix} 1+uv^T & ue_2^T & ue_3^T & \cdots & ue_n^T \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \det\left(I_n + v^T u\right) &= \det(T) \\ &= \det([T]_\beta) \\ &= 1 + uv^T. \end{aligned}$$

❑

Now, we begin to present some examples of automorphisms of $\mathbb{C}^n$, $n \geq 2$.

Example 3.2 Nonsingular affine linear transformation.

Given $A \in GL(n;\mathbb{C})$ and $b \in \mathbb{C}^n$. Then the nonsingular affine linear transformation $T(z) = Az + b$ is obviously an automorphism of $\mathbb{C}^n$ with inverse $T^{-1}(w) = A^{-1}(w-b)$, and $J_T(z) = \det A$.

Example 3.3 Shears.

Let $t \in \mathbb{C}$ and $v \in \mathbb{C}^n$. Define

$$F_t(z) = z + tf(\Lambda z)v,$$

where $\Lambda : \mathbb{C}^n \to \mathbb{C}^k$ is a linear map for some $1 \leq k < n$ satisfying $\Lambda v = 0$, and $f : \mathbb{C}^k \to \mathbb{C}$ is an entire function. Then F_t is an automorphism of $\mathbb{C}^n$.

Proof. (1) F_t is injective.

Suppose $F_t(z) = F_t(z')$, then we have

$$\begin{aligned} \Lambda z &= \Lambda(F_t(z)) - tf(\Lambda z)\Lambda v \\ &= \Lambda(F_t(z)) = \Lambda(F_t(z')) \\ &= \Lambda z'. \end{aligned}$$

Therefore,

$$\begin{aligned} z &= F_t(z) - tf(\Lambda z)v \\ &= F_t(z') - tf(\Lambda z')v \\ &= z'. \end{aligned}$$

Hence, F_t is injective.

(2) F_t is surjective.

Given $w \in \mathbb{C}^n$, let

$$z = w - tf(\Lambda w)v\,.$$

Then

$$\begin{aligned} F_t(z) &= z + tf(\Lambda z)v \\ &= w - tf(\Lambda w)v + tf(\Lambda z)v \\ &= w. \end{aligned}$$

Hence, F_t is surjective.

(3) The inverse of F_t:

From (2), it is obviously that $F_t^{-1}(w) = w - tf(\Lambda w)v$.

(4) The Jacobian matrix and Jacobian of F_t:

Let $g(z) = f \circ \Lambda(z)$. Then $F_t(z) = z + tg(z)v$. We have

$$\begin{aligned} F'(z) &= I_n + \begin{pmatrix} t\frac{\partial g}{\partial z_1}(z)v_1 & \cdots & t\frac{\partial g}{\partial z_n}(z)v_1 \\ \vdots & \ddots & \vdots \\ t\frac{\partial g}{\partial z_1}(z)v_n & \cdots & t\frac{\partial g}{\partial z_n}(z)v_n \end{pmatrix} \\ &= I_n + t(v_1, \ldots, v_n)^T \left(\frac{\partial g}{\partial z_1}(z), \ldots, \frac{\partial g}{\partial z_n}(z)\right). \end{aligned}$$

Write $\Lambda(z) = \left(\sum_{j=1}^{n} a_{1j}z_j, \ldots, \sum_{j=1}^{n} a_{kj}z_j\right)$, $\nabla g(z) = \left(\frac{\partial g}{\partial z_1}(z), \ldots, \frac{\partial g}{\partial z_n}(z)\right)$ and

$w = \Lambda z$. Then

$$\begin{aligned}
\nabla g(z) &= \nabla f(\Lambda z) D\Lambda(z) \\
&= \nabla f(w) D\Lambda(z) \\
&= \left(\tfrac{\partial f}{\partial w_1}(w), \ldots, \tfrac{\partial f}{\partial w_k}(w)\right) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix},
\end{aligned}$$

where $D\Lambda(z)$ denotes the Jacobian matrix of Λ at z. By Lemma 3.1, we have

$$\begin{aligned}
J_{F_t}(z) &= \det\left(I_n + t(v_1, \ldots, v_n)^T \left(\tfrac{\partial g}{\partial z_1}(z), \ldots, \tfrac{\partial g}{\partial z_n}(z)\right)\right) \\
&= 1 + t\left(\tfrac{\partial g}{\partial z_1}(z), \ldots, \tfrac{\partial g}{\partial z_n}(z)\right)(v_1, \ldots, v_n)^T \\
&= 1 + t\left(\sum_{i=1}^{k} \frac{\partial f}{\partial w_i}(w) a_{i1}, \ldots, \sum_{i=1}^{k} \frac{\partial f}{\partial w_i}(w) a_{in}\right)(v_1, \ldots, v_n)^T \\
&= 1 + \sum_{i=1}^{k}\sum_{j=1}^{n} a_{ij} \frac{\partial f}{\partial w_i}(\Lambda z) v_j.
\end{aligned}$$

(5) Obviously, $\{\, F_t \,|\, t \in \mathbb{C} \,\}$ is a one parameter subgroup of automorphisms of $\mathbb{C}^n$, i.e., F_0 is the identity and $F_s \circ F_t = F_{s+t}$ for all $s, t \in \mathbb{C}$. ❑

Example 3.4 General shears.

Let $t \in \mathbb{C}$ and let $v \in \mathbb{C}^n$ with $||v|| = 1$. Define

$$G_t(z) = z + \left(e^{tg(\Lambda z)} - 1\right)\langle z, v\rangle v,$$

where Λ is as above with $\Lambda v = 0$, $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$, and $g : \mathbb{C}^k \to \mathbb{C}$ is an entire function. Then G_t is an automorphism of $\mathbb{C}^n$.

Proof. (1) G_t is injective.

Suppose $G_t(z) = G_t(z')$. Since $z = G_t(z) - \left(e^{tg(\Lambda z)} - 1\right)\langle z, v\rangle v$, we have

$$\begin{aligned}
\Lambda z &= \Lambda(G_t(z)) - \left(e^{tg(\Lambda z)} - 1\right)\langle z, v\rangle \Lambda v \\
&= \Lambda\left(G_t(z)\right) \\
&= \Lambda(G_t(z')) \\
&= \Lambda(G_t(z')) - \left(e^{tg(\Lambda z')} - 1\right)\langle z', v\rangle \Lambda v \\
&= \Lambda z'.
\end{aligned}$$

Hence

$$\begin{aligned}
\langle G_t(z), v\rangle &= \langle z, v\rangle + \left(e^{tg(\Lambda z)} - 1\right)\langle z, v\rangle\langle v, v\rangle \\
&= e^{tg(\Lambda z)}\langle z, v\rangle.
\end{aligned}$$

So

$$\begin{aligned}
\langle z, v\rangle &= e^{-tg(\Lambda z)}\langle G_t(z), v\rangle \\
&= e^{-tg(\Lambda z')}\langle G_t(z'), v\rangle \\
&= \langle z', v\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
z &= G_t(z) - \left(e^{tg(\Lambda z)} - 1\right)\langle z, v\rangle v \\
&= G_t(z') - \left(e^{tg(\Lambda z')} - 1\right)\langle z', v\rangle v \\
&= z'.
\end{aligned}$$

Hence, G_t is injective.

(2) G_t is surjective.

Given $w \in \mathbb{C}^n$. Let

$$z = w + \left(e^{-tg(\Lambda w)} - 1\right)\langle w, v\rangle v.$$

Then

$$\begin{aligned}
\langle z, v\rangle &= \left\langle w + \left(e^{-tg(\Lambda w)} - 1\right)\langle w, v\rangle v, v\right\rangle \\
&= \langle w, v\rangle + \left(e^{-tg(\Lambda w)} - 1\right)\langle w, v\rangle \\
&= e^{-tg(\Lambda w)}\langle w, v\rangle.
\end{aligned}$$

So

$$\begin{aligned}
G_t(z) &= z + \left(e^{tg(\Lambda z)} - 1\right)\langle z, v\rangle v \\
&= w + \left(e^{-tg(\Lambda w)} - 1\right)\langle w, v\rangle v + \left(e^{tg(\Lambda w)} - 1\right)e^{-tg(\Lambda w)}\langle w, v\rangle v \\
&= w.
\end{aligned}$$

Hence, G_t is surjective.

(3) The inverse of G_t:

From (2), it is obviously that $G_t^{-1}(w) = w + \left(e^{-tg(\Lambda w)} - 1\right)\langle w, v\rangle v$.

(4) The Jacobian matrix and Jacobian of G_t:

Let $f(z) = g \circ \Lambda(z)$. Then $G_t(z) = z + \left(e^{tf(z)} - 1\right)\langle z, v\rangle v$. We have

$$\begin{aligned} G_t'(z) &= I_n + (e^{tf(z)} - 1)\left[t\langle z, v\rangle \begin{pmatrix} \frac{\partial f}{\partial z_1}(z)v_1 & \cdots & \frac{\partial f}{\partial z_n}(z)v_1 \\ \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial z_1}(z)v_n & \cdots & \frac{\partial f}{\partial z_n}(z)v_n \end{pmatrix} + \begin{pmatrix} \overline{v}_1 v_1 & \cdots & \overline{v}_n v_1 \\ \vdots & \ddots & \vdots \\ \overline{v}_1 v_n & \cdots & \overline{v}_n v_n \end{pmatrix} \right] \\ &= I_n + (e^{tf(z)} - 1)\, t\, \langle z, v\rangle\, v^T \nabla f(z) + (e^{tf(z)} - 1)\, v^T \overline{v} \\ &= I_n + (e^{tf(z)} - 1) v^T \left(t \langle z, v\rangle \nabla f(z) + \overline{v}\right). \end{aligned}$$

where $v = (v_1, \ldots, v_n)$ and $\overline{v} = (\overline{v}_1, \ldots, \overline{v}_n)$. Write Λz as in Example 3.3. Then

$$\begin{aligned} \nabla f(z) &= \nabla g(\Lambda z) D\Lambda(z) \\ &= \nabla g(w) D\Lambda(z) \\ &= \left(\frac{\partial g}{\partial w_1}(w), \ldots, \frac{\partial g}{\partial w_k}(w)\right) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix}. \end{aligned}$$

By Lemma 3.1, we have

$$\begin{aligned} J_{G_t}(z) &= 1 + (e^{tf(z)} - 1)\left(t \langle z, v\rangle \nabla f(z) + \overline{v}\right) v^T \\ &= 1 + (e^{tf(z)} - 1)\left(t \langle z, v\rangle \nabla f(z) v^T + 1\right) \\ &= 1 + (e^{tf(z)} - 1)\left(t \langle z, v\rangle \left(\sum_{i=1}^{k} \frac{\partial g}{\partial w_i}(w) a_{i1}, \ldots, \sum_{i=1}^{k} \frac{\partial g}{\partial w_i}(w) a_{in}\right)(v_1, \ldots, v_n)^T + 1\right) \\ &= 1 + (e^{tg(\Lambda z)} - 1)\left(t \langle z, v\rangle \left(\sum_{i=1}^{k}\sum_{j=1}^{n} a_{ij} \frac{\partial g}{\partial w_i}(\Lambda z) v_j\right) + 1\right). \end{aligned}$$

❑

Example 3.5 *Let $t \in \mathbb{C}$ and let $v \in \mathbb{C}^{2n}$. Define $S_t : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ by*

$$S_t(z) = z + th\big(\omega(z, v)\big)v,$$

where h is an entire function on $\mathbb{C}$ and

$$\omega = \sum_{j=1}^{n} dz_j \wedge dz_{n+j}$$

is the complex symplectic form on $\mathbb{C}^{2n}$. Then S_t is an automorphism of $\mathbb{C}^{2n}$. Such S_t is also called a shear.

Proof. (1) ω is bilinear form on $\mathbb{C}^{2n}$ and $\omega(z, w) = -\omega(w, z)$, for all $z, w \in \mathbb{C}^{2n}$. In particular, $\omega(z, z) = 0$ for all $z \in \mathbb{C}^{2n}$.

If $z, w \in \mathbb{C}^{2n}$, then

$$\begin{aligned} \omega(z, w) &= \sum_{j=1}^{n} (dz_j \wedge dz_{n+j})(z, w) \\ &= \sum_{j=1}^{n} \begin{vmatrix} dz_j(z) & dz_j(w) \\ dz_{n+j}(z) & dz_{n+j}(w) \end{vmatrix} \\ &= \sum_{j=1}^{n} \begin{vmatrix} z_j & w_j \\ z_{n+j} & w_{n+j} \end{vmatrix} \\ &= \sum_{j=1}^{n} (z_j w_{n+j} - z_{n+j} w_j). \end{aligned}$$

So ω is bilinear on $\mathbb{C}^{2n}$. Also,

$$\begin{aligned} \omega(z, w) &= \sum_{j=1}^{n} (z_j w_{n+j} - z_{n+j} w_j) \\ &= -\sum_{j=1}^{n} (z_{n+j} w_j - z_j w_{n+j}) \\ &= -\omega(w, z) \end{aligned}$$

for all $z, w \in \mathbb{C}^{2n}$. In particular, $\omega(z, z) = 0$ for all $z \in \mathbb{C}^{2n}$.

(2) S_t is injective.

Suppose $S_t(z) = S_t(z')$. Then

$$\omega(z,v) = \omega\big(S_t(z),v\big) = \omega\big(S_t(z',v)\big) = \omega\big(z',v\big),$$

and we get

$$\begin{aligned} z &= S_t(z) - th\big(\omega(z,v)\big)v \\ &= S_t(z') - th\big(\omega(z',v)\big)v \\ &= z'. \end{aligned}$$

Hence, S_t is injective.

(3) S_t is surjective.

Given $w \in \mathbb{C}^{2n}$. Let

$$z = w - th\big(\omega(w,v)\big)v.$$

Then

$$\begin{aligned} \omega(z,v) &= \omega(w,v) + th\big(\omega(w,v)\big)\omega(v,v) \\ &= \omega(w,v). \end{aligned}$$

So

$$\begin{aligned} S_t(z) &= z + th\big(\omega(z,v)\big)v \\ &= w - th\big(\omega(w,v)\big)v + th\big(\omega(z,v)\big)v \\ &= w. \end{aligned}$$

Hence, S_t is surjective.

(4) The inverse of S_t:

From (3), it is obviously that $S_t^{-1}(w) = w - th\big(\omega(w,v)\big)v$.

(5) The Jacobian of S_t:

The Jacobian matrix of S_t is

$$S_t'(z) = I_{2n} + tv^T u,$$

where $u = \Big(h'\big(\omega(z,v)\big)w_{n+1}\,, \ldots,\ h'\big(\omega(z,v)\big)w_{2n}\,,\ -h'\big(\omega(z,v)\big)w_1\,, \ldots,\ -h'\big(\omega(z,v)\big)w_n \Big)$ and $v = (v_1, v_2, \ldots, v_{2n})$ are two vectors in $\mathbb{C}^{2n}$. By Lemma 3.1, we have

$$J_{S_t}(z) = 1 + t\sum_{j=1}^{n} h'\big(w(z,v)\big)(v_j w_{n+j} - v_{n+j} w_j).$$

❑

Example 3.6 *Let $a_1, \ldots, a_n$ be nonnegative integers, $c_1, \ldots, c_n$ be complex numbers such that $\sum_{j=1}^{n} c_j a_j = 0$ and $f : \mathbb{C} \to \mathbb{C}$ be entire. Define $f_j : \mathbb{C}^n \to \mathbb{C}$ by*

$$f_j(z) = z_j e^{c_j f(z_1^{a_1} \cdots z_n^{a_n})}$$

for all $j = 1, 2, \ldots, n$. Then $F = (f_1, f_2, \ldots, f_n)$ is an automorphism of $\mathbb{C}^n$.

(1) F is injective.

Suppose $F(z) = F(z')$. Denote $z^a = z_1^{a_1} \cdots z_n^{a_n}$, $z'^a = {z'_1}^{a_1} \cdots {z'_n}^{a_n}$. Then

$$z_j e^{c_j f(z^a)} = z'_j e^{c_j f(z'^a)}$$

for $1 \leq j \leq n$. We have

$$\prod_{j=1}^{n} z_j^{a_j} e^{a_j c_j f(z^a)} = \prod_{j=1}^{n} {z'_j}^{a_j} e^{a_j c_j f(z'^a)}.$$

Since $\sum_{j=1}^{n} c_j a_j = 0$, we get

$$\begin{aligned} z^a &= z^a e^{f(z^a) \sum_{j=1}^{n} c_j a_j} \\ &= z^a \prod_{j=1}^{n} e^{a_j c_j f(z^a)} \\ &= z'^a \prod_{j=1}^{n} e^{a_j c_j f(z'^a)} \\ &= z'^a e^{f(z'^a) \sum_{j=1}^{n} c_j a_j} \\ &= z'^a. \end{aligned}$$

which implies $f(z^a) = f(z'^a)$. Hence $z_j = z'_j$ for $1 \leq j \leq n$, i.e., $z = z'$. Hence F is injective.

(2) F is surjective.

Given $w \in \mathbb{C}^n$. Let

$$z_j = w_j e^{-c_j f(w^a)}.$$

As in (1), we have

$$w^a = z^a e^{\left(\sum_{j=1}^n c_j a_j\right) f(w^a)} = z^a.$$

Therefore

$$\begin{aligned} F(z) &= (z_1 e^{c_1 f(z^a)}, \ldots, z_n e^{c_n f(z^a)}) \\ &= (w_1, \ldots, w_n). \end{aligned}$$

Hence F is surjective.

(3) The inverse of F:

From (2), it is obviously that $F^{-1}(w) = (w_1 e^{-c_1 f(w^a)}, \ldots, w_n e^{-c_n f(w^a)})$.

(4) The Jacobian matrix and Jacobian of F:

Clearly,

$$F'(z) = \begin{pmatrix} e^{c_1 f(z^a)} + c_1 z_1 e^{c_1 f(z^a)} f'(z^a) a_1 \frac{z^a}{z_1} & \cdots & c_1 z_1 e^{c_1 f(z^a)} f'(z^a) a_n \frac{z^a}{z_n} \\ \vdots & \ddots & \vdots \\ c_n z_n e^{c_n f(z^a)} f'(z^a) a_1 \frac{z^a}{z_1} & \cdots & e^{c_n f(z^a)} + c_n z_n e^{c_n f(z^a)} f'(z^a) a_n \frac{z^a}{z_n} \end{pmatrix}.$$

Hence

$$\begin{aligned} J_F(z) &= e^{\left(\sum_{j=1}^n c_j\right) f(z^a)} \det \left[I_n + f'(z^a) \begin{pmatrix} z_1 c_1 a_1 \frac{z^a}{z_1} & \cdots & z_1 c_1 a_n \frac{z^a}{z_n} \\ \vdots & \ddots & \vdots \\ z_n c_n a_1 \frac{z^a}{z_1} & \cdots & z_n c_n a_n \frac{z^a}{z_n} \end{pmatrix} \right] \\ &= e^{\left(\sum_{j=1}^n c_j\right) f(z^a)} \det \left(I_n + f'(z^a) v^T u \right), \end{aligned}$$

where $v = (z_1 c_1, \ldots, z_n c_n)$ and $u = (a_1 \frac{z^a}{z_1}, \ldots, a_n \frac{z^a}{z_n})$. By Lemma 3.1, we have

$$\begin{aligned} J_F(z) &= e^{\left(\sum_{j=1}^n c_j\right) f(z^a)} (1 + f'(z^a) u v^T) \\ &= e^{\left(\sum_{j=1}^n c_j\right) f(z^a)} (1 + z^a \textstyle\sum_{j=1}^n c_j a_j) \\ &= e^{\left(\sum_{j=1}^n c_j\right) f(z^a)} \\ &= \frac{w_1 \cdots w_n}{z_1 \cdots z_n}. \end{aligned}$$

Example 3.7 Lower triangular mappings.

Let $F = (f_1, \ldots, f_n)$ be defined by

$$\begin{cases} f_1(z) &= c_1 z_1 \\ f_j(z) &= c_j z_j + h_j(z_1, \ldots, z_{j-1}), \end{cases}$$

where c_j are constants and each h_j is a holomorphic function of $(z_1, \ldots, z_{j-1})$ for $2 \le j \le n$. Then F is an automorphism of $\mathbb{C}^n$ if $c_j \neq 0$ for all $j = 1, 2, \ldots, n$.

Proof. (1) F is injective.

Suppose $F(z) = F(z')$. By the definition of F, it is easy to show that $z = z'$. So F is injective.

(2) F is surjective.

Given $w \in \mathbb{C}^n$. Let

$$\begin{cases} z_1 &= \frac{1}{c_1} w_1 \\ z_j &= \frac{1}{c_j} w_j - \frac{1}{c_j} h_j(z_1, \ldots, z_{j-1}), \qquad 2 \le j \le n. \end{cases}$$

Then $F(z) = w$. Hence F is surjective.

(3) The inverse of F:

From (2), it is obviously that $F^{-1}(w) = (g_1(w), \ldots, g_n(w))$ where

$$\begin{cases} g_1(w) &= \frac{1}{c_1} w_1 \\ g_j(w) &= \frac{1}{c_j} w_j - \frac{1}{c_j} h_j(g_1(w), \ldots, g_{j-1}(w)), \qquad 2 \le j \le n. \end{cases}$$

(4) The Jacobian matrix and Jacobian of F:

Clearly,

$$F'(z) = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ \frac{\partial h_2}{\partial z_1}(z) & c_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{\partial h_n}{\partial z_1}(z) & \cdots & \frac{\partial h_n}{\partial z_{n-1}}(z) & c_n \end{pmatrix}.$$

Hence

$$J_F(z) = \prod_{j=1}^{n} c_j.$$

Therefore, F is an automorphism of $\mathbb{C}^n$. ❑

Example 3.8 *Let $F_a : \mathbb{C}^n \to \mathbb{C}^n$ be defined by*

$$F(z) = F_a(z) = (\, z_1^2 + a_1 z_2, \ldots, z_{n-1}^2 + a_{n-1} z_n \,,\, a_n z_1 \,),$$

where $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ satisfying $0 < |\, a_j \,| < 1$ for $1 \le j \le n$.

Proof. (1) F is injective.

Suppose $F(z) = F(w)$. Then

$$\begin{cases} z_j^2 + a_j z_{j+1} = w_j^2 + a_j w_{j+1}, & 1 \le j \le n-1 \\ a_n z_1 = a_n w_1. \end{cases}$$

So $z_j = w_j$ for $1 \le j \le n$, i.e., $z = w$.

(2) F is surjective.

Given $w \in \mathbb{C}^n$. Let

$$\begin{cases} z_1 = a_n^{-1} w_n \\ z_j = a_{j-1}^{-1}(w_{j-1} - z_{j-1}^2), & 2 \le j \le n. \end{cases}$$

Then $F(z) = w$. Hence F is surjective.

(3) The inverse of F:

From (2), it is obviously that $F^{-1}(w) = (g_1(w), \ldots, g_n(w))$ where

$$\begin{cases} g_1(w) = a_n^{-1} w_n \\ g_j(w) = a_{j-1}^{-1}(w_{j-1} - g_{j-1}^2(w)), & 2 \le j \le n. \end{cases}$$

(4) The Jacobian matrix and Jacobian of F:

Clearly,

$$F'(z) = \begin{pmatrix} 2z_1 & a_1 & 0 & \cdots & 0 \\ 0 & 2z_2 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & 2z_{n-1} & a_{n-1} \\ a_n & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Hence

$$J_F(z) = (-1)^{n+1} \prod_{j=1}^{n} a_j.$$

Therefore, F is an automorphism of $\mathbb{C}^n$. ❑

In the next section, we will use the class of automorphisms in Example 3.8 to generate a class of Fatou-Bieberbach domains in $\mathbb{C}^n$.

4 A Class of n-Dimensional Complex Henon Maps and Fatou-Bieberbach Domains in $\mathbb{C}^n$

It is well-known that any proper simply connected domain in the complex plane $\mathbb{C}$ is biholomorphic to the unit disk D in $\mathbb{C}$ which is the remarkable Riemann mapping theorem. Hence, there are no proper subdomains of $\mathbb{C}$ can be biholomorphic to $\mathbb{C}$. Therefore, we may ask the same question for domains in $\mathbb{C}^n$, $n \geq 2$. The problem becomes very complicated and difficult, and is a very interesting research area for a long time.

The first proper subdomain in $\mathbb{C}^2$ which is biholomorphic to $\mathbb{C}^2$ was discovered by Fatou and Bieberbach [1]. So we call such domain as a Fatou-Bieberbach domain. In this section, we will follow the approach in [2, 5, 7] to use the complex analytic dynamics of n complex variables by iterating the Henon map to construct a class of Fatou-Bieberbach domains in $\mathbb{C}^n$ which generalizes the case $n = 2$ in [5] and $n = 3$ in [2].

Definition 4.1 *A Fatou-Bieberbach domain is a proper subdomain Ω of $\mathbb{C}^n$ such that Ω is biholomorphic to $\mathbb{C}^n$.*

Definition 4.2 *An n-dimensional complex Henon map is a mapping of the form*

$$F(z) = F_a(z) = (\, z_1^2 + a_1 z_2\, , \ldots ,\, z_{n-1}^2 + a_{n-1} z_n\, ,\, a_n z_1\,),$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $a = (a_1, \ldots, a_n)$ satisfying $0 < |\, a_j\,| < 1$ for $1 \leq j \leq n$.

As we have seen in Example 3.8, F is an automorphism of $\mathbb{C}^n$.

In the rest of this discussion, we shall use the notation F^j to denote the j-fold composition $F \circ F \circ \cdots \circ F$, and let $\max\limits_{1 \leq j \leq n} |a_j| = \alpha$, $\min\limits_{1 \leq j \leq n} |a_j| = \beta$.

Lemma 4.3 *Given* $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$*, where* $0 < |\, a_j \,| < 1$ *for* $1 \leq j \leq n$*. Given* b *such that* $0 < \alpha < b < 1$*. Then there exist* $\varepsilon > 0$ *and* $c > 0$ *such that if* $||z|| < \varepsilon$*, we have*

$$||F^j(z)|| < cb^j$$

for all $j = 1, 2, 3, \ldots$

***Proof*.** We will prove the lemma by induction. Since $\alpha < b$, so we can choose $0 < c < 1$ such that

$$2c^2b^2 + 4\alpha cb + \alpha^2 < b^2$$

Using the continuity of F and $F(0) = 0$, we can find $\varepsilon > 0$ such that if $||z|| < \varepsilon$, we have $||F(z)|| < cb$. Suppose, for any $||z|| < \varepsilon$, we have $||F^j(z)|| < cb^j$. If $F^j(z) = w = (w_1, \ldots, w_n)$, then

$$||F^{j+1}(z)|| = ||F(w)||$$

$$= \sqrt{|\, w_1^2 + a_1 w_2 \,|^2 + |\, w_2^2 + a_2 w_3 \,|^2 + \cdots + \left|\, w_{n-1}^2 + a_{n-1} w_n \,\right|^2 + |\, a_n w_1 \,|^2}$$

$$\leq \sqrt{|\, w_1 \,|^4 + \cdots + |\, w_{n-1} \,|^4 + 2\alpha(|\, w_1 \,|^2 \,|\, w_2 \,| + \cdots + |\, w_{n-1} \,|^2 \,|\, w_n \,|) + \alpha^2 ||w||^2}$$

$$\leq \sqrt{||w||^4 + 2\alpha ||w||^3 + \alpha^2 ||w||^2}$$

$$< \sqrt{c^2 b^{2j} + 2\alpha c b^j + \alpha^2}\, ||w||$$

$$< \sqrt{b^2}\, ||w|| < cb^{j+1}$$

So, by induction, the lemma is proved. ❑

Definition 4.4 *The basin of attraction of* F *is defined to be*

$$\Omega = \left\{ z \in \mathbb{C}^n \,\middle|\, \lim_{n \to \infty} F^j(z) = 0 \right\}$$

Lemma 4.3 says that $B(0, \varepsilon) \subseteq \Omega$. In particular, Ω is nonempty. From now on, we assume that for the given $a \in \mathbb{C}^n$ in Lemma 4.3, there exists $b \in \mathbb{R}$ with $0 < b^2 < |a_j| < b < 1$ for $1 \leq j \leq n$. Note that if $\alpha^2 < \beta$ or, $|\, a_j \,|^2 < |\, a_i \,|$ for all $i \neq j$, then such b can be found. Let F be the n-dimensional complex Henon map defined as above.

Theorem 4.5 *The basin of attraction Ω of F is a Fatou-Bieberbach domain in $\mathbb{C}^n$. Moreover, the basin of attraction Ω is not the product of a k-dimensional Fatou-Bieberbach domain and $\mathbb{C}^{n-k}$ for any $1 \leq k \leq n-1$.*

Before proving the theorem, we first prove some lemmas.

Lemma 4.6 *The basin of attraction Ω of F is an open subset of $\mathbb{C}^n$. In fact,*

$$\Omega = \bigcup_{j=1}^{\infty} F^{-j}\big(B(0,\varepsilon)\big),$$

where ε is as in Lemma 4.3.

Proof. It is enough to show that $\Omega = \bigcup_{j=1}^{\infty} F^{-j}\big(B(0,\varepsilon)\big)$. Given $z \in \Omega$, then $\lim_{j\to\infty} F^j(z) = 0$. Therefore, there exists $j_0 \in \mathbb{N}$ such that $||F^{j_0}(z)|| < \varepsilon$, i.e., $F^{j_0}(z) \in B(0,\varepsilon)$. Hence,

$$z \in F^{-j_0}\big(B(0,\varepsilon)\big) \subseteq \bigcup_{j=1}^{\infty} F^{-j}\big(B(0,\varepsilon)\big).$$

So we have

$$\Omega \subseteq \bigcup_{j=1}^{\infty} F^{-j}\big(B(0,\varepsilon)\big).$$

Conversely, given $z \in \bigcup_{j=1}^{\infty} F^{-j}\big(B(0,\varepsilon)\big)$, then $z \in F^{-j_0}\big(B(0,\varepsilon)\big)$ for some j_0. So $F^{j_0}(z) \in B(0,\varepsilon)$. By Lemma 4.3, we have

$$||F^j(z)|| = ||F^{j-j_0}(F^{j_0}(z))|| < cb^{j-j_0}$$

for any $j > j_0$. Hence, $\lim_{j\to\infty} F^j(z) = 0$, i.e., $z \in \Omega$ ❑

Now, denote

$$A = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & a_{n-1} \\ a_n & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

which is the complex Jacobian matrix of F at $z = 0$ from Example 3.8. Clearly,

$$A^{-1} = \begin{pmatrix} 0 & \cdots & \cdots & 0 & a_n^{-1} \\ a_1^{-1} & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n-1}^{-1} & 0 \end{pmatrix}.$$

Lemma 4.7 *Let $z \in \mathbb{C}^n$. Then, we have*

$$||A^{-1}F(z) - z|| \leq \beta^{-1}||z||^2, \text{ and } ||A||^{-1} = \beta^{-1}.$$

Proof. It is clear that

$$\begin{aligned} A^{-1}F(z) - z &= (z_1, a_1^{-1}z_1^2 + z_2, \ldots, a_{n-1}^{-1}z_{n-1}^2 + z_n) - (z_1, \ldots, z_n) \\ &= (0, a_1^{-1}z_1^2, \ldots, a_{n-1}^{-1}z_{n-1}^2). \end{aligned}$$

So

$$\begin{aligned} ||A^{-1}F(z) - z|| &= \sqrt{|a_1|^{-2}|z_1|^4 + |a_2|^{-2}|z_2|^4 + \cdots + |a_{n-1}|^{-2}|z_{n-1}|^4} \\ &\leq \beta^{-1}(|z_1|^2 + |z_2|^2 + \cdots + |z_{n-1}|^2) \\ &\leq \beta^{-1}||z||^2. \end{aligned}$$

Also,

$$A^{-1}z = (a_n^{-1}z_n, a_1^{-1}z_1, a_2^{-1}z_2, \ldots, a_{n-1}^{-1}z_{n-1}),$$

so

$$||A^{-1}z|| \leq \beta^{-1}||z||.$$

Moreover, if $\beta = \min_{1\leq j\leq n} |\, a_j \,| = a_{j_0}$ and $z_0 = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in the j_0-th component, then

$$||A^{-1}z_0|| = \beta^{-1}||z_0||.$$

So

$$||A^{-1}|| = \sup_{z\neq 0} \frac{||A^{-1}z||}{||z||} = \beta^{-1}.$$

❑

Lemma 4.8 *The sequence $\{A^{-j}F^j\}$ converges uniformly on compact subsets of Ω.*

Proof. It suffices to show that, given $z_0 \in \Omega$, $\{A^{-j}F^j\}$ converges uniformly on $B(z_0;\eta)$ for some $\eta > 0$. Since, if this is done, then given a compact set $K \subseteq \Omega$, for any $z \in K$, choose $\eta_z > 0$ such that $\{A^{-j}F^j\}$ converges uniformly on $B(z,\eta_z)$. Since $\{B(z;\eta_z)|z \in K\}$ is an open covering of K, by compactness of K, we have $K \subseteq \bigcup_{i=1}^{m} B(z_i,\eta_i)$ for some $m \in \mathbb{N}$, where $\eta_i = \eta_{z_i}$, $1 \le i \le m$. Hence $\{A^{-j}F^j\}$ converges uniformly on K. Now, given $z_0 \in \Omega$, we have $\lim_{j\to\infty} F^j(z_0) = 0$, so we can choose $j_0 \in \mathbb{N}$ such that $F^{j_0}(z) \in B(0;\varepsilon)$. By the continuity of F^{j_0}, we can find $\eta > 0$ such that $F^{j_0}(z) \in B(0;\varepsilon)$ for all $z \in B(z_0,\eta)$.

Now, for all $j > j_0$ and $z \in B(z_0;\eta)$,

$$
\begin{aligned}
&||A^{-(j+1)}F^{j+1}(z) - A^{-j}F^j(z)||\\
&= ||A^{-j}(A^{-1}F)(F^j(z)) - A^{-j}F^j(z)||\\
&= ||A^{-j}(A^{-1}F - I)(F^j(z))||, \quad \text{where } I \text{ is the identity mapping of } \mathbb{C}^n\\
&\le ||A^{-j}||\,||(A^{-1}F - I)(F^j(z))||\\
&\le ||A^{-j}|| \cdot \beta^{-1} \cdot ||F^j(z)||^2, \quad \text{by Lemma 4.7}\\
&= \beta^{-(j+1)}||F^j(z)||^2, \quad \text{by Lemma 4.7}\\
&= \beta^{-(j+1)}||F^{j-j_0}(F^{j_0}(z))||^2\\
&\le \beta^{-(j+1)}(cb^{j-j_0})^2, \quad \text{by Lemma 4.3}\\
&= (c^2\beta^{-1}b^{-2j_0})\left(\frac{b^2}{\beta}\right)^j.
\end{aligned}
$$

Set $M = c^2\beta^{-1}b^{-2j_0}$. Then M is a fixed constant and we have

$$||A^{-(j+1)}F^{j+1}(z) - A^{-j}F^j(z)|| \le M\left(\frac{b^2}{\beta}\right)^j$$

for all $j > j_0$. By the choice of b, we have $0 < \frac{b^2}{\beta} < 1$, so

$$\lim_{j\to\infty}\left(\frac{b^2}{\beta}\right)^j = 0.$$

Therefore, $\{A^{-j}F^j\}$ converges uniformly on $B(z_0;\eta)$ which proves the lemma. ❑

Lemma 4.9 *F is an automorphism of Ω. In particular, the automorphism group of Ω containing F^j, for $j = 0, 1, 2, \ldots$.*

Proof. Since we have known that $F : \mathbb{C}^n \to \mathbb{C}^n$ is an automorphism. So it enough to show that $F(\Omega) \subseteq \Omega$. Given $z \in \Omega$, we have

$$\lim_{j\to\infty} F^j(F(z)) = \lim_{j\to\infty} F^{j+1}(z) = 0,$$

so $F(z) \in \Omega$. Hence, $F(\Omega) \subseteq \Omega$. Therefore, F is an automorphism of Ω. ❑

Lemma 4.10 *Let* $V = \left\{ z \in \mathbb{C}^n \,\middle|\, |z_1| > 100\,,\ |z_j| < \frac{\sqrt{|z_1|}}{2} \text{ for } 2 \le j \le n-1\,,\ |z_n| < \frac{|z_1|}{8} \right\}$. *Then* $\Omega \cap V = \emptyset$.

Proof. If $z \in V$ and $F(z) = w$, then

$$\begin{aligned} w &= (w_1, w_2, \ldots, w_n) \\ &= (z_1^2 + a_1 z_2, \ldots, z_{n-1}^2 + a_{n-1} z_n, a_n z_1). \end{aligned}$$

$$\begin{aligned} |w_1| &= |z_1^2 + a_1 z_2| \ge |z_1|^2 - |a_1| \cdot |z_2| > \tfrac{9}{16}|z_1|^2 > 8|z_1| > 100. \\ |w_2| &= |z_2^2 + a_2 z_3| \le |z_2|^2 + |a_2| \cdot |z_3| < \tfrac{|z_1|}{4} + \tfrac{\sqrt{|z_1|}}{2} < \tfrac{|z_1|}{3} < \tfrac{\sqrt{|w_1|}}{2}. \\ &\vdots \\ |w_{n-1}| &= |z_{n-1}^2 + a_{n-1} z_n| \le |z_{n-1}|^2 + |a_{n-1}| \cdot |z_n| < \tfrac{|z_1|}{4} + \tfrac{|z_1|}{8} = \tfrac{3}{8}|z_1| < \tfrac{\sqrt{|w_1|}}{2}. \\ |w_n| &= |a_n z_1| \le |a_n| \cdot |z_1| < \tfrac{|w_1|}{8}. \end{aligned}$$

Thus, we see that F maps a point $z \in V$ to a point w of the same type with larger modulus and whose first entry is at least eightfold as large. It follows that $F^j(z) \to \infty$ as $j \to \infty$, i.e., $z \notin \Omega$. Hence $\Omega \cap V = \emptyset$. ❑

Now, we can prove the main theorem. Denote $\lim_{j\to\infty} A^{-j} F^j = \Phi$.

Proof of Theorem 4.5.

By Lemma 4.8, $\Phi : \Omega \to \mathbb{C}^n$ is a holomorphic map. So it's enough to show that Φ is bijective. Clearly, the complex Jacobian matrix of $A^{-1}F$ at 0 is the $n \times n$ identity matrix, so is $A^{-j}F^j$ for all $j = 0, 1, 2, \ldots$. It follows that the complex

Jacobian matrix of Φ at 0 is the identity map. By the inverse function Theorem, Φ is biholomorphic from a neighborhood W of 0 onto $\Phi(W)$.

Now, to prove that Φ is injective. If z, $z' \in \Omega$ such that $\Phi(z) = \Phi(z')$, then

$$\lim_{j\to\infty} F^j(z) = \lim_{j\to\infty} F^j(z') = 0.$$

So we can find $j_0 \in \mathbb{N}$, such that $F^{j_0}(z) \in W$ and $F^{j_0}(z') \in W$. By $\Phi(z) = \Phi(z')$, we have

$$\lim_{j\to\infty} A^{-j} F^j(z) = \lim_{j\to\infty} A^{-j} F^j(z'),$$

then

$$A^{-j_0} \lim_{j\to\infty} A^{-(j-j_0)} F^{j-j_0}(F^{j_0}(z)) = A^{-j_0} \lim_{j\to\infty} A^{-(j-j_0)} F^{j-j_0}(F^{j_0}(z')).$$

Hence

$$A^{-j_0}\Phi(F^{j_0}(z)) = A^{-j_0}\Phi(F^{j_0}(z')).$$

Since A is nonsingular, we get

$$\Phi(F^{j_0}(z)) = \Phi(F^{j_0}(z')).$$

Using the injectivity of Φ on W and $F^{j_0}(z)$, $F^{j_0}(z') \in W$, we conclude that

$$F^{j_0}(z) = F^{j_0}(z').$$

Moreover, F^{j_0} is an automorphism of Ω. We have $z = z'$.

Next, to prove that Φ is surjective. By Lemma 4.8,

$$\lim_{j\to\infty} A^{-j-1} F^{j+1} = \Phi$$

uniformly on compact subsets of Ω, that is,

$$A^{-1}\left(\lim_{j\to\infty} A^{-j} F^j\right) F = \Phi$$

uniformly on compact subsets of Ω. Therefore, we obtain

$$A^{-1}\Phi F = \Phi\,.$$

Let $\Phi(\Omega) = S$, then we must show that $S = \mathbb{C}^n$. Obviously,

$$A^{-1}(S) = A^{-1}(\Phi(\Omega)) = A^{-1}\left(\Phi\left(F(\Omega)\right)\right) = \Phi(\Omega) = S,$$

where we use $F(\Omega) = \Omega$ in Lemma 4.9. On the other hand, since Φ is a biholomorphism from W onto $\Phi(W)$. It follows that $\Phi(W)$ is a neighborhood of $\Phi(0) = 0$ and $\Phi(W) \subseteq S$. So we can find $r > 0$ such that $B(0;r) \subseteq S$. Hence $A^{-1}(B(0;r)) \subseteq A^{-1}(S) = S$. Now, we claim that $B(0;\frac{r}{\alpha}) \subseteq S$. Let $w \in B(0;\frac{r}{\alpha})$, and choose $z = (a_1 w_2, \ldots, a_{n-1} w_n, a_n w_1)$. Then $||z|| \leq \alpha ||w|| < r$, so $w = A^{-1}z \in S$. Therefore, $B(0;\frac{r}{\alpha}) \subseteq S$. By repeating the process, we conclude that S contains $B\left(0;\frac{r}{\alpha^j}\right)$ for all $j \in \mathbb{N}$. So

$$\bigcup_{j=1}^{\infty} B\left(0;\frac{r}{\alpha^j}\right) \subseteq S.$$

Since $0 < \alpha < 1$, we have $\displaystyle\bigcup_{j=1}^{\infty} B\left(0;\frac{r}{\alpha^j}\right) = \mathbb{C}^n$. Hence $S = \mathbb{C}^n$ which implies that Φ is surjective. So we have proved that Φ is a biholomorphism from Ω onto $\mathbb{C}^n$.

Finally, by Lemma 4.10, it is easy to see that Ω is a proper subdomain of $\mathbb{C}^n$, i.e., Ω is a Fatou-Bieberbach domain in $\mathbb{C}^n$, and Ω is not the product of a Fatou-Bieberbach domain in $\mathbb{C}^k$ and $\mathbb{C}^{n-k}$. Therefore, the Theorem is proved. ❑

If $a_j = a$ for all $1 \leq j \leq n$, and $0 < |\,a\,| < 1$, then $|\,a_j\,|^2 < |\,a_i\,|$ for all $i \neq j$ is clearly satisfied. Therefore, we have the following corollary.

Corollary 4.11 *Let*

$$F(z) = (\, z_1^2 + a z_2\,,\; z_2^2 + a z_3\,,\; \ldots\,,\; z_{n-1}^2 + a z_n\,,\; a z_1\,)\,.$$

Then the basin of attraction Ω of F is a Fatou-Bieberbach domain $\mathbb{C}^n$.

Remark. Corollary 4.11 generalizes the results in [2, 5].

For general function $F \in \mathrm{Aut}(\mathbb{C}^n)$, the following two theorems have been proved in [7].

Theorem 4.12 *Let $F \in \mathrm{Aut}(\mathbb{C}^n)$ and $p \in \mathbb{C}^n$ be a fixed point of F. Denote the eigenvalues of $A = F'(p)$ by $\lambda_1, \ldots, \lambda_n$ satisfy $0 < |\lambda_n| \le \cdots \le |\lambda_2| \le |\lambda_1|$ and $|\lambda_1|^2 \le |\lambda_n|$. Let Ω be the basin of attraction of F at p, $\Phi = \lim_{j\to\infty} A^{-j}F^j$. Then $\Phi : \Omega \to \mathbb{C}^n$ is a biholomorphic mapping.*

Theorem 4.13 *Let $F \in \mathrm{Aut}(\mathbb{C}^n)$ and $p \in \mathbb{C}^n$ be a fixed point of F. Denote the eigenvalues of $A = F'(p)$ by $\lambda_1, \ldots, \lambda_n$ satisfy $|\lambda_j| < 1$. Let Ω be the basin of attraction of F to p. Then there exists a biholomorphic mapping Φ from Ω onto $\mathbb{C}^n$.*

Apply Theorem 4.13 and Lemma 4.10, we have the following corollary.

Corollary 4.14 *Let $F(z) = (z_1^2 + a_1 z_2, \ldots, z_{n-1}^2 + a_{n-1} z_n, a_n z_1)$, where $a = (a_1, \ldots, a_n)$ satisfying $0 < |a_j| < 1$ for $1 \le j \le n$. Then the basin of attraction of F is a Fatou-Bieberbach domain.*

Note that our main results in Theorem 4.5 and Corollary 4.14 are by no mean a special case of Theorem 4.12 and Theorem 4.13. Since they give no information about the properness of Ω. For example, if F is a nonsingular linear transformation of $\mathbb{C}^n$ with $0 < |\lambda_j| < 1$ for all $1 \le j \le n$, then the basin of attraction of F is $\mathbb{C}^n$ which is not a Fatou-Bieberbach domain in $\mathbb{C}^n$.

Finally, we include two examples in [7] of Fatou-Bieberbach domains with some interesting properties.

Example 4.15 *Let $F(z, w) = (\alpha w, \alpha z + w^2)$, where $0 < |\alpha| < 1$. Then the basin of attraction Ω of F is a Fatou-Biberbach domain. Moreover, the intersection of Ω with every complex line is bounded.*

Proof. Obviously, $F \in \mathrm{Aut}(\mathbb{C}^2)$, and $F'(0) = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$ with eigenvalues $\pm\alpha$. Let Ω be the basin of attraction of F. By Theorem 4.13, Ω is biholomorphic to $\mathbb{C}^2$. Let

$$S = \{ (z, w) \in \mathbb{C}^2 \mid |w| > 1 + 2|\alpha| + |z| \}.$$

If $(z,w) \in S$ and $F(z,w) = (u,v)$, then

$$\begin{aligned}|\,v\,| &= |\,\alpha z + w^2\,| \ge |\,w\,|^2 - |\,\alpha z\,| > |\,w\,|(1 + 2\,|\,\alpha\,| + |\,z\,|) - |\,\alpha z\,| \\ &> |\,w\,| + |\,w\alpha\,| > 1 + 2\,|\,\alpha\,| + |\,w\alpha\,| = 1 + 2\,|\,\alpha\,| + |\,u\,|\,.\end{aligned}$$

Hence $F(z,w) \in S$, i.e., $F(S) \subseteq S$. Therofore $S \subseteq \mathbb{C}^2 - \Omega$ and Ω is a Fatou-Bieberbach domain. Now let L be a complex line in $\mathbb{C}^2$ parameterized by

$$\begin{cases} z &= a + b\lambda \\ w &= c + d\lambda \end{cases}$$

where $a, b, c, d \in \mathbb{C}$ are fixed constants and $\lambda \in \mathbb{C}$. We may assume $bd \neq 0$. Then

$$\begin{aligned}|\,v\,| &= |\,\alpha(a + b\lambda) + (c + d\lambda)^2\,| \\ &> 1 + 2\,|\,\alpha\,| + |\,\alpha(c + d\lambda)\,| \\ &> 1 + 2\,|\,\alpha\,| + |\,u\,|\end{aligned}$$

if $|\,\lambda\,|$ is large enough. So if $|\,\lambda\,|$ is large, then $F(z,w) \in S \subseteq \mathbb{C}^2 - \Omega$, i.e., $L \cap \Omega$ is bounded. ❑

Example 4.16 *There is a collection of pairwise disjoint Fatou-Bieberbach domains in* $\mathbb{C}^2$.

Let $F(z,w) = \big(z + w, \frac{1}{2}(1 - w - e^{z+w})\big)$ *and* $p_m = (2m\pi i, 0)$, *where* $m \in \mathbb{Z}$. *Let* $\Omega_m = \{(z,w) \in \mathbb{C}^2 \mid \lim F^j(z,w) = p_m\}$ *be the basin of attraction of* F *at* p_m. *Then* $\{\Omega_m \mid m \in \mathbb{Z}\}$ *is a collection of pairwise disjoint Fatou-Bieberbach domains in* $\mathbb{C}^2$ *satisfying* $\Omega_m = \Omega_0 + p_m$ *for all* $m \in \mathbb{Z}$.

Proof. Given $m \in \mathbb{Z}$, the eigenvalues of $F'(p_m)$ are $\pm 1/\sqrt{2}$. By Theorem 4.13, each Ω_m is biholomorphic to $\mathbb{C}^2$. Clearly, $\Omega_n \cap \Omega_m = \emptyset$ if $n \neq m$, so that Ω_m is a proper subdomain of $\mathbb{C}^2$. Hence, $\{\Omega_m \mid m \in \mathbb{Z}\}$ is a collection of pairwise disjoint Fatou-Bieberbach domains. Finally, since $F((z,w) + p_m) = F(z,w) + p_m$, by induction, we have

$$F^j((z,w) + p_m) = F^j(z,w) + p_m$$

for all $j \in \mathbb{N}$. Hence,

$$\lim_{j\to\infty} F^j((z,w) + p_m) = \lim_{j\to\infty} F^j(z,w) + p_m\,.$$

So $\lim_{j\to\infty} F^j((z,w)+p_m) = p_m$ if, and only if $\lim_{j\to\infty} F^j(z,w) = 0$ which implies

$$\Omega_m = \Omega_0 + p_m$$

for all $m \in \mathbb{Z}$. ❑