Chapter 3

On Flowe-Harris' Proof of the Determinant Formula of the Generalized Vandermonde Matrix

In this chapter we will provide a modified proof of the determinant formula of $D\mathbb{V}_{\{q;u_1,u_2,\cdots,u_q\}}$, introduced in Chapter 1; this formula has been given by R. P. Flowe and G. A. Harris [6], Theorem 1.1. However, we use in our proof a different induction step.

3.1 The Determinant

Let $v_1, v_2, \cdots, v_q \in \mathbb{R}$ with multiplicities $u_1, u_2, \cdots, u_q$, respectively, and $\sum_{i=1}^{q} u_i = n$, $c_j = (1, v_j, v_j^2, \cdots, v_j^{n-1})^T$,

$$D\mathbb{V}_{\{q;u_1,u_2,\cdots,u_q\}} := [c_1, c_1^{'}, \cdots, c_1^{(u_1-1)}; \cdots; c_q, c_q^{'}, \cdots, c_q^{(u_q-1)}].$$

A superscript on c_j in $D\mathbb{V}_{\{q;u_1,u_2,\cdots,u_q\}}$ denotes the number of applications of $\frac{\partial}{\partial v_j}$ to c_j.

Theorem 3.1.1

$$\det D\mathbb{V}_{\{q;u_1,u_2,\cdots,u_q\}} = \left[\prod_{i=1}^{q}(\prod_{j=0}^{u_i-1} j!)\right] \times \left[\prod_{1\le j<i\le q}(v_i - v_j)^{u_i u_j}\right]. \qquad (3.1.1)$$

Proof. We use mathematical induction on n, the size of the matrix $D\mathbb{V}_{\{q;u_1,u_2,\cdots,u_q\}}$.
(1) The case $n = 2$: There are two subcases to discuss:
Subcase 1:

$$\det D\mathbb{V}_{\{1;2\}} = \det \begin{bmatrix} 1 & 0 \\ v_1 & 1 \end{bmatrix} = 1.$$

Subcase 2:

$$\det D\mathbb{V}_{\{2;1,1\}} = \det \begin{bmatrix} 1 & 1 \\ v_1 & v_2 \end{bmatrix} = v_2 - v_1.$$

Assume $n = k$, the formula (3.1.1) holds.
(2) Consider $n = k+1$ with $k > 1$. Assume that formula (3.1.1) holds for $n = k$. Now we consider a matrix of the form $D\mathbb{V}_{\{q;u_1,u_2,\cdots,u_q\}}$ with size $k+1$ and $\sum_{i=1}^{q} u_i = k+1$. Let us do the following steps to change the matrix $D\mathbb{V}_{\{q;u_1,u_2,\cdots,u_q\}} = [B_1, B_2, \cdots, B_q]$, where for all $1 \leq t \leq q$,

$$B_t = [B_t(i,j)] = \begin{cases} 0, & i < j; \\ (j-1)!, & i = j; \\ \frac{(i-1)!}{(i-j)!} v_t^{i-j}, & otherwise. \end{cases}$$

i.e.

$$B_t = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ v_t & 1 & 0 & \cdots & 0 \\ v_t^2 & 2v_t & 2! & \cdots & 0 \\ v_t^3 & 3v_t^2 & 3 \times 2v_t & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & (u_t - 1)! \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_t^k & kv_t^{k-1} & k(k-1)v_t^{k-2} & \cdots & k(k-1)\cdots[k+1-(u_t-1)]v_t^{k+1-u_t} \end{bmatrix}.$$

Step 1. In $D\mathbb{V}_{\{q;u_1,u_2,\cdots,u_q\}} = [B_1, B_2, \cdots, B_q]$, ($kth$ row)$\times(-v_1)$ adds to $(k+1)th$ row, ($(k-1)th$ row)$\times(-v_1)$ adds to $(k)th$ row, $\cdots$, (1st row)$\times(-v_1)$ adds to 2nd row, then $D\mathbb{V}_{\{q;u_1,u_2,\cdots,u_q\}}$ becomes $D\mathbb{V}^{(0)}_{\{q;u_1,u_2,\cdots,u_q\}} = [B_1^{(0)}, B_2^{(0)}, \cdots, B_q^{(0)}]$, where

$$B_1^{(0)} = [B_1^{(0)}(i,j)] = \begin{cases} 0, & i = 1, j \geq 2; \\ 0, & i < j; \\ (j-1)!, & i = j; \\ [\frac{(i-1)!}{(i-j)!} - \frac{(i-2)!}{(i-j-1)!}]v_1^{i-j}, & otherwise. \end{cases}$$

i.e.

$$B_1^{(0)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & v_1 & 2! & \cdots & 0 \\ 0 & v_1^2 & (3\times 2 - 2!)v_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & (u_1-1)! \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & v_1^{k-1} & [k(k-1)-(k-1)(k-2)]v_1^{k-2} & \cdots & \{k(k-1)\cdots[k+1-(u_1-1)]-(k-1)(k-2)\cdots(k+1-u_1)\}v_1^{k-(u_1-1)} \end{bmatrix},$$

and for all $2 \leq t \leq q$,

$$B_t^{(0)} = [B_t^{(0)}(i,j)] = \begin{cases} 0, & i < j; \\ v_t^{i-1} - v_t^{i-2}v_1, & j = 1, i \geq 2; \\ (j-1)!, & i = j; \\ \frac{(i-1)!}{(i-j)!}v_t^{i-j} - \frac{(i-2)!}{(i-j-1)!}v_t^{i-j-1}v_1, & otherwise. \end{cases}$$

i.e.

$$
B_t^{(0)} = \left[\begin{array}{cccccc}
1 & 0 & 0 & \cdots & 0 \\
v_t - v_1 & 1 & 0 & \cdots & 0 \\
v_t^2 - v_t v_1 & 2v_t - v_1 & 2! & \cdots & 0 \\
v_t^3 - v_t^2 v_1 & 3v_t^2 - 2v_t v_1 & 3 \times 2v_t - 2! v_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & (u_t - 1)! \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
v_t^k - v_t^{k-1} v_1 & k v_t^{k-1} - (k-1) v_t^{k-2} v_1 & k(k-1) v_t^{k-2} - (k-1)(k-2) v_t^{k-3} v_1 & \cdots & \{k(k-1)\cdots[k+1-(u_t-1)]\} v_t^{k+1-u_t} - [(k-1)(k-2)\cdots(k+1-u_t)] v_t^{k+1-u_t-1} v_1
\end{array}\right],
$$

note that $D\mathbb{V}^{(0)}_{\{q;u_1,u_2,\cdots,u_q\}}$ is not the form of the generalized Vandermonde matrix. Then we expand the determinant along the 1st column, thus we obtain

$$
\det D\mathbb{V}_{\{q;u_1,u_2,\cdots,u_q\}} = \det D\mathbb{V}^{(0)}_{\{q;u_1,u_2,\cdots,u_q\}} = \det D\mathbb{V}^{(1)}_{\{q;u_1,u_2,\cdots,u_q\}},
$$

where $D\mathbb{V}^{(1)}_{\{q;u_1,u_2,\cdots,u_q\}} = [B_1^{(1)}, B_2^{(1)}, \cdots, B_q^{(1)}]$, and

$$
B_1^{(1)} = [B_1^{(1)}(i,j)] = \begin{cases} 0, & i < j; \\ j!, & i = j; \\ [\frac{i!}{(i-j)!} - \frac{(i-1)!}{(i-j-1)!}] v_1^{i-j}, & \textit{otherwise}. \end{cases}
$$

i.e.

$$
B_1^{(1)} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
v_1 & 2! & \cdots & 0 \\
v_1^2 & (3 \times 2 - 2!)v_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & (u_1 - 1)! \\
\vdots & \vdots & \vdots & \vdots \\
v_1^{k-1} & (k(k-1)-(k-1)(k-2))v_1^{k-2} & \cdots & \{k(k-1)\cdots[k+1-(u_1-1)]-(k-1)(k-2)\cdots(k+1-u_1)\}v_1^{k-(u_1-1)}
\end{bmatrix},
$$

for all $2 \le t \le q$,

$$
B_t^{(1)} = [B_t^{(1)}(i,j)] = \begin{cases}
0, & i < j-1; \\
v_t^i - v_t^{i-1}v_1, & j = 1; \\
(j-1)!, & i = j-1; \\
\frac{i!}{(i-j+1)!}v_t^{i-j+1} - \frac{(i-1)!}{(i-j)!}v_t^{i-j}v_1, & otherwise.
\end{cases}
$$

i.e.

$$
B_t^{(1)} = \begin{bmatrix}
v_t - v_1 & 1 & 0 \\
v_t^2 - v_t v_1 & 2v_t - v_1 & 2! \\
v_t^3 - v_t^2 v_1 & 3v_t^2 - 2v_t v_1 & 3 \times 2v_t - 2!v_1 \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
v_t^k - v_t^{k-1}v_1 & kv_t^{k-1} - (k-1)v_t^{k-2}v_1 & k(k-1)v_t^{k-2} - (k-1)(k-2)v_t^{k-3}v_1
\end{bmatrix}
$$

$$
\begin{matrix}
\cdots & 0 \\
\cdots & 0 \\
\cdots & 0 \\
\vdots & \vdots \\
\vdots & (u_t - 1)! \\
\vdots & \vdots \\
\cdots & \{k(k-1)\cdots[k+1-(u_t-1)]\}v_t^{k+1-u_t} - [(k-1)(k-2)\cdots(k+1-u_t)]v_t^{k+1-u_t-1}v_1
\end{matrix} \Bigg],
$$

note that $D\mathbb{V}^{(1)}_{\{q;u_1,u_2,\cdots,u_q\}}$ is not the form of the generalized Vandermonde matrix.

Step 2. In this step, we will change $D\mathbb{V}^{(1)}_{\{q;u_1,u_2,\cdots,u_q\}}$ to the form of the generalized Vandermonde matrix. Let $B_1^{(1)} = [B_{11}^{(1)}, B_{12}^{(1)}, \cdots, B_{1(u_1-1)}^{(1)}]$, for all $2 \le t \le q$, $B_t^{(1)} = [B_{t1}^{(1)}, B_{t2}^{(1)}, \cdots, B_{t(u_t)}^{(1)}]$, then we define $B_1^{(2)} = [B_{11}^{(2)}, B_{12}^{(2)}, \cdots, B_{1(u_1-1)}^{(2)}]$ and for all $1 \le t \le q$, $B_t^{(2)} = [B_{t1}^{(2)}, B_{t2}^{(2)}, \cdots, B_{t(u_t)}^{(2)}]$. Because

$$B_1^{(1)} = [B_1^{(1)}(i,j)] = \begin{cases} 0, & i < j; \\ j!, & i = j; \\ [\frac{i!}{(i-j)!} - \frac{(i-1)!}{(i-j-1)!}]v_1^{i-j}, & otherwise. \end{cases}$$

$$= \begin{cases} 0, & i < j; \\ j \times (j-1)!, & i = j; \\ j \times \frac{(i-1)!}{(i-j)!}v_1^{i-j}, & otherwise. \end{cases}$$

then

$$B_1^{(2)} := [\frac{B_{11}^{(1)}}{1}, \frac{B_{12}^{(1)}}{2}, \cdots, \frac{B_{1(u_1-1)}^{(1)}}{(u_1-1)}]$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ v_1 & 1 & 0 & \cdots & 0 \\ v_1^2 & 2v_1 & 2! & \cdots & 0 \\ v_1^3 & 3v_1^2 & 3\times 2v_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & ((u_1-1)-1)! \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_1^{k-1} & (k-1)v_1^{k-2} & (k-1)(k-2)v_1^{k-3} & \cdots & \{(k-1)(k-2)\cdots[k-((u_1-1)-1)]\}v_1^{k-(u_1-1)} \end{bmatrix},$$

note that $B_1^{(2)}$ vanishes if $u_1 = 1$.
Second, for all $2 \le t \le q$, $B_t^{(2)}(i,1) = \frac{B_t^{(1)}(i,1)}{v_t - v_1}$, $1 \le i \le k$, and for all $2 \le j \le u_t$,

$$B_t^{(2)}(i,j) := \frac{B_t^{(2)}(i,j-1) \times [-(j-1)]) + B_t^{(1)}(i,j)}{v_t - v_1}$$

$$= \begin{cases} \frac{0\times[-(j-1)]+0}{v_t-v_1}, & i < j-1; \\ \frac{(j-2)!\times[-(j-1)]+(j-1)!}{v_t-v_1}, & i = j-1; \\ \frac{\frac{(i-1)!}{(i-j+1)!}v_t^{i-j+1}\times[-(j-1)]+[\frac{i!}{(i-j+1)!}v_t^{i-j+1} - \frac{(i-1)!}{(i-j)!}v_t^{i-j}v_1]}{v_t-v_1}, & otherwise. \end{cases}$$

$$= \begin{cases} 0, & i < j-1; \\ 0, & i = j-1; \\ \frac{(i-1)!}{(i-j)!} v^{i-j}, & otherwise. \end{cases}$$

so

$$B_t^{(2)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ v_t & 1 & 0 & \cdots & 0 \\ v_t^2 & 2v_t & 2! & \cdots & 0 \\ v_t^3 & 3v_t^2 & 3 \times 2v_t & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & (u_t-1)! \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_t^{k-1} & (k-1)v_t^{k-2} & (k-1)(k-2)v_t^{k-3} & \cdots & \{(k-1)(k-2)\cdots[k-(u_t-1)]\}v_t^{k-u_t} \end{bmatrix}.$$

Then

$$D\mathbb{V}^{(2)}_{\{q;u_1,u_2,\cdots,u_q\}} := [B_1^{(2)}, B_2^{(2)}, \cdots, B_q^{(2)}] = D\mathbb{V}_{\{q;u_1-1,u_2,\cdots,u_q\}},$$

where $(u_1-1)+\sum_{i=2}^{q} u_i = k$, so by assumption, we get

$$\det D\mathbb{V}^{(2)}_{\{q;u_1,u_2,\cdots,u_q\}} = \left[\prod_{j=0}^{u_1-2} j!\right] \times \left[\prod_{i=2}^{q}(\prod_{j=0}^{u_i-1} j!)\right] \times \left[\prod_{2\le j\le q}(v_j - v_1)^{(u_1-1)u_j}\right]$$

$$\times \left[\prod_{2\le i<j\le q}(v_j - v_i)^{u_i u_j}\right].$$

Hence

$$\begin{aligned} &\det D\mathbb{V}_{\{q;u_1,u_2,\cdots,u_q\}} \\ &= \det D\mathbb{V}^{(0)}_{\{q;u_1,u_2,\cdots,u_q\}} \\ &= \det D\mathbb{V}^{(1)}_{\{q;u_1,u_2,\cdots,u_q\}} \\ &= (\prod_{j=1}^{u_1-1} j) \times \prod_{j=2}^{q}(v_j-v_1)^{u_j} \times \det D\mathbb{V}^{(2)}_{\{q;u_1,u_2,\cdots,u_q\}} \\ &= (\prod_{j=0}^{u_1-1} j) \times \prod_{j=2}^{q}(v_j-v_1)^{u_j} \times \left[\prod_{j=0}^{u_1-2} j!\right] \times \left[\prod_{i=2}^{q}(\prod_{j=0}^{u_i-1} j!)\right] \\ &\times \left[\prod_{2\le j\le q}(v_j - v_1)^{(u_1-1)u_j}\right] \times \left[\prod_{2\le i<j\le q}(v_j - v_i)^{u_i u_j}\right] \end{aligned}$$

$$= \left[\prod_{i=1}^{q}(\prod_{j=0}^{u_i-1} j!)\right] \times \left[\prod_{1\le i<j\le q} (v_j - v_i)^{u_i u_j}\right].$$

This completes the proof.

3.2 The Example

Now let us take a look at the following example to see how the steps work.

Example 3.2.1 Let $q = 3, u_1 = 2, u_2 = 3, u_3 = 5$, then

$$D\mathbb{V}_{\{3;2,3,5\}} = [B_1, B_2, B_3]$$

$$= \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
v_1 & 1 & v_2 & 1 & 0 & v_3 & 1 & 0 & 0 & 0 \\
v_1^2 & 2v_1 & v_2^2 & 2v_2 & 2! & v_3^2 & 2v_3 & 2! & 0 & 0 \\
v_1^3 & 3v_1^2 & v_2^3 & 3v_2^2 & 3\times 2v_2 & v_3^3 & 3v_3^2 & 3\times 2v_3 & 3! & 0 \\
v_1^4 & 4v_1^3 & v_2^4 & 4v_2^3 & 4\times 3v_2^2 & v_3^4 & 4v_3^3 & 4\times 3v_3^2 & 4\times 3\times 2v_3 & 4! \\
v_1^5 & 5v_1^4 & v_2^5 & 5v_2^4 & 5\times 4v_2^3 & v_3^5 & 5v_3^4 & 5\times 4v_3^3 & 5\times 4\times 3v_3^2 & 5\times 4\times 3\times 2v_3 \\
v_1^6 & 6v_1^5 & v_2^6 & 6v_2^5 & 6\times 5v_2^4 & v_3^6 & 6v_3^5 & 6\times 5v_3^4 & 6\times 5\times 4v_3^3 & 6\times 5\times 4\times 3v_3^2 \\
v_1^7 & 7v_1^6 & v_2^7 & 7v_2^6 & 7\times 6v_2^5 & v_3^7 & 7v_3^6 & 7\times 6v_3^5 & 7\times 6\times 5v_3^4 & 7\times 6\times 5\times 4v_3^3 \\
v_1^8 & 8v_1^7 & v_2^8 & 8v_2^7 & 8\times 7v_2^6 & v_3^8 & 8v_3^7 & 8\times 7v_3^6 & 8\times 7\times 6v_3^5 & 8\times 7\times 6\times 5v_3^4 \\
v_1^9 & 9v_1^8 & v_2^9 & 9v_2^8 & 9\times 8v_2^7 & v_3^9 & 9v_3^8 & 9\times 8v_3^7 & 9\times 8\times 7v_3^6 & 9\times 8\times 7\times 6v_3^5
\end{bmatrix}.$$

Step 1. $[9th\ row]\times(-v_1)$ *adds to* $10th\ row$,$[8th\ row]\times(-v_1)$ *adds to* $9th\ row$,$\cdots$,$(1st\ row)\times(-v_1)$ *adds to* $2nd\ row$, then $D\mathbb{V}_{\{3;2,3,5\}}$ becomes $D\mathbb{V}^{(0)}_{\{3;2,3,5\}} = [B_1^{(0)}, B_2^{(0)}, B_3^{(0)}]$, where

$$B_1^{(0)} = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & v_1 \\
0 & v_1^2 \\
0 & v_1^3 \\
0 & v_1^4 \\
0 & v_1^5 \\
0 & v_1^6 \\
0 & v_1^7 \\
0 & v_1^8
\end{bmatrix},$$

$$
B_2^{(0)} = \begin{bmatrix}
1 & 0 & 0 \\
v_2 - v_1 & 1 & 0 \\
v_2^2 - v_2 v_1 & 2v_2 - v_1 & 2! \\
v_2^3 - v_2^2 v_1 & 3v_2^2 - 2v_2 v_1 & 3 \times 2v_2 - 2! v_1 \\
v_2^4 - v_2^3 v_1 & 4v_2^3 - 3v_2^2 v_1 & 4 \times 3v_2^2 - 3 \times 2v_2 v_1 \\
v_2^5 - v_2^4 v_1 & 5v_2^4 - 4v_2^3 v_1 & 5 \times 4v_2^3 - 4 \times 3v_2^2 v_1 \\
v_2^6 - v_2^5 v_1 & 6v_2^5 - 5v_2^4 v_1 & 6 \times 5v_2^4 - 5 \times 4v_2^3 v_1 \\
v_2^7 - v_2^6 v_1 & 7v_2^6 - 6v_2^5 v_1 & 7 \times 6v_2^5 - 6 \times 5v_2^4 v_1 \\
v_2^8 - v_2^7 v_1 & 8v_2^7 - 7v_2^6 v_1 & 8 \times 7v_2^6 - 7 \times 6v_2^5 v_1 \\
v_2^9 - v_2^8 v_1 & 9v_2^8 - 8v_2^7 v_1 & 9 \times 8v_2^7 - 8 \times 7v_2^6 v_1
\end{bmatrix},
$$

$$
B_3^{(0)} = \left[\begin{matrix}
1 & 0 & 0 \\
v_3 - v_1 & 1 & 0 \\
v_3^2 - v_3 v_1 & 2v_3 - v_1 & 2! \\
v_3^3 - v_3^2 v_1 & 3v_3^2 - 2v_3 v_1 & 3 \times 2v_3 - 2! v_1 \\
v_3^4 - v_3^3 v_1 & 4v_3^3 - 3v_3^2 v_1 & 4 \times 3v_3^2 - 3 \times 2v_3 v_1 \\
v_3^5 - v_3^4 v_1 & 5v_3^4 - 4v_3^3 v_1 & 5 \times 4v_3^3 - 4 \times 3v_3^2 v_1 \\
v_3^6 - v_3^5 v_1 & 6v_3^5 - 5v_3^4 v_1 & 6 \times 5v_3^4 - 5 \times 4v_3^3 v_1 \\
v_3^7 - v_3^6 v_1 & 7v_3^6 - 6v_3^5 v_1 & 7 \times 6v_3^5 - 6 \times 5v_3^4 v_1 \\
v_3^8 - v_3^7 v_1 & 8v_3^7 - 7v_3^6 v_1 & 8 \times 7v_3^6 - 7 \times 6v_3^5 v_1 \\
v_3^9 - v_3^8 v_1 & 9v_3^8 - 8v_3^7 v_1 & 9 \times 8v_3^7 - 8 \times 7v_3^6 v_1
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
3! & 0 \\
4 \times 3 \times 2v_3 - 3! v_1 & 4! \\
5 \times 4 \times 3v_3^2 - 4 \times 3 \times 2v_3 v_1 & 5 \times 4 \times 3 \times 2v_3 - 4! v_1 \\
6 \times 5 \times 4v_3^3 - 5 \times 4 \times 3v_3^2 v_1 & 6 \times 5 \times 4 \times 3v_3^2 - 5 \times 4 \times 3 \times 2v_3 v_1 \\
7 \times 6 \times 5v_3^4 - 6 \times 5 \times 4v_3^3 v_1 & 7 \times 6 \times 5 \times 4v_3^3 - 6 \times 5 \times 4 \times 3v_3^2 v_1 \\
8 \times 7 \times 6v_3^5 - 7 \times 6 \times 5v_3^4 v_1 & 8 \times 7 \times 6 \times 5v_3^4 - 7 \times 6 \times 5 \times 4v_3^3 v_1 \\
9 \times 8 \times 7v_3^6 - 8 \times 7 \times 6v_3^5 v_1 & 9 \times 8 \times 7 \times 6v_3^5 - 8 \times 7 \times 6 \times 5v_3^4 v_1
\end{matrix}\right],
$$

note that $D\mathbb{V}_{\{3;2,3,5\}}^{(0)}$ is not the form of the generalized Vandermonde matrix. Then we expand the determinant along the 1st column, thus we obtain

$$
\det D\mathbb{V}_{\{3;2,3,5\}} = \det D\mathbb{V}_{\{3;2,3,5\}}^{(0)} = \det D\mathbb{V}_{\{3;2,3,5\}}^{(1)},
$$

where $D\mathbb{V}^{(1)}_{\{3;2,3,5\}} = [B^{(1)}_1, B^{(1)}_2, B^{(1)}_3]$, and

$$B^{(1)}_1 = \begin{bmatrix} 1 \\ v_1 \\ v_1^2 \\ v_1^3 \\ v_1^4 \\ v_1^5 \\ v_1^6 \\ v_1^7 \\ v_1^8 \end{bmatrix},$$

$$B^{(1)}_2 = \begin{bmatrix} v_2 - v_1 & 1 & 0 \\ v_2^2 - v_2 v_1 & 2v_2 - v_1 & 2! \\ v_2^3 - v_2^2 v_1 & 3v_2^2 - 2v_2 v_1 & 3 \times 2v_2 - 2! v_1 \\ v_2^4 - v_2^3 v_1 & 4v_2^3 - 3v_2^2 v_1 & 4 \times 3v_2^2 - 3 \times 2v_2 v_1 \\ v_2^5 - v_2^4 v_1 & 5v_2^4 - 4v_2^3 v_1 & 5 \times 4v_2^3 - 4 \times 3v_2^2 v_1 \\ v_2^6 - v_2^5 v_1 & 6v_2^5 - 5v_2^4 v_1 & 6 \times 5v_2^4 - 5 \times 4v_2^3 v_1 \\ v_2^7 - v_2^6 v_1 & 7v_2^6 - 6v_2^5 v_1 & 7 \times 6v_2^5 - 6 \times 5v_2^4 v_1 \\ v_2^8 - v_2^7 v_1 & 8v_2^7 - 7v_2^6 v_1 & 8 \times 7v_2^6 - 7 \times 6v_2^5 v_1 \\ v_2^9 - v_2^8 v_1 & 9v_2^8 - 8v_2^7 v_1 & 9 \times 8v_2^7 - 8 \times 7v_2^6 v_1 \end{bmatrix},$$

$$B_3^{(1)} = \begin{bmatrix} v_3 - v_1 & 1 & 0 & 0 & 0 \\ v_3^2 - v_3v_1 & 2v_3 - v_1 & 2! & 0 & 0 \\ v_3^3 - v_3^2v_1 & 3v_3^2 - 2v_3v_1 & 3\times 2v_3 - 2!v_1 & 3! & 0 \\ v_3^4 - v_3^3v_1 & 4v_3^3 - 3v_3^2v_1 & 4\times 3v_3^2 - 3\times 2v_3v_1 & 4\times 3\times 2v_3 - 3!v_1 & 4! \\ v_3^5 - v_3^4v_1 & 5v_3^4 - 4v_3^3v_1 & 5\times 4v_3^3 - 4\times 3v_3^2v_1 & 5\times 4\times 3v_3^2 - 4\times 3\times 2v_3v_1 & 5\times 4\times 3\times 2v_3 - 4!v_1 \\ v_3^6 - v_3^5v_1 & 6v_3^5 - 5v_3^4v_1 & 6\times 5v_3^4 - 5\times 4v_3^3v_1 & 6\times 5\times 4v_3^3 - 5\times 4\times 3v_3^2v_1 & 6\times 5\times 4\times 3v_3^2 - 5\times 4\times 3\times 2v_3v_1 \\ v_3^7 - v_3^6v_1 & 7v_3^6 - 6v_3^5v_1 & 7\times 6v_3^5 - 6\times 5v_3^4v_1 & 7\times 6\times 5v_3^4 - 6\times 5\times 4v_3^3v_1 & 7\times 6\times 5\times 4v_3^3 - 6\times 5\times 4\times 3v_3^2v_1 \\ v_3^8 - v_3^7v_1 & 8v_3^7 - 7v_3^6v_1 & 8\times 7v_3^6 - 7\times 6v_3^5v_1 & 8\times 7\times 6v_3^5 - 7\times 6\times 5v_3^4v_1 & 8\times 7\times 6\times 5v_3^4 - 7\times 6\times 5\times 4v_3^3v_1 \\ v_3^9 - v_3^8v_1 & 9v_3^8 - 8v_3^7v_1 & 9\times 8v_3^7 - 8\times 7v_3^6v_1 & 9\times 8\times 7v_3^6 - 8\times 7\times 6v_3^5v_1 & 9\times 8\times 7\times 6v_3^5 - 8\times 7\times 6\times 5v_3^4v_1 \end{bmatrix},$$

note that $D\mathbb{V}^{(1)}_{\{3;2,3,5\}}$ is not the form of the generalized Vandermonde matrix.

Step 2. In this step, we will change $D\mathbb{V}^{(1)}_{\{3;2,3,5\}}$ to the form of the generalized Vandermonde matrix. Let $B_2^{(1)} = [B_{21}^{(1)}, B_{22}^{(1)}, B_{23}^{(1)}]$ and $B_3^{(1)} = [B_{31}^{(1)}, B_{32}^{(1)}, B_{33}^{(1)}, B_{34}^{(1)}, B_{35}^{(1)}]$. At first we define $B_1^{(2)} = B_1^{(1)}$, $B_2^{(2)} = [B_{21}^{(2)}, B_{22}^{(2)}, B_{23}^{(2)}]$ and $B_3^{(2)} = [B_{31}^{(2)}, B_{32}^{(2)}, B_{33}^{(2)}, B_{34}^{(2)}, B_{35}^{(2)}]$, Second, when $t = 2$, $B_2^{(2)}(i,1) = \frac{B_2^{(1)}(i,1)}{v_2 - v_1}$, $1 \le i \le 9$, and for all $2 \le j \le 3$,

$$B_2^{(2)}(i,j) := \frac{B_2^{(2)}(i,j-1) \times [-(j-1)]) + B_2^{(1)}(i,j)}{v_2 - v_1};$$

and when $t = 3$, $B_3^{(2)}(i,1) = \frac{B_3^{(1)}(i,1)}{v_3 - v_1}$, $1 \le i \le 9$, and for all $2 \le j \le 5$,

$$B_3^{(2)}(i,j) := \frac{B_3^{(2)}(i,j-1) \times [-(j-1)]) + B_3^{(1)}(i,j)}{v_3 - v_1}.$$

So

$$B_2^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ v_2 & 1 & 0 \\ v_2^2 & 2v_2 & 2! \\ v_2^3 & 3v_2^2 & 3\times 2v_2 \\ v_2^4 & 4v_2^3 & 4\times 3v_2^2 \\ v_2^5 & 5v_2^4 & 5\times 4v_2^3 \\ v_2^6 & 6v_2^5 & 6\times 5v_2^4 \\ v_2^7 & 7v_2^6 & 7\times 6v_2^5 \\ v_2^8 & 8v_2^7 & 8\times 7v_2^6 \end{bmatrix},$$

$$B_3^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ v_3 & 1 & 0 & 0 & 0 \\ v_3^2 & 2v_3 & 2! & 0 & 0 \\ v_3^3 & 3v_3^2 & 3\times 2v_3 & 3! & 0 \\ v_3^4 & 4v_3^3 & 4\times 3v_3^2 & 4\times 3\times 2v_3 & 4! \\ v_3^5 & 5v_3^4 & 5\times 4v_3^3 & 5\times 4\times 3v_3^2 & 5\times 4\times 3\times 2v_3 \\ v_3^6 & 6v_3^5 & 6\times 5v_3^4 & 6\times 5\times 4v_3^3 & 6\times 5\times 4\times 3v_3^2 \\ v_3^7 & 7v_3^6 & 7\times 6v_3^5 & 7\times 6\times 5v_3^4 & 7\times 6\times 5\times 4v_3^3 \\ v_3^8 & 8v_3^7 & 8\times 7v_3^6 & 8\times 7\times 6v_3^5 & 8\times 7\times 6\times 5v_3^4 \\ v_3^9 & 9v_3^8 & 9\times 8v_3^7 & 9\times 8\times 7v_3^6 & 9\times 8\times 7\times 6v_3^5 \end{bmatrix}.$$

Then

$$D\mathbb{V}^{(2)}_{\{3;2,3,5\}} := [B_1^{(2)}, B_2^{(2)}, B_3^{(2)}] = D\mathbb{V}_{\{3;1,3,5\}},$$

so by assumption, we get

$$\det D\mathbb{V}^{(2)}_{\{3;2,3,5\}} = (\prod_{j=0}^{2} j!) \times (\prod_{j=0}^{4} j!) \times (v_2 - v_1)^3 \times (v_3 - v_1)^5$$
$$\times (v_3 - v_2)^{3\times 5}.$$

Hence

$\det D\mathbb{V}_{\{3;2,3,5\}}$

$= \det D\mathbb{V}^{(0)}_{\{3;2,3,5\}}$

$= \det D\mathbb{V}^{(1)}_{\{3;2,3,5\}}$

$= (v_2 - v_1)^3 \times (v_3 - v_1)^5 \times \det D\mathbb{V}^{(2)}_{\{3;2,3,5\}}$

$= (\prod_{j=0}^{1} j) \times (v_2 - v_1)^3 \times (v_3 - v_1)^5$

$$\times (\prod_{j=0}^{2} j!) \times (\prod_{j=0}^{4} j!) \times (v_2 - v_1)^3 \times (v_3 - v_1)^5 \times (v_3 - v_2)^{3\times 5}$$
$$= (\prod_{j=0}^{1} j!) \times (\prod_{j=0}^{2} j!) \times (\prod_{j=0}^{4} j!) \times (v_3 - v_2)^{3\times 5} \times (v_3 - v_1)^{2\times 5} \times (v_2 - v_1)^{2\times 3}.$$