1 Introduction.

The Theory of Graph coloring has been developed for more than 150 years. More recently, graph coloring was applied to many operations research, design of experiments and computer science. For example, graph coloring is used in committee-scheduling problem to model avoidance of conflicts. Similarly, in a university we want to assign time slots for final examinations so that two courses with a common student have different slots.

Let $X = \{x_1, x_2 \ldots, x_n\}$ be a set of sources of power supply such that the work time of any source is one quantum of time and all sources working for any given quantum of time turn on and turn off synchronously.

Consider the following general constraints on their common work:

(1)let $\mathfrak{C} = \{C_1, C_2 \ldots, C_k\}$, $C_i \subseteq X$, $i = 1, 2 \ldots, k$, $k \geq 1$, be a family of subsets of X such that at least two sources from every C_i work for the same quantum of time;

(2)let $\mathfrak{D} = \{D_1, D_2 \ldots, D_m\}$, $D_j \subseteq X$, $j = 1, 2 \ldots, m$, $m \geq 1$, be a family of subsets of X such that at least two sources from every D_j work for different quanta of time.

Call the set X with such constraints a *system* and denote it by $H = (X, \mathfrak{C}, \mathfrak{D})$. Suppose that system H is active("working", "alive") during any quantum of time if at least one source is working for this time.

We consider the following problem: how can we schedule the system H in such a way that the time of working(which may be understood also as the alive time of the whole system) is longest?

A *mixed hypergraph* is a triple $H = (X, \mathfrak{C}, \mathfrak{D})$, where X is a vertex set and each of $\mathfrak{C}$, $\mathfrak{D}$ is a list of subsets of X with at least two elements, the $\mathfrak{C}$-edges and $\mathfrak{D}$-edges, respectively. The concept of mixed hypergraph was inroduced by Vitaly Voloshin [11]. For example, let $H^* = (X, \mathfrak{C}, \mathfrak{D})$ be a mixed hypergraph where $X = \{a, b, c, d, e, f\}$, $\mathfrak{C} = \{\{c, d\}, \{a, e, f\}\}$, $\mathfrak{D} = \{\{b, f\}, \{a, b, e\}, \{a, b, c, d, e, f\}\}$. Then $\{c, d\}, \{a, e, f\}$ are $\mathfrak{C}$-edges and $\{b, f\}, \{a, b, e\}, \{a, b, c, d, e, f\}$ are $\mathfrak{D}$-edges. For convenience, we would use cd, aef as $\mathfrak{C}$-edges or $\mathfrak{D}$-edges instead of $\{c, d\}, \{a, e, f\}$. We introduce coloring rules which is different from traditional graph coloring. A *proper k-coloring* of a mixed hypergraph is a function from the vertex set to a set of k colors so that each $\mathfrak{C}$-edge has two vertices with a common color and each $\mathfrak{D}$-edge has two vertices with distinct colors. A *strict k-coloring* is a proper k-coloring using all k colors. In other words, a mixed hypergraph has a strict k-coloring if there exists an onto function $c : X \longrightarrow \{1, \ldots, k\}$ such that each $\mathfrak{C}$-edge has at least two vertices assigned a common value and each $\mathfrak{D}$-edge has at least two vertices assigned distinct values.

A mixed hypergraph is *strict k-colorable* if it has a strict k-coloring. For example, consider H^* again with coloring c $\mathbf{c}(a) = 1$, $\mathbf{c}(b) = 2$, $\mathbf{c}(c) = 1$, $\mathbf{c}(d) = 1$, $\mathbf{c}(e) = 1$, $\mathbf{c}(f) = 3$. Then, H^* is strict 3-colorable. The *feasible set* of H is $\{k$: H has a *strict k-coloring*$\}$ and is denoted by $S(H)$. The minimum number of feasible set of H is its *lower chromatic number* $\chi(H)$; the maximum number is its *upper chromatic number* $\overline{\chi}(H)$. If $\overline{\chi}(H)$ equals k, H is *maximum k-colorable*. If the feasible set of H is an empty set, it is uncolorable. For example, $H_1 = (X_1, \mathfrak{C}_1, \mathfrak{D}_1)$ where $X_1 = \{1, 2\}, \mathfrak{C}_1 = \{12\}, \mathfrak{D}_1 = \{12\}$. It is easy to check that H_1 could not be colored. A mixed hyprgraph $H_{l,m}$ is called $(l,m) - uniform$ if every $\mathfrak{C}$-edge of X is a l-subset and every $\mathfrak{D}$-edge is a m-subset of X. We use n to denote $|X|$ for the mixed hypergraph $H = (X, \mathfrak{C}, \mathfrak{D})$. Every proper k-coloring induces a partition of vertex set into color classes. These partitions are defined *feasible partitions*. The number of feasible partitions into k colors is denoted by r_k. Then the integer vector $R(H) = (r_1, r_2, \ldots, r_n)$ is called the *chromatic spectrum* of H. A mixed hypergraph has a *gap* if $\chi(H) < k < \overline{\chi}(H)$ and $r_k = 0$.

The theory of mixed hypergraph is growing rapidly. It has been used to solve problems in such areas as list-coloring of graphs[9], integer programming[4, 9], scheduling and molecular biology[10], coloting of block designs[1, 5, 6, 7, 8], and a variety of applied areas[10]. Other coloring problems considered from a different opinion can be rephrased in terms of mixed hypergraph[3].

In this paper, we talk about the *gap* of a special kind of mixed hypergraph $H_{l,m}$. We define the "k-partition" to make the theorem proved more easily. A "k-partition" of a mixed hypergraph $H = (X, \mathfrak{C}, \mathfrak{D})$ is a partition $\{A_1, A_2, \ldots, A_k\}$ of X if H has a strict k-coloring c with $c(A_i) = i;\ i = 1, 2 \ldots, k$. We discuss about the gap of $H_{l,m}$, because there need to be a uniform of sources and they want to avoid special life working hours.

A mixed hypergraph H without D-edges has lower chromatic number 1. In this case, color classed in a strict coloring can be combined to form a strict coloring using fewer colors, and thus $S(H) = \{1, \ldots, \overline{\chi}(H)\}$. Similarly, $\overline{\chi}(H) = n$ if and only if H has no C-edges. In this case, color classes in a strict coloring can be partitioned to form a proper coloring using more colors, and thus $S(H) = \{\chi(H), \ldots, n\}$. These two results can be applied in my first two cases of mixed hypergraph $H_{l,m}$ where $m = 0$ and $l = 0$, respectively. Then two special situations where $l = 2$ or $m = 2$ must be discussed independently, which is discussed in section 3. The main ideal is showed in section 4, we would refer to a algorithm to create a pap in mixed hypergraph $H_{l,m}$ where $l > 2$ and $m > 2$.

The smallest mixed hypergraphs with gap has been found. But it is still hard to know how could we use the best way to create a gap in other special kinds of mixed hypergraphs. The meaning of best way is that using minimum points or minimum edges. For example, I can't promise that my way is the best. So you can do this direction, if you are interested in my research after reading.

2 Coloring of a specific mixed hypergraph

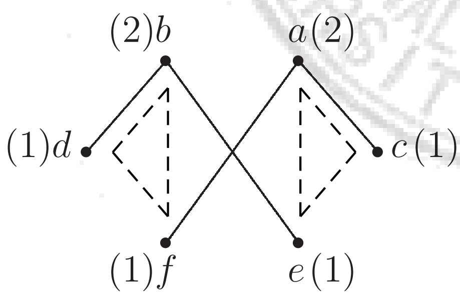
We now give you a example to help you understand many definitions of mixed hypergraph.

Example 2.1 *Let* $H = (X, \mathfrak{C}, \mathfrak{D})$ *be a mixed hypergraph where* $X = \{a, b, c, d, e, f\}$*,* $\mathfrak{C} = \{ace, bdf\}$*,* $\mathfrak{D} = \{ac, af, bd, be\}$*.*

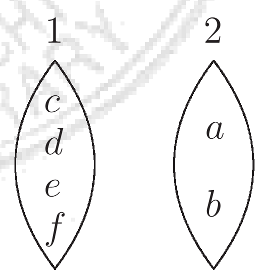
We show the coloring situations and introduce many definitions of mixed hypergraph.

"Strict 2-coloring on H" - - - - - - $\mathfrak{C}$-edge ———— $\mathfrak{D}$-edge

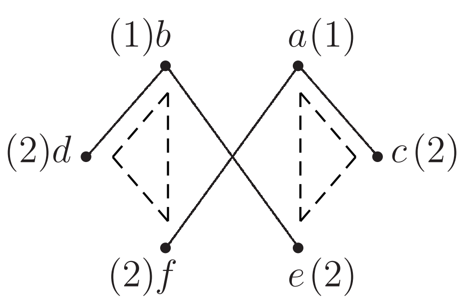
Coloring on H



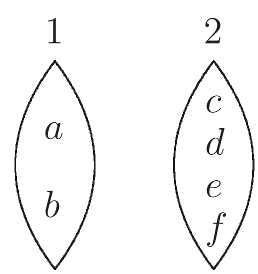
Coloring Partition



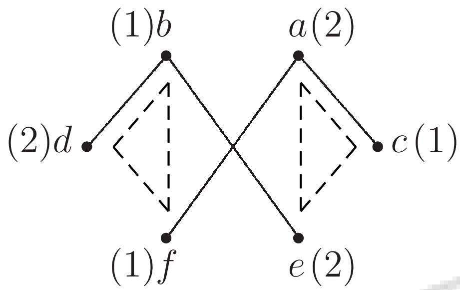
Coloring on H



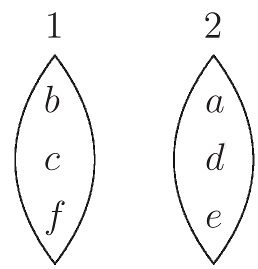
Coloring Partition



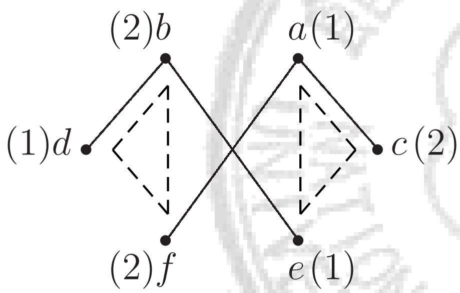
Coloring on H



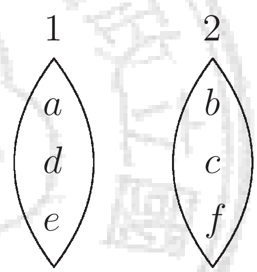
Coloring Partition



Coloring on H



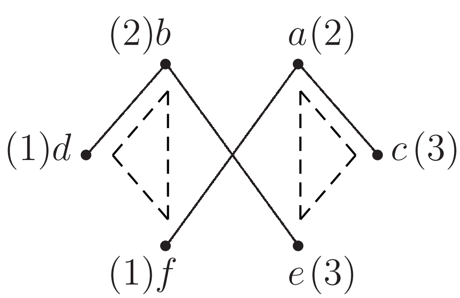
Coloring Partition



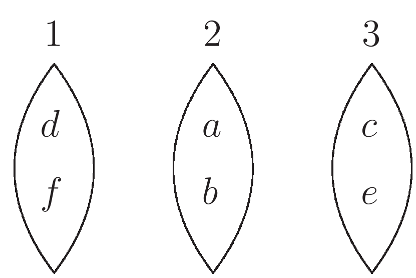
So $r_2 = 4$, H is 2-colorable.

"Strict 3-coloring on H"

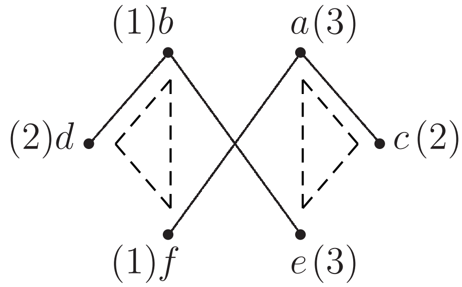
Coloring on H



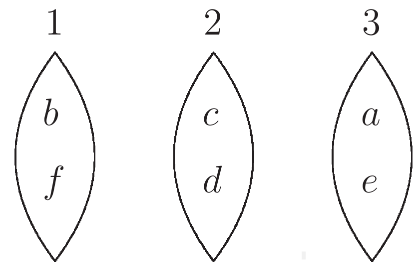
Coloring Partition



Coloring on H



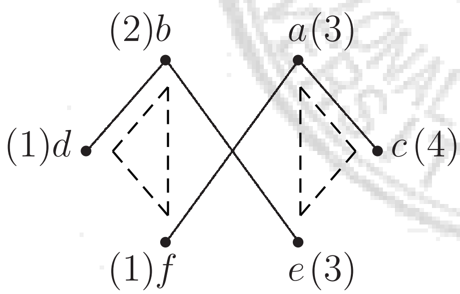
Coloring Partition



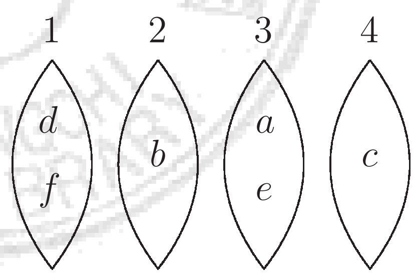
Actually, we count $r_3 = 48$, so H is 3-colorable. We show all coloring situations in the appendix 1.

"Strict 4-coloring on H"

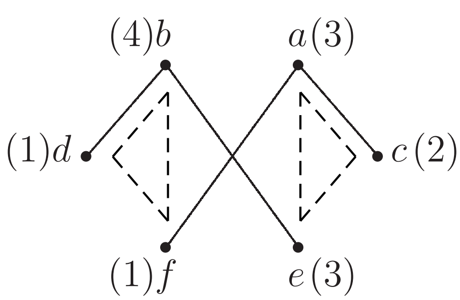
Coloring on H



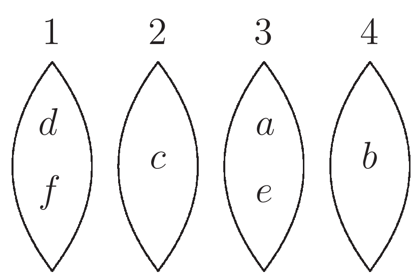
Coloring Partition



Coloring on H



Coloring Partition



In fact, it is easy to figure out $r_4 = 96$, so H is 4-colorable. We give you all coloring ways on H in the appendix 2.

At least, it is not impossible existing a strict 5-coloring on H, because if there is a strict 5-coloring on H, there must be 3 different colors in $\{a, c, e\}$ or $\{b, d, f\}$. Then it would contradict to $\mathfrak{C}$-edge ace or $\mathfrak{C}$-edge bdf. It is the same reason why H is not 6-colorable.

Now we know that $S(H) = \{2, 3, 4\}$, $\chi(H) = 2$, $\overline{\chi}(H) = 4$, $R(H) = (0, 4, 48, 96, 0, 0)$, and H is maximum 4-colorable without gap.

Actually, it is not easy to create a gap in a mixed hypergraph. We give a mixed hypergraph with gap. Let $H^* = (X, \mathfrak{C}, \mathfrak{D})$ with $X = \{a, b, c, d, e, f\}$, $\mathfrak{C} = \{acd, aef, bcd, bef, def, cdf\}$, $\mathfrak{D} = \{ab, ad, af, bc, be, cf, de\}$ has a gap. We would give a proof in section 3.

Later we begin to discuss the existence of gap in $H_{l,m}$ where $l \geq 2$ and $m \geq 2$.

3 The situation of gap in special case

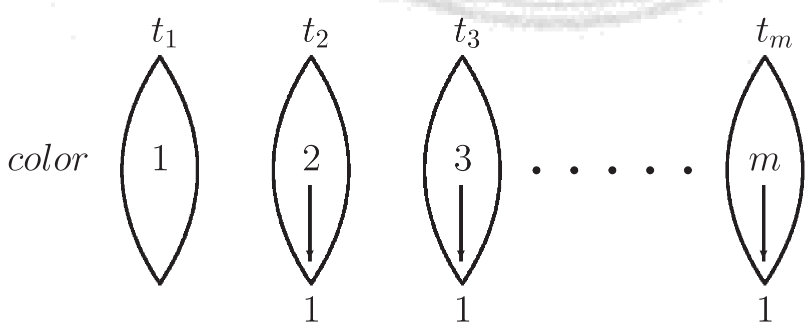
Definition 3.1 *If there is no $\mathfrak{D}$-edge in $\mathfrak{D}$ in a mixed hypergraph, we denote it $H = (X, \mathfrak{C}, \phi)$. And if there is no $\mathfrak{C}$-edge in $\mathfrak{C}$ in a mixed hypergraph, we denote it $H = (X, \phi, \mathfrak{D})$.*

Later we will discuss the gap of these two kinds of special mixed hypergraph.

Theorem 3.2 *$H = (X, \mathfrak{C}, \phi)$ has no gap.*

Proof. If H is maximum m-colorable, let c be a coloring on H.

$T = \{t_1, t_2, \ldots, t_m\}$ is a partition of H and setting $c(t_i) = i$ completes a strict m-coloring.

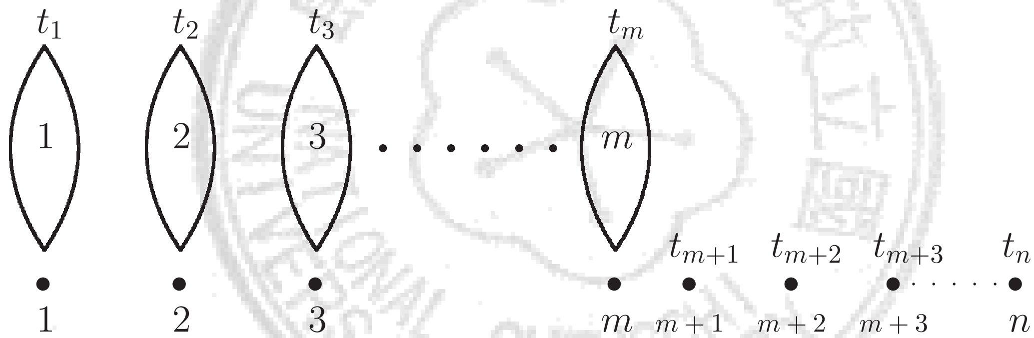


We can find that if all vertices in t_i have the same color ($i \in [\,m\,]$), and this coloring c can accord with all C-edges. Set $c(t_i) = 1\,(i = 2 \sim m)$ step by step, then the feasible set of H is $\{1, 2, 3, \ldots, m\}$. $H = (X, \mathfrak{C}, \phi)$ has no gap. ❑

In like manner, a mixed hypergraph only without $\mathfrak{C}$-edges doesn't has a pap, either.

Theorem 3.3 *$H = (X, \phi, \mathfrak{D})$ has no gap.*

Proof. If $\chi(H)$ equal m,there is a coloring c on H. Set $T = \{t_1, t_2, \ldots, t_m\}$ is a partition of H and setting $c(t_i) = i$ completes a strict m-coloring.



We can see that if we pick out some vertex in t_1 and let the single vertex be t_{m+1} and $c'(t_{m+1}) = m + 1$. It is a new partition and a new coloring c', which still accords with all D-edges. We can do the same thing step by step until $|t_1| = 1$. Then we can do this kind of job (from t_1 to t_m)to make a new partition $\{t_1^{'}, t_2^{'}, t_3^{'} \ldots, t_n^{'}\}$ and $|t_i^{'}| = 1$ for $i = 1 \sim m$. Then we know $S(H) = \{m, m+1, m+2, \ldots, n\}$, then $H = (X, \phi, \mathfrak{D})$ has no gap. ❑

We would introduce a construction *contracting* to keep $S(H)$ of the mixed hypergraph the same in a special case, thought eliminating points and edges.

Construction 3.4 *(Contracting)*

$H = (X, \mathfrak{C}, \mathfrak{D})$ *is a mixed hypergraph*$(\,|\,\mathfrak{C}\,| = s, |\,\mathfrak{D}\,| = t\,)$. *If there is a* $C_i = \{x_k, x_l\}$ *for some* i *in* $\mathfrak{C}$, *then we can get* $H_1 = (X_1, \mathfrak{C}^1, \mathfrak{D}^1)$, *where*

1. $X_1 = (X \backslash \{x_k, x_l\}) \cup \{\,y\,\}$, y *is a new vertex.*
2. $x_k \in C_i$ *or* $x_l \in C_i$, $C_i \in \mathfrak{C}$ *and* $i = 1 \sim s$, *then* $C_i^1 = (C_i \backslash \{x_k, x_l\} \cup \{\,y\,\}$ *otherwise* $C_i^1 = C_i$
3. $x_k \in D_i$ *or* $x_l \in D_i$, $D_i \in \mathfrak{D}$ *and* $i = 1 \sim t$, *then* $D_i^1 = (D_i \backslash \{x_k, x_l\} \cup \{\,y\,\}$ *otherwise* $D_i^1 = D_i$

Theorem 3.5 *Let* H *be a mixed hypergraph with feasible set* S. *If the size of each* C*-edge in* H *is 2, then the mixed hypergraph* H_1 *obtained from* H *via construction3.4 has feasible set* S.

Proof. We can assume that $C_1 = \{x_1, x_2\}$, $|\,X\,| = n$.Now c is a proper coloring of H if and only if c_1 is a proper coloring of H_1 by setting $c(x_1) = c(x_2) = c_1(y)$ and $c(x_j) = c_1(x_j)(j = 3 \sim n)$, then the number of colors used is the same. ❑

Lemma 3.6 *A mixed hypergraph* $H = (X, \mathfrak{C}, \mathfrak{D})$ *has no gap if the size of all* C*-edges is 2.*

Proof. We prove the statement by induction on $|\,\mathfrak{C}\,|$.

Basic step: $|\mathfrak{C}| = 0$. $H = (X, \phi, \mathfrak{D})$ has no gap by theorem 3.3.

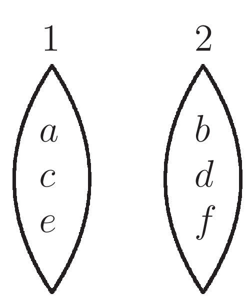
Induction step: We suppose that the claim holds for $|\mathfrak{C}| < n$. If H has n C-edges, then the mixed hypergraph H_1 is obtained from H via construction 3.4. By the induction hypothesis, H_1 has no gap. And $S(H) = S(H_1)$ by Theorem 3.5, so H has no gap. Then A mixed hypergraph $H = (X, \mathfrak{C}, \mathfrak{D})$ has no gap if the size of all C-edges is 2. ❑

But if the size of each D-edge in H is 2, it has a pap. We can give an example that this kind of H has a pap.

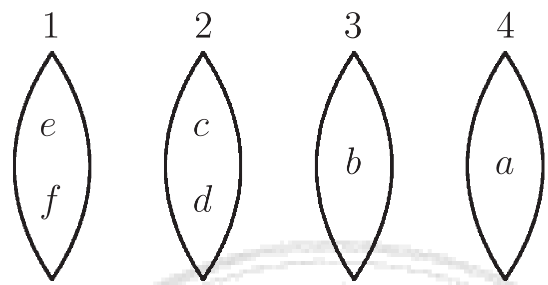
Example 3.7 *The feasible set of* $H^* = (X, \mathfrak{C}, \mathfrak{D})$ *where* $X = \{a, b, c, d, e, f\}$, $\mathfrak{C} = \{acd, aef, bcd, bef, def, cdf\}$, *and* $\mathfrak{D} = \{ab, ad, af, bc, be, cf, de\}$ *is* $\{2, 4\}$.

Proof. Each partition of H^* with 2-coloring and 4-coloring is unique, and they are showed below.

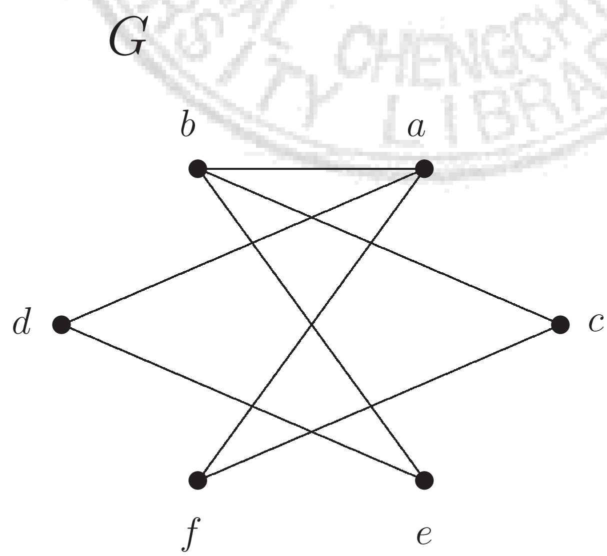
$The\ partition\ of\ H^*\ with\ 2-coloring:$



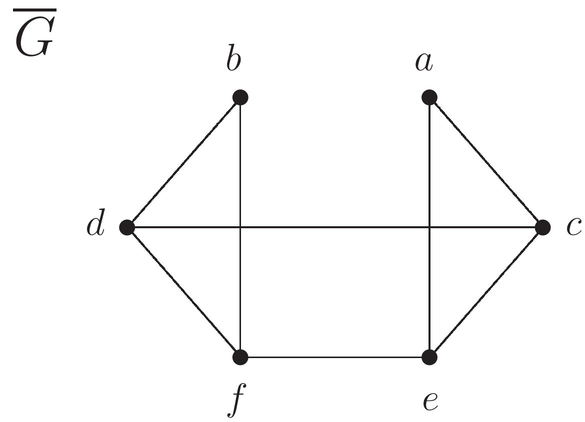
$The\ partition\ of\ H^*\ with\ 4-coloring:$



Then we show why H^* is not 3-colorable. The partition of H^* with 3-coloring has three kind of sets $\{4, 1, 1\}$, $\{3, 2, 1\}$ and $\{2, 2, 2\}$. First we start from a graph G with vertex set $\{a, b, c, d, e, f\}$ and edge set $\mathfrak{D} = \{ab, ad, af, bc, be, cf, de\}$.

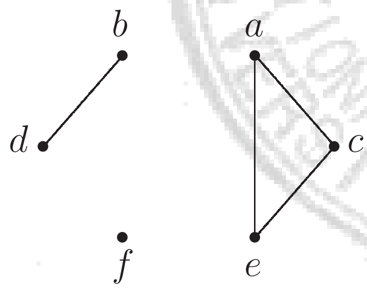


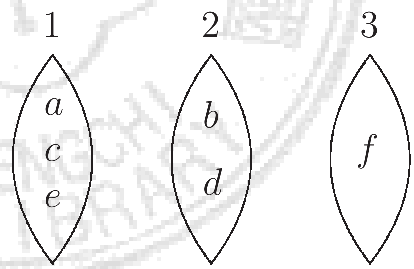
Then we can get the complement of graph $\overline{G}$.



Because there is no K_4 clique in $\overline{G}$, the partition of H with 3 coloring must be not the kind of set $\{4, 1, 1\}$. There are two K_3 cliques in $\overline{G}$, so there are six situations discussed in the partition of set$\{3, 2, 1\}$.

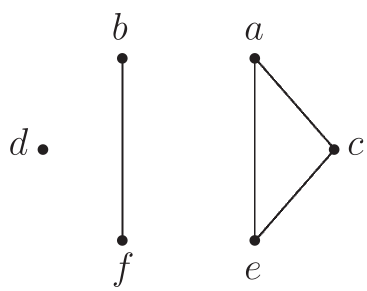
Situation in $\overline{G}$ Corresponding Partition

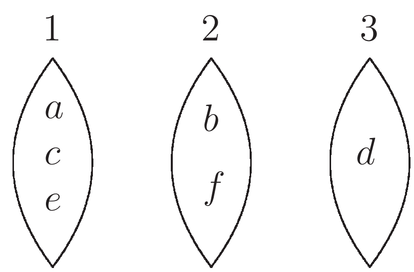




This kind of partition contradict $\mathfrak{C}$-edge, bef

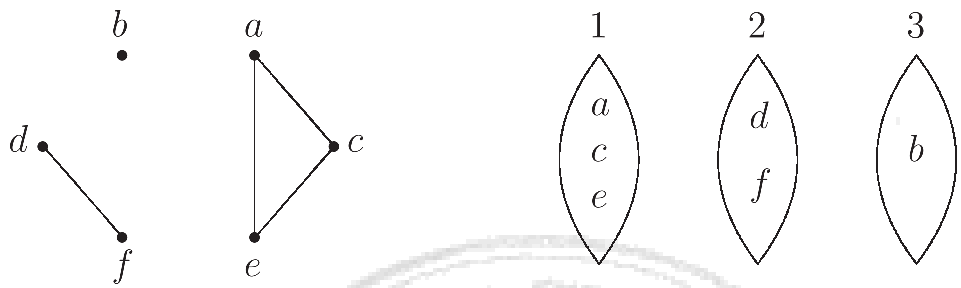
Situation in $\overline{G}$ Corresponding Partition





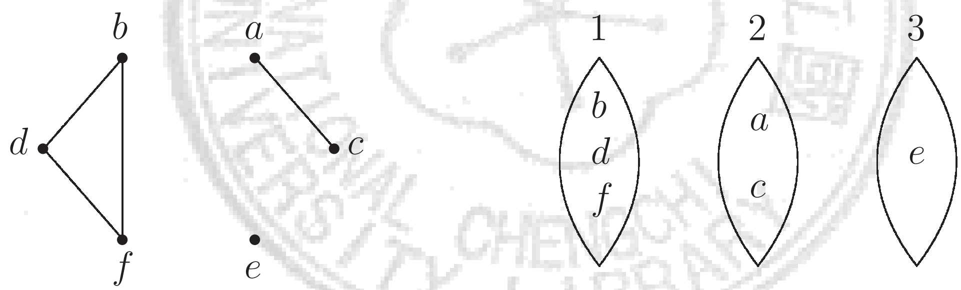
This kind of partition contradict $\mathfrak{C}$-edge, def

Situation in $\overline{G}$ Corresponding Partition



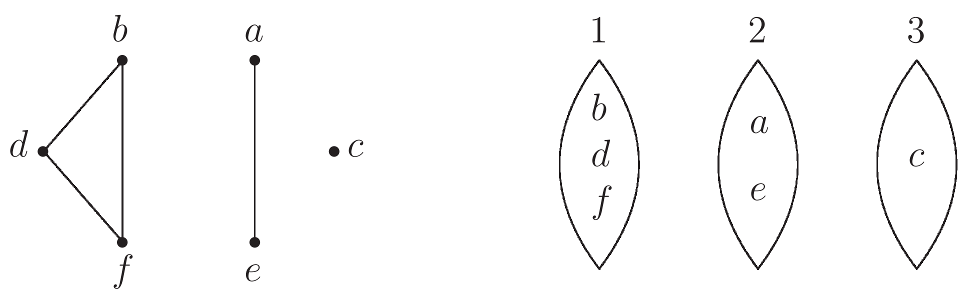
This kind of partition contradict $\mathfrak{D}$-edge, bef

Situation in $\overline{G}$ Corresponding Partition



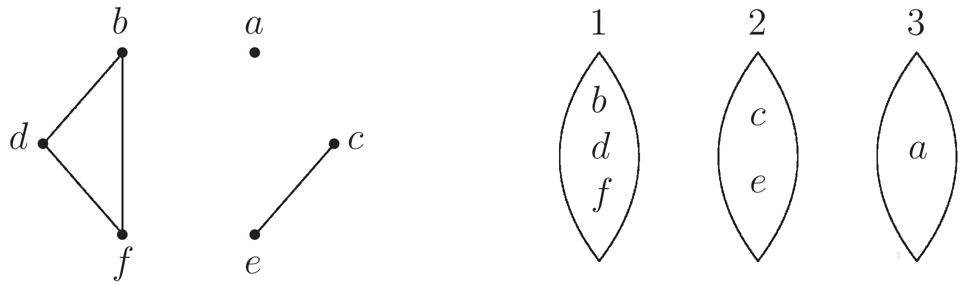
This kind of partition contradict $\mathfrak{C}$-edge, aef

Situation in $\overline{G}$ Corresponding Partition



This kind of partition contradict $\mathfrak{C}$-edge, acd

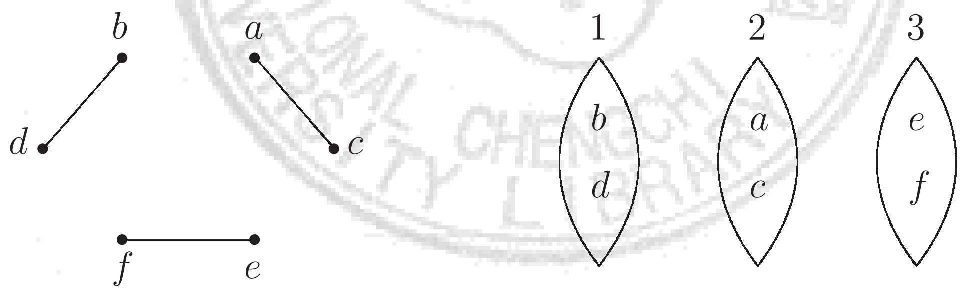
Situation in $\overline{G}$ Corresponding Partition



This kind of partition contradict $\mathfrak{C}$-edge, aef

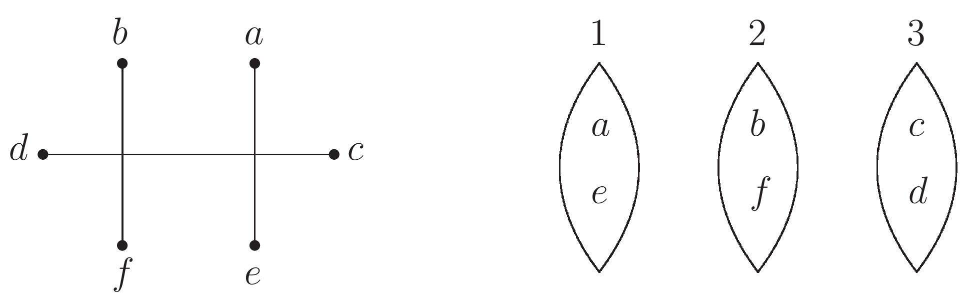
So the partition of H with 3 coloring must be not the kind of set$\{3, 2, 1\}$. Now there are two kind of perfect matching in $\overline{G}$, so there are two situations discussed in the partition of set$\{2, 2, 2\}$.

Situation in $\overline{G}$ Corresponding Partition



This kind of partition contradict C-edge, cdf

Situation in $\overline{G}$ Corresponding Partition



This kind of partition contradict C-edge, def

So the partition of H with 3 coloring must be not the kind of set $\{2, 2, 2\}$. We can conclude that H^* is not a 3-colorable mixed hypergraph.

❑

We later can use this example to construct $(k, 2)$-uniform mixed hypergraph with gap.

4 Algorithm of gap in (l, m)-uniform mixed hypergraph

Definition 4.1 *Mixed hypergraph H is called (l, m)-uniform and denoted $H_{l,m}$ if every $\mathfrak{C}$-edge is a l-subset of X, and every $\mathfrak{D}$-edge is a m-subset of X.*

Theorem 4.2 *If H is (2,k)-uniform, then H has no gap.*

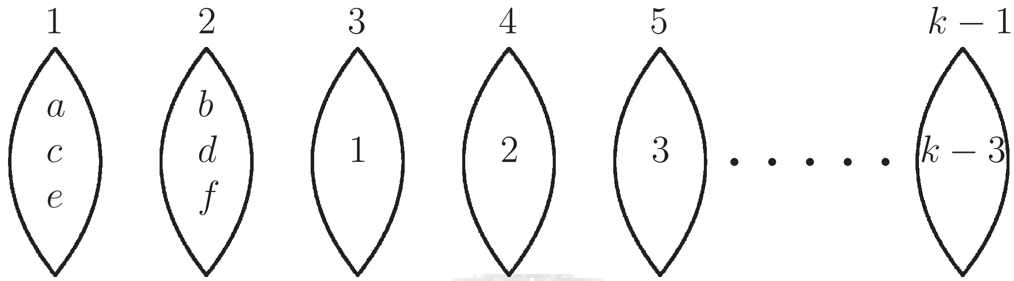
***Proof*.** This can be proved by Lemma 3.6 ❑

Construction 4.3 *We make a mixed hypergraph $H \in H_{k,2}(k \geq 3)$ with vertex set $\{a, b, c, d, e, f, 1, \ldots, k-3\}$. Let T be the set of the form (i, j) for $i, j \in \{1, 2, \ldots, k-3\}$. Let U be the set of the form $\{(i, a), (i, b), (i, c), (i, d), (i, e), (i, f)\}$. $\mathfrak{C} = \{(1, \ldots, k-3, acd), (1, \ldots, k-3, aef), (1, \ldots, k-3, bcd), (1, \ldots, k-3, bef), (1, \ldots, k-3, def), (1, \ldots, k-3, cdf)\}$, $\mathfrak{D} = \{ab, ad, af, bc, be, cf, de\} \cup T \cup U$.*

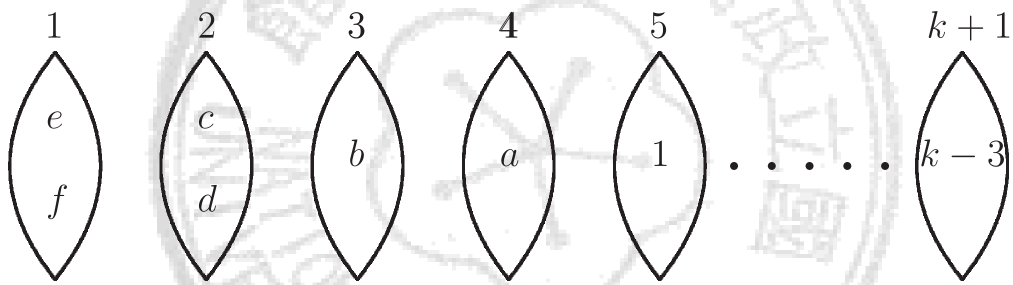
Lemma 4.4 *The feasible set of the (k,2)-unifoerm mixed hypergraph H in Construction 4.3 is $\{k-1, k+1\}$, then H has a pap.*

***Proof*.** We can easily see that each partition of $H_{k,2}$ with $(k-1)$-coloring and $(k+1)$-coloring is unique, and they are showed below.

The partition of H^* *with* $(k-1)-$*coloring* :



The partition of H^* *with* $(k+1)-$*coloring* :



❑

Definition 4.5 *We define a mixed hypergraph* H_{2t} *with vertex set* $\{a_1, \ldots, a_t, b_1, \ldots, b_t\}$.

Construction 4.6 *Let* $H_{2t} = (X, \mathfrak{C}, \mathfrak{D})$ *be a* $(n, 2p)$*-uniform mixed hypergraph where* $n \geq 3, p \geq 2, t = (2n-3)(p-1)+1$.

1. $\mathfrak{C} = \{a_i b_i a_{l_1} a_{l_2} \ldots a_{l_{n-2}}\} \cup \{a_i b_i b_{l_1} b_{l_2} \ldots b_{l_{n-2}}\}$, *where* $i, l_1, l_2, \ldots, l_{n-2}$ $\in \{1, 2, \ldots, t\}$ *are pairwise distinct.*

2. $\mathfrak{D} = \{a_{j_1} a_{j_2} \ldots a_{j_p} b_{j_1} b_{j_2} \ldots b_{j_p}\}$ *where* $j_1, j_2, \ldots, j_p \in \{1, 2, \ldots, t\}$ *are pairwise distinct.*

Theorem 4.7 *The feasible set of the mixed hypergraph* H_{2t} *in construction* 4.6 *is* $\{2, \ldots, 2n-4, 2n-2, \ldots, t\}$.

Proof. Let c be an arbitrary coloring of H_{2t}. There are two cases of coloring c.

case 1. If $c(a_i) \neq c(b_i)$ for some i, without loss of generality, we can let $i = 1$.This kind of C-edges $(a_1 b_1 a_{l_1} \ldots a_{l_{n-2}})$, $l_1, \ldots, l_{n-2} \in \{2, 3, \ldots, t\}$, forces the coloring of the points $\{a_1, \ldots, a_t, b_1\}$ to use at most $n-1$ colors. Because if there are more than n colors in $\{a_1, \ldots, a_t, b_1\}$, it will contradict one of $\mathfrak{C}$-edges, $(a_1 b_1 a_{l_1} \ldots a_{l_{n-2}})$ where $l_1, \ldots, l_{n-2} \in \{2, 3, \ldots, t\}$. Other kind of C-edges $(a_1 b_1 b_{l_1} \ldots b_{l_{n-2}})$ where $l_1, \ldots, l_{n-2} \in \{2, 3, \ldots, t\}$, forces the coloring of the points $\{b_1, \ldots, b_t, a_1\}$ to use at most $n-1$ colors. The reason is the same. When $c(a_i) \neq c(b_i)$ for some i, H_{2t} has at most $2n-4$ colors. Then we give a coloring c to complete a strict $(2n-4)$-coloring.

$$
\begin{array}{lll}
\textit{set} & c(a_i) = 1, i \in [\, t-(n-3)\,] & c(b_1) = 2, i \in [\, t-(n-3)\,] \\
 & c(a_{t-(n-3)+1}) = 3 & c(b_{t-(n-3)+1}) = 4 \\
 & c(a_{t-(n-3)+2}) = 5 & c(b_{t-(n-3)+2}) = 6 \\
 & \vdots & \vdots \\
 & c(a_t) = 2n-5 & c(b_t) = 2n-4
\end{array}
$$

This is a strict $(2n-4)$-coloring of H_{2t}. Then we do a job that $c(a_j) = 1$, $j = \{t-n+4, t-n+5, \ldots, t\}$ step by step and that

$c(b_j) = 2$, $j = \{t-n+4, t-n+5, \ldots, t\}$ step by step. We can easily know that H_{2t} has a feasible set $\{2, 3, \ldots, 2n-4\}$ when $c(a_i) \neq c(b_i)$ for some i.

case 2. If $c(a_i) = c(b_i)$ for $i \in \{1, 2, \ldots, t\}$, all D-edges force that H_{2t} has at least $2n-2$ colors by pigeonhole principle. Then we give a coloring c to complete a strict $(2n-2)$-coloring.

set

$$\begin{cases} c(a_1) = 1 & c(b_1) = 1 \\ \quad\vdots & \quad\vdots \\ c(a_{p-1}) = 1 & c(b_{p-1}) = 1 \end{cases}$$

$$\begin{cases} c(a_p) = 2 & c(b_p) = 2 \\ \quad\vdots & \quad\vdots \\ c(a_{2(p-1)}) = 2 & c(b_{2(p-1)}) = 2 \end{cases}$$

$$\begin{array}{ll} \quad\vdots & \quad\vdots \end{array}$$

$$\begin{cases} c(a_{(2n-4)(p-1)+1}) = 2n-3 & c(b_{(2n-4)(p-1)+1}) = 2n-3 \\ \quad\vdots & \quad\vdots \\ c(a_{(2n-3)(p-1)}) = 2n-3 & c(b_{(2n-3)(p-1)}) = 2n-3 \end{cases}$$

$$c(a_{(2n-3)(p-1)+1}) = 2n-2 \quad c(b_{(2n-3)(p-1)+1}) = 2n-2$$

$$note : t = (2n-3)(p-1)+1$$

This is a strict $(2n-2)$-coloring of H_{2t}. We can recolor H_{2t} step by step to form the following coloring.

$$
\begin{array}{lll}
set & c(a_1)=1 & c(b_1)=1 \\
 & c(a_2)=2 & c(b_2)=2 \\
 & c(a_3)=3 & c(b_3)=3 \\
 & \vdots & \vdots \\
 & c(a_t)=t & c(b_t)=t \\
 & & note: t=(2n-3)(p-1)+1
\end{array}
$$

We can easily know that H_{2t} has a feasible set $\{2n-2\,,\,\ldots\,,(2n-3)(p-1)+1\}$ when $c(a_i)=c(b_i)$ for all i.

Then the feasible set of mixed hypergraph H_{2t} in construction4.6 is $\{2,\ldots,2n-4,2n-2,\ldots,t\}$. ❑

Construction 4.8 *Let $H_{2t}=(X,\mathfrak{C},\mathfrak{D})$ be a $(n,2p-1)$-uniform mixed hypergraph where $n\geq 3, p\geq 2, t=(2n-3)(p-1)+1$.*

1. $\mathfrak{C}=\{a_ib_ia_{l_1}a_{l_2}\ldots a_{l_{n-2}}\}\bigcup\{a_ib_ib_{l_1}b_{l_2}\ldots b_{l_{n-2}}\}$ *where* $i,l_1,l_2,\ldots,l_{n-2}\in\{1,2,\ldots,t\}$ *are pairwise distinct.*
2. $\mathfrak{D}=\{a_{j_1}a_{j_2}\ldots a_{j_p}b_{j_1}b_{j_2}\ldots b_{j_{p-1}}\}$ *where* $j_1,j_2,\ldots,j_p\in\{1,2,\ldots,t\}$ *are pairwise distinct.*

Theorem 4.9 *The feasible set of the mixed hypergraph H_{2t} in construction* 4.8 *is* $\{2, \ldots, 2n-4, 2n-2, \ldots, t\}$.

Proof. Let c be an arbitrary coloring of H_{2t}. There are two cases of coloring c.

case 1. If $c(a_i) \neq c(b_i)$ for some i, without loss of generality, we can let $i = 1$.This kind of C-edges $(a_1 b_1 a_{l_1} \ldots a_{l_{n-2}})$, $l_1, \ldots, l_{n-2} \in \{2, 3, \ldots, t\}$, forces the coloring of the points $\{a_1, \ldots, a_t, b_1\}$ to use at most $n-1$ colors. Because if there are more than n colors in $\{a_1, \ldots, a_t, b_1\}$, it will contradict one of $\mathfrak{C}$-edges, $(a_1 b_1 a_{l_1} \ldots a_{l_{n-2}})$ where $l_1, \ldots, l_{n-2} \in \{2, 3, \ldots, t\}$. Other kind of C-edges $(a_1 b_1 b_{l_1} \ldots b_{l_{n-2}})$ where $l_1, \ldots, l_{n-2} \in \{2, 3, \ldots, t\}$, forces the coloring of the points $\{b_1, \ldots, b_t, a_1\}$ to use at most $n-1$ colors. The reason is the same. When $c(a_i) \neq c(b_i)$ for some i, H_{2t} has at most $2n-4$ colors. Then we give a coloring c to complete a strict $(2n-4)$-coloring.

$$
\begin{array}{lll}
\textit{set} & c(a_i) = 1, i \in [\,t-(n-3)\,] & c(b_1) = 2, i \in [\,t-(n-3)\,] \\
& c(a_{t-(n-3)+1}) = 3 & c(b_{t-(n-3)+1}) = 4 \\
& c(a_{t-(n-3)+2}) = 5 & c(b_{t-(n-3)+2}) = 6 \\
& \vdots & \vdots \\
& c(a_t) = 2n-5 & c(b_t) = 2n-4
\end{array}
$$

This is a strict $(2n-4)$-coloring of H_{2t}. Then we do a job that $c(a_j) = 1$, $j = \{t-n+4, t-n+5, \ldots, t\}$ step by step and that

$c(b_j) = 2$, $j = \{t-n+4, t-n+5, \ldots, t\}$ step by step. We can easily know that H_{2t} has a feasible set $\{2, 3, \ldots, 2n-4\}$ when $c(a_i) \neq c(b_i)$ for some i.

case 2. If $c(a_i) = c(b_i)$ for $i \in \{1, 2, \ldots, t\}$, all D-edges force that H_{2t} has at least $2n-2$ colors by pigeonhole principle. Then we give a coloring c to complete a strict $(2n-2)$-coloring.

set

$$\begin{cases} c(a_1) = 1 & c(b_1) = 1 \\ \vdots & \vdots \\ c(a_{p-1}) = 1 & c(b_{p-1}) = 1 \end{cases}$$

$$\begin{cases} c(a_p) = 2 & c(b_p) = 2 \\ \vdots & \vdots \\ c(a_{2(p-1)}) = 2 & c(b_{2(p-1)}) = 2 \end{cases}$$

$$\vdots \qquad\qquad \vdots$$

$$\begin{cases} c(a_{(2n-4)(p-1)+1}) = 2n-3 & c(b_{(2n-4)(p-1)+1}) = 2n-3 \\ \vdots & \vdots \\ c(a_{(2n-3)(p-1)}) = 2n-3 & c(b_{(2n-3)(p-1)}) = 2n-3 \end{cases}$$

$$c(a_{(2n-3)(p-1)+1}) = 2n-2 \quad c(b_{(2n-3)(p-1)+1}) = 2n-2$$

$$note : t = (2n-3)(p-1)+1$$

This is a strict $(2n-2)$-coloring of H_{2t}. We can recolor H_{2t} step by step to form the following coloring.

$$
\begin{array}{lll}
set & c(a_1) = 1 & c(b_1) = 1 \\
 & c(a_2) = 2 & c(b_2) = 2 \\
 & c(a_3) = 3 & c(b_3) = 3 \\
 & \vdots & \vdots \\
 & c(a_t) = t & c(b_t) = t \\
 & & note : t = (2n-3)(p-1)+1
\end{array}
$$

We can easily know that H_{2t} has a feasible set $\{2n-2\,, \ \ldots\ , (2n-3)(p-1)+1\}$ when $c(a_i) = c(b_i)$ for all i.

Then the feasible set of mixed hypergraph H_{2t} in construction4.6 is $\{2, \ldots, 2n-4, 2n-2, \ldots, t\}$. ❑

We can control the length of gap.

Corollary 4.10 *The feasible set of mixed hypergraph H_{2t} in construction 4.6 and construction 4.8 is $\{2, \ldots, 2n-4, 2n+s-3, \ldots, t\}$, if $t = (2n-4+s)(p-1)+1$ and $s \geq 1$.*

Proof. Let c be an arbitrary coloring of H_{2t}. There are two cases of coloring c.

case.1 If $c(a_i) \neq c(b_i)$ for some i, H_{2t} has a feasible set $\{2, \ldots, 2n-4\}$ by the proof in theorem 4.7 and theorem 4.9.

case.2 If $c(a_i) = c(b_i)$ for $i \in \{1, 2, \ldots, t\}$, the pigeonhole principle forces that H_{2t} has at least $(2n - 4 + s) + 1$ colors.

So it is possible to control the size of gap in mixed hypergraph H_{2t}. ❑

Theorem 4.11 *For each integer $l \geq 3$, $m \geq 2$, there exists a (l, m)-uniform mixed hypergraph $H = (X, \mathfrak{C}, \mathfrak{D})$ whose feasible set contains a gap.*

Proof. This can be proved by theorem 4.7 and theorem 4.9. ❑

Example 4.12 *Now we can construct a $(4, 3)$-uniform mixed hypergraph with gap.*

We can use Construct 4.8.Then $n = 4$ and $p = 2$ imply $t = 6$. Now we start this work with $H_{2 \cdot 6}$ where

1. $X = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$

2. $\mathfrak{C} = \{ (a_1b_1a_2a_3), (a_1b_1a_2a_4), (a_1b_1a_2a_5), (a_1b_1a_2a_6), (a_1b_1a_3a_4),$
$(a_1b_1a_3a_5), (a_1b_1a_3a_6), (a_1b_1a_4a_5), (a_1b_1a_4a_6), (a_1b_1a_5a_6),$
$(a_1b_1b_2b_3), (a_1b_1b_2b_4), (a_1b_1b_2b_5), (a_1b_1b_2b_6), (a_1b_1b_3b_4),$
$(a_1b_1b_3b_5), (a_1b_1b_3b_6), (a_1b_1b_4b_5), (a_1b_1b_4b_6), (a_1b_1b_5b_6),$
$(a_2b_2a_1a_3), (a_2b_2a_1a_4), (a_2b_2a_1a_5), (a_2b_2a_1a_6), (a_2b_2a_3a_4),$
$(a_2b_2a_3a_5), (a_2b_2a_3a_6), (a_2b_2a_4a_5), (a_2b_2a_4a_6), (a_2b_2a_5a_6),$
$(a_2b_2b_1b_3), (a_2b_2b_1b_4), (a_2b_2b_1b_5), (a_2b_2b_1b_6), (a_2b_2b_3b_4),$
$(a_2b_2b_3b_5), (a_2b_2b_3b_6), (a_2b_2b_4b_5), (a_2b_2b_4b_6), (a_2b_2b_5b_6),$
$(a_3b_3a_2a_1), (a_3b_3a_2a_4), (a_3b_3a_2a_5), (a_3b_3a_2a_6), (a_3b_3a_1a_4),$
$(a_3b_3a_1a_5), (a_3b_3a_1a_6), (a_3b_3a_4a_5), (a_3b_3a_4a_6), (a_3b_3a_5a_6),$
$(a_3b_3b_2b_1), (a_3b_3b_2b_4), (a_3b_3b_2b_5), (a_3b_3b_2b_6), (a_3b_3b_1b_4),$
$(a_3b_3b_1b_5), (a_3b_3b_1b_6), (a_3b_3b_4b_5), (a_3b_3b_4b_6), (a_3b_3b_5b_6),$
$(a_4b_4a_2a_3), (a_4b_4a_2a_1), (a_4b_4a_2a_5), (a_4b_4a_2a_6), (a_4b_4a_3a_1),$
$(a_4b_4a_3a_5), (a_4b_4a_3a_6), (a_4b_4a_1a_5), (a_4b_4a_1a_6), (a_4b_4a_5a_6),$
$(a_4b_4b_2b_3), (a_4b_4b_2b_1), (a_4b_4b_2b_5), (a_4b_4b_2b_6), (a_4b_4b_3b_1),$
$(a_4b_4b_3b_5), (a_4b_4b_3b_6), (a_4b_4b_1b_5), (a_4b_4b_1b_6), (a_4b_4b_5b_6),$
$(a_5b_5a_2a_3), (a_5b_5a_2a_4), (a_5b_5a_2a_1), (a_5b_5a_2a_6), (a_5b_5a_3a_4),$
$(a_5b_5a_3a_1), (a_5b_5a_3a_6), (a_5b_5a_4a_1), (a_5b_5a_4a_6), (a_5b_5a_1a_6),$
$(a_5b_5b_2b_3), (a_5b_5b_2b_4), (a_5b_5b_2b_1), (a_5b_5b_2b_6), (a_5b_5b_3b_4),$
$(a_5b_5b_3b_1), (a_5b_5b_3b_6), (a_5b_5b_4b_1), (a_5b_5b_4b_6), (a_5b_5b_1b_6),$
$(a_6b_6a_2a_3), (a_6b_6a_2a_4), (a_6b_6a_2a_5), (a_6b_6a_2a_1), (a_6b_6a_3a_4),$
$(a_6b_6a_3a_5), (a_6b_6a_3a_1), (a_6b_6a_4a_5), (a_6b_6a_4a_1), (a_6b_6a_5a_1),$
$(a_6b_6b_2b_3), (a_6b_6b_2b_4), (a_6b_6b_2b_5), (a_6b_6b_2b_1), (a_6b_6b_3b_4),$
$(a_6b_6b_3b_5), (a_6b_6b_3b_1), (a_6b_6b_4b_5), (a_6b_6b_4b_1), (a_6b_6b_5b_1) \}$

3. $\mathfrak{D} = \{ (a_1b_1a_2), (a_1b_1a_3), (a_1b_1a_4), (a_1b_1a_5), (a_1b_1a_6), (a_2b_2a_1),$
$(a_2b_2a_3), (a_2b_2a_4), (a_2b_2a_5), (a_2b_2a_6), (a_3b_3a_2), (a_3b_3a_1),$
$(a_3b_3a_4), (a_3b_3a_5), (a_3b_3a_6), (a_4b_4a_2), (a_4b_4a_3), (a_4b_4a_1),$
$(a_4b_4a_5), (a_4b_4a_6), (a_5b_5a_2), (a_5b_5a_3), (a_5b_5a_4), (a_5b_5a_1),$
$(a_5b_5a_6), (a_6b_6a_2), (a_6b_6a_3), (a_6b_6a_4), (a_6b_6a_5), (a_6b_6a_1) \}$

Now we give every coloring situations below.

$4-coloring \quad c(a_i) \neq c(b_i)$, for $i \in \{1, 2, \ldots, 6\}$

	a_1	a_2	a_3	a_4	a_5	a_6
color	1	1	1	1	1	3
	b_1	b_2	b_3	b_4	b_5	b_6
color	2	2	2	2	2	4

$3-coloring \quad c(a_i) \neq c(b_i)$, for $i \in \{1, 2, \ldots, 6\}$

	a_1	a_2	a_3	a_4	a_5	a_6
color	1	1	1	1	1	1
	b_1	b_2	b_3	b_4	b_5	b_6
color	2	2	2	2	2	4

$2-coloring \quad c(a_i) \neq c(b_i)$, for $i \in \{1, 2, \ldots, 6\}$

	a_1	a_2	a_3	a_4	a_5	a_6
color	1	1	1	1	1	1
	b_1	b_2	b_3	b_4	b_5	b_6
color	2	2	2	2	2	2

$6-coloring \quad c(a_i)=c(b_i)$, for $i \in \{1,2,\dots,6\}$

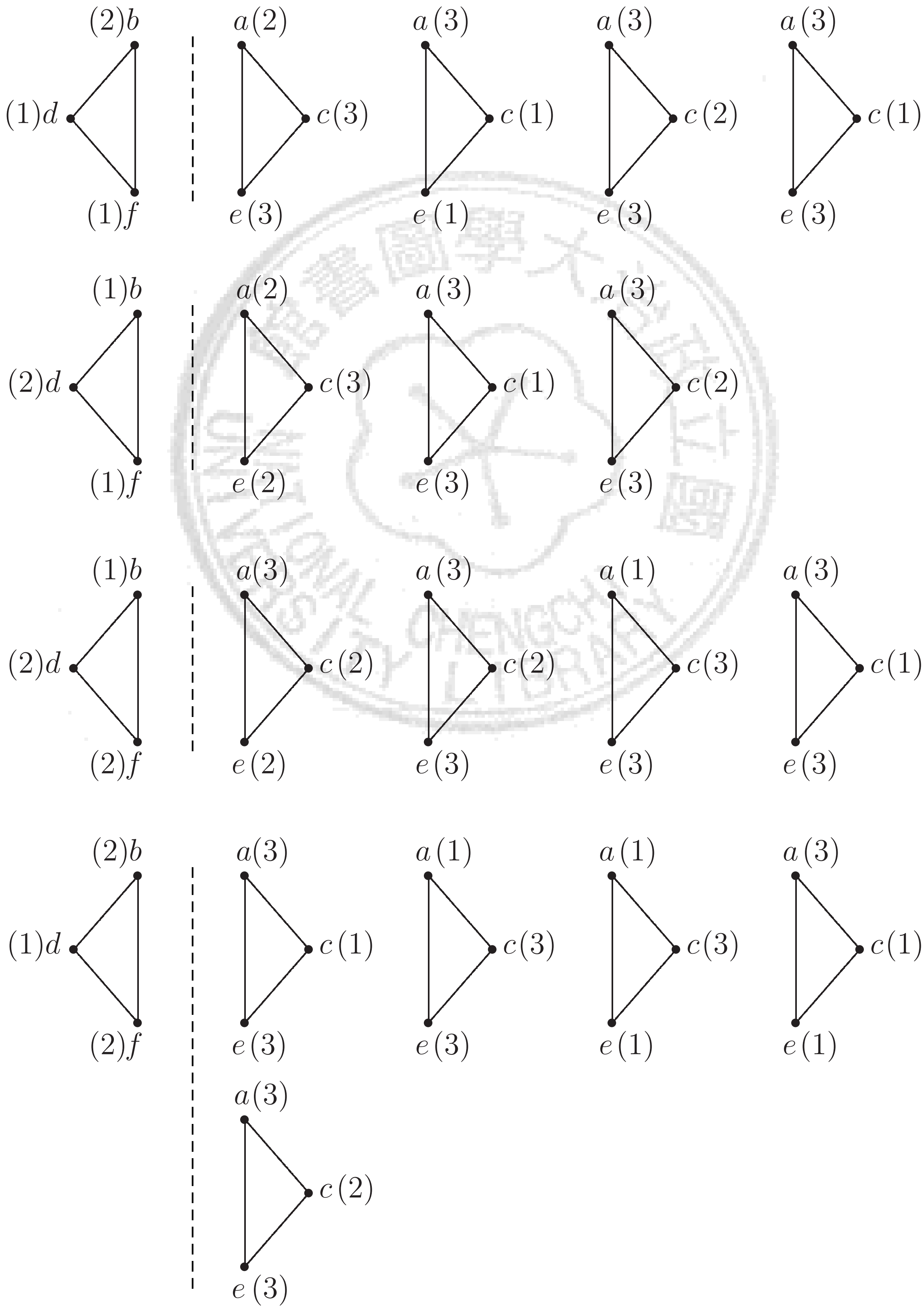
	a_1	a_2	a_3	a_4	a_5	a_6
color	1	2	3	4	5	6
	b_1	b_2	b_3	b_4	b_5	b_6
color	1	2	3	4	5	6

Then the feasible of this kind of (l,m)-uniform mixed hypergraph $H_{2\cdot 6}$ is $\{2,3,4,6\}$. In other words, it has a pap.

We are not sure this is the best way to create a gap in $H_{l,m}$. We are always look for the best possible in turns of number of vertices and number of $\mathfrak{C}$, $\mathfrak{D}$-edges.

5 Appendix 1

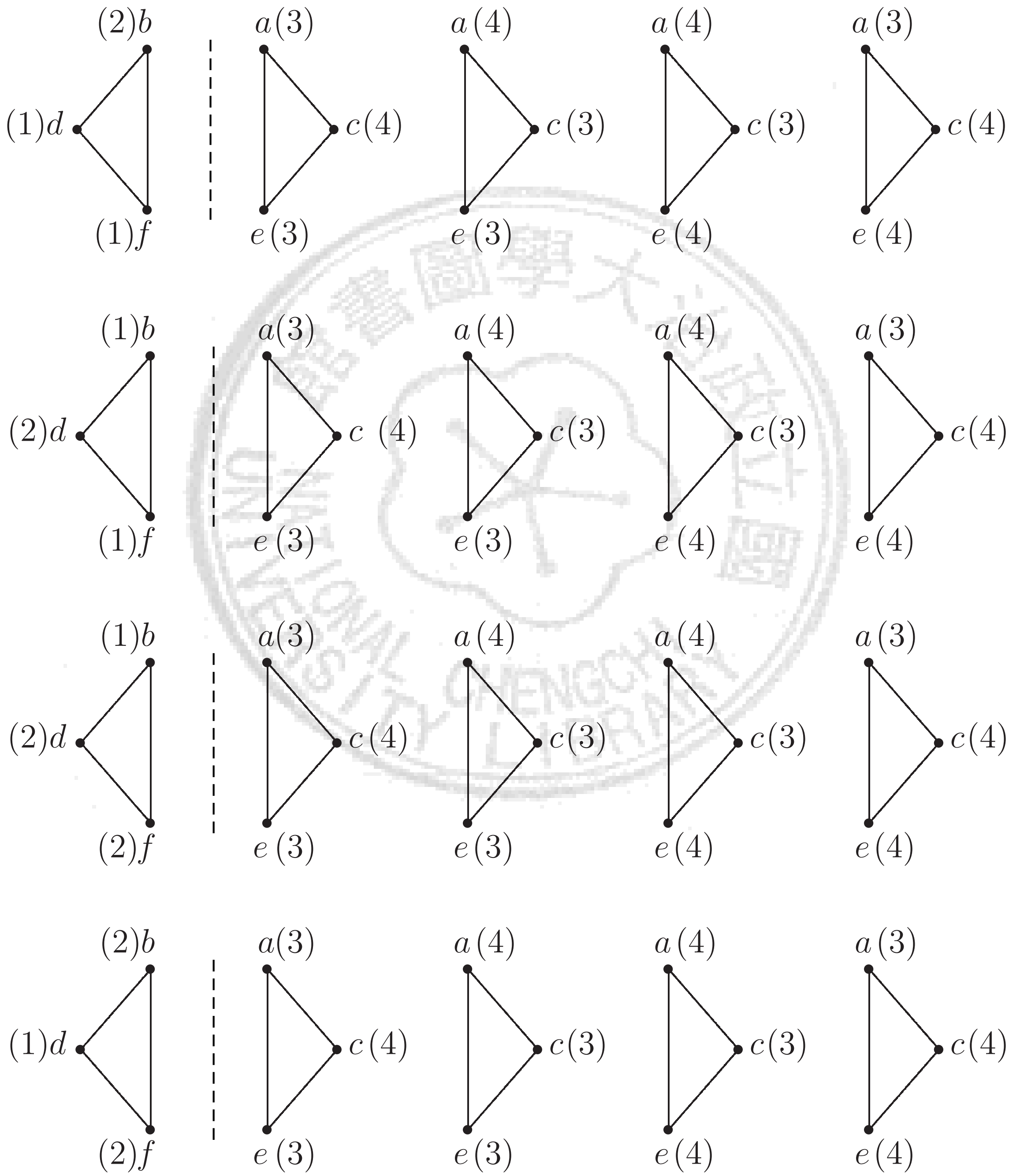
Strict 3-coloringColoring on H



There are 16 ways to color mixed hypergraph H when set $\{b, d, f\}$ be colored 1 and 2,and there are three kind of coloring situations, (1,2), (1,3) , (2,3),in set $\{b, d, f\}$. So $r_3 = 16 \times 3 = 48$.

6 Appendix 2

Strict 4-coloringColoring on H



There are 16 ways coloring H when the set$\{b, d, f\}$ is colored $(1, 2)$ and the set $\{a, c, e\}$ is colored $(3, 4)$. Then there still exists 5 other coloring situations : $(1, 2) - (3, 4)$, $(1, 3) - (2, 4)$, $(2, 3) - (1, 4)$,

$(2,3)-(1,4)$, $(2,4)-(1,3)$, $(3,4)-(1,2)$ colored in set $\{b, d, f\}$ and set $\{a, c, e\}$, respectively. So $r_4 = 96$.