# Part II

# Various Decompositions of Some Generalized Vandermonde Matrices

# Chapter 4

# On a Special Generalized Vandermonde Matrix and Its LU Factorization

In Part II of this thesis we consider various decompositions of certain generalized Vandermonde matrices with emphases on their LU factorizations and 1-banded factorizations. In a recent paper [15], H. Oruç and G. M. Phillips obtained an explicit formula of the LU factorization of the classical Vandermonde matrix $V$ and expressed the matrices $L$ and $U$ as a product of 1-banded matrices, and later Sheng-liang Yang [21] gave a simpler alternative approach and proofs of their results.

In this chapter we will investigate the LU and 1-banded factorizations of a special generalized Vandermonde matrix namely $V_{\{2;1,n-1\}}$ which is the transpose of $\mathbb{V}_{\{2;1,n-1\}}$ introduced in Chapter 1, i.e.

$$V_{\{2;1,n-1\}} = \begin{bmatrix} 1 & v_1 & v_1^2 & \cdots & v_1^{n-1} \\ 1 & v_2 & v_2^2 & \cdots & v_2^{n-1} \\ 0 & v_2 & 2v_2^2 & \cdots & (n-1)v_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & v_2 & 2^{n-2}v_2^2 & \cdots & (n-1)^{n-2}v_2^{n-1} \end{bmatrix}.$$

As mentioned in Chapter 1 $V_{\{q;u_1,u_2,\cdots,u_q\}}$ has also occurred in Yang-Holtti's paper [23]: Their main results are a non-unique decomposition of the (GP) block matrix(Theorem 1, p.56) and its determinant formula, but not the determination of its inverse.

Related to the above-mentioned paper [21], Yang's another paper [22] treated the LU factorizations of two special classes of generalized Vandermonde matrices, different from ours. Yang's paper [22] prompts us to investigate the feasibility of the LU factorization and the 1-banded factorization of the above-defined matrix $V_{\{2;1,n-1\}}$. Our two main results are Theorem 4.1.1 and Theorem 4.2.1 on the LU and the bidiagonal(1-banded) factorizations of $V_{\{2;1,n-1\}}$.

As an application of our result Theorem 4.2.1, we give in our last Section 4 the closed-form formula of $V_{\{2;1,n-1\}}^{-1}$ as product of triangular matrices; such closed-form formulae are possible thank to the fact that the explicit formula of the inverse of a tridiagonal matrix has been calculated in the literature (see e.g. [5]).

## 4.1 The LU Factorization of $V_{\{2;1,n-1\}}$

The main goal of this section is to obtain an explicit formula of the LU factorization of the special generalized Vandermonde matrix $V_{\{2;1,n-1\}}$. Our first main result is the following theorem:

**Theorem 4.1.1** *$V_{\{2;1,n-1\}}$ can be factorized as $V_{\{2;1,n-1\}} = L_n U_n$, where $L_n = [L_n(i,j)]$ is a lower triangular matrix with unit main diagonal and $U_n = [U_n(i,j)]$ is an upper triangular matrix, whose entries are defined as follows:*

$$L_n(i,j) = 0, i < j;\ L_n(i,i) = \frac{a_{i,i}v_2 - a_{i-1,i-1}v_1}{v_2 - v_1} = 1, i \geq 1;\ L_n(2,1) = 1;\ L_n(i,1) = 0, i \geq 3;$$

$$L_n(i,j) = \frac{a_{i,j}v_2 - a_{i-1,j-1}v_1}{v_2 - v_1},\ a_{i,j} = (j-1)a_{i-1,j} + a_{i-1,j-1}, i \geq 3, j \geq 2, i > j,$$

*and*

$$U_n(i,j) = 0, i > j;\ U_n(1,1) = 1;\ U_n(1,j) = v_1^{j-1}, j \geq 2;$$

$$U_n(i,j) = (i-2)!v_2^{i-2}(v_2 - v_1)[\sum_{m=0}^{j-i} \binom{i-2+m}{i-2} v_2^m v_1^{(j-i)-m}], j \geq i \geq 2.$$

*where* $a_{i,i} = 1, i \geq 0;\ a_{i,1} = 0, i \geq 2;\ a_{i,2} = 1, i \geq 3;\ a_{i+1,i} = \sum_{k=1}^{i-1} k, i \geq 2.$

**Proof.** We use mathematical induction on $n$, the size of $V_{\{2;1,n-1\}}$.

(1) The case $n = 2$ :

$$L_2U_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & v_1 \\ 0 & v_2 - v_1 \end{bmatrix} = \begin{bmatrix} 1 & v_1 \\ 1 & v_2 \end{bmatrix} = V_{\{2;1,1\}}.$$

(2) The case $k \Rightarrow k+1$ with $k > 1$ : Assume $V_{\{2;1,k-1\}} = L_kU_k$ holds, we want to prove $V_{\{2;1,k\}} = L_{k+1}U_{k+1}$, i.e. $\forall_{1 \leq d \leq k+1}$,

$$V_{\{2;1,k\}}(k+1, d) = \sum_{f=1}^{k+1} L_{k+1}(k+1, f)U_{k+1}(f, d) \quad (4.1.1),$$

and

$$V_{\{2;1,k\}}(d, k+1) = \sum_{f=1}^{k+1} L_{k+1}(d, f)U_{k+1}(f, k+1) \quad (4.1.1').$$

First, it is easy to verify that (4.1.1) holds for $d = 1, 2$, by definitions.

Next we show (4.1.1) for $3 \leq d \leq k$. To this end, we start with our assumption:

$$V_{\{2;1,k-1\}}(k, d) = \sum_{f=1}^{k} L_k(k, f)U_k(f, d), 3 \leq d \leq k.$$

We have by definitions:

$(d-1)^{k-2}v_2^{d-1} = V_{\{2;1,k-1\}}(k, d)$

$= \dfrac{v_2}{v_2 - v_1} \times (v_2^{d-1} - v_1^{d-1}) + \sum_{f=3}^{d-1} L_{k,f}U_{f,d} + \dfrac{a_{k,d}v_2 - a_{k-1,d-1}v_1}{v_2 - v_1} \times (d-2)!v_2^{d-2}(v_2 - v_1)$

$= v_2 \times (v_2^{d-2} + v_2^{d-3}v_1 + \cdots + v_2v_1^{d-3} + v_1^{d-2})$

$+ \sum_{f=3}^{d-1} \dfrac{a_{k,f}v_2 - a_{k-1,f-1}v_1}{v_2 - v_1} \times [(f-2)!v_2^{f-2}(v_2 - v_1) \sum_{m=0}^{d-f} \binom{f-2+m}{f-2} v_2^m v_1^{(d-f)-m}]$

$+ (a_{k,d}v_2 - a_{k-1,d-1}v_1) \times (d-2)!v_2^{d-2}$

$= v_2 \times (v_2^{d-2} + v_2^{d-3}v_1 + \cdots + v_2v_1^{d-3} + v_1^{d-2}) + \sum_{f=3}^{d-1} a_{k,f}(f-2)!v_2^{f-1}[\sum_{m=0}^{d-f} \binom{f-2+m}{f-2} v_2^m v_1^{(d-f)-m}]$

$- v_2v_1[\sum_{m=0}^{d-3} \binom{1+m}{1} v_2^m v_1^{(d-3)-m}] - \sum_{f=4}^{d-1} a_{k-1,f-1}(f-2)!v_2^{f-2}v_1[\sum_{m=0}^{d-f} \binom{f-2+m}{f-2} v_2^m v_1^{(d-f)-m}]$

$+ a_{k,d} \times (d-2)!v_2^{d-1} - a_{k-1,d-1} \times (d-2)!v_2^{d-2}v_1.$

Next, by partially differentiating to $v_2$ on both sides, we obtain

$(d-1)^{k-1}v_2^{d-2} = (v_2^{d-2} + v_2^{d-3}v_1 + \cdots + v_2v_1^{d-3} + v_1^{d-2})$
$+ v_2 \times [(d-2)v_2^{d-3} + (d-3)v_2^{d-4}v_1 + \cdots + 2v_2v_1^{d-4} + v_1^{d-3}]$
$+ \sum_{f=3}^{d-1} a_{k,f}(f-1)!v_2^{f-2}[\sum_{m=0}^{d-f} \binom{f-2+m}{f-2} v_2^m v_1^{(d-f)-m}]$
$+ \sum_{f=3}^{d-1} a_{k,f}(f-2)!v_2^{f-1}[\sum_{m=0}^{d-f} \binom{f-2+m}{f-2} m v_2^{m-1} v_1^{(d-f)-m}]$
$- v_1[\sum_{m=0}^{d-3} \binom{1+m}{1} v_2^m v_1^{(d-3)-m}] - v_2v_1[\sum_{m=0}^{d-3} \binom{1+m}{1} m v_2^{m-1} v_1^{(d-3)-m}]$
$- \sum_{f=4}^{d-1} a_{k-1,f-1}(f-2)!(f-2)v_2^{f-3}v_1[\sum_{m=0}^{d-f} \binom{f-2+m}{f-2} v_2^m v_1^{(d-f)-m}]$
$- \sum_{f=4}^{d-1} a_{k-1,f-1}(f-2)!v_2^{f-2}v_1[\sum_{m=0}^{d-f} \binom{f-2+m}{f-2} m v_2^{m-1} v_1^{(d-f)-m}]$
$+ a_{k,d} \times (d-1)!v_2^{d-2} - a_{k-1,d-1} \times (d-2)!(d-2)v_2^{d-3}v_1,$

and, by multiplying $v_2$ to both sides, we yield the left hand side of (4.1.1):

$(d-1)^{k-1}v_2^{d-1} = V_{\{2;1,k\}}(k+1,d) = (v_2^{d-1} + v_2^{d-2}v_1 + \cdots + v_2^2v_1^{d-3} + v_2v_1^{d-2})$
$+ [(d-2)v_2^{d-1} + (d-3)v_2^{d-2}v_1 + \cdots + 2v_2^3v_1^{d-4} + v_2^2v_1^{d-3}]$
$+ \sum_{f=3}^{d-1} a_{k,f}(f-1)!v_2^{f-1}[\sum_{m=0}^{d-f} \binom{f-2+m}{f-2} v_2^m v_1^{(d-f)-m}]$
$+ \sum_{f=3}^{d-1} a_{k,f}(f-2)!v_2^{f-1}[\sum_{m=0}^{d-f} \binom{f-2+m}{f-2} m v_2^{m} v_1^{(d-f)-m}]$
$- v_2v_1[\sum_{m=0}^{d-3} \binom{1+m}{1} v_2^m v_1^{(d-3)-m}] - v_2v_1[\sum_{m=0}^{d-3} \binom{1+m}{1} m v_2^{m} v_1^{(d-3)-m}]$
$- \sum_{f=4}^{d-1} a_{k-1,f-1}(f-2)!(f-2)v_2^{f-2}v_1[\sum_{m=0}^{d-f} \binom{f-2+m}{f-2} v_2^m v_1^{(d-f)-m}]$
$- \sum_{f=4}^{d-1} a_{d-1,f-1}(f-2)!v_2^{f-2}v_1[\sum_{m=0}^{d-f} \binom{f-2+m}{f-2} m v_2^{m} v_1^{(d-f)-m}]$
$+ a_{k,d} \times (d-1)!v_2^{d-1} - a_{k-1,d-1} \times (d-2)!(d-2)v_2^{d-2}v_1.(4.1.2)$

On the other hand, the right hand side of (4.1.1) is

$\sum_{f=1}^{k+1} L_{k+1}(k+1,f)U_{k+1}(f,d)$
$= \frac{v_2}{v_2 - v_1} \times (v_2^{d-1} - v_1^{d-1}) + \sum_{f=3}^{d-1} L_{k+1,f}U_{f,d} + \frac{a_{k+1,d}v_2 - a_{k,d-1}v_1}{v_2 - v_1} \times (d-2)!v_2^{d-2}(v_2 - v_1)$
$= v_2 \times (v_2^{d-2} + v_2^{d-3}v_1 + \cdots + v_2v_1^{d-3} + v_1^{d-2})$

$$+\sum_{f=3}^{d-1}\frac{a_{k+1,f}v_2-a_{k,f-1}v_1}{v_2-v_1}\times[(f-2)!v_2^{f-2}(v_2-v_1)\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]$$
$$+(a_{k+1,d}v_2-a_{k,d-1}v_1)\times(d-2)!v_2^{d-2}$$
$$=v_2\times(v_2^{d-2}+v_2^{d-3}v_1+\cdots+v_2v_1^{d-3}+v_1^{d-2})$$
$$+\sum_{f=3}^{d-1}a_{k+1,f}(f-2)!v_2^{f-1}[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]-v_2v_1[\sum_{m=0}^{d-3}\binom{1+m}{1}v_2^m v_1^{(d-3)-m}]$$
$$-\sum_{f=4}^{d-1}a_{k,f-1}(f-2)!v_2^{f-2}v_1[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]$$
$$+a_{k+1,d}\times(d-2)!v_2^{d-1}-a_{k,d-1}\times(d-2)!v_2^{d-2}v_1$$
$$=v_2\times(v_2^{d-2}+v_2^{d-3}v_1+\cdots+v_2v_1^{d-3}+v_1^{d-2})$$
$$+\sum_{f=3}^{d-1}[(f-1)a_{k,f}+a_{k,f-1}](f-2)!v_2^{f-1}[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]$$
$$-v_2v_1[\sum_{m=0}^{d-3}\binom{1+m}{1}v_2^m v_1^{(d-3)-m}]-\sum_{f=4}^{d-1}[(f-2)a_{k-1,f-1}+a_{k-1,f-2}](f-2)!v_2^{f-2}v_1[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]$$
$$+[(d-1)a_{k,d}+a_{k,d-1}]\times(d-2)!v_2^{d-1}-[(d-2)a_{k-1,d-1}+a_{k-1,d-2}]\times(d-2)!v_2^{d-2}v_1$$
$$=v_2\times(v_2^{d-2}+v_2^{d-3}v_1+\cdots+v_2v_1^{d-3}+v_1^{d-2})$$
$$+\sum_{f=3}^{d-1}a_{k,f}(f-1)!v_2^{f-1}[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]$$
$$+v_2^2[\sum_{m=0}^{d-3}\binom{1+m}{1}v_2^m v_1^{(d-3)-m}]+\sum_{f=4}^{d-1}a_{k,f-1}(f-2)!v_2^{f-1}[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]$$
$$-v_2v_1[\sum_{m=0}^{d-3}\binom{1+m}{1}v_2^m v_1^{(d-3)-m}]-\sum_{f=4}^{d-1}(f-2)a_{k-1,f-1}(f-2)!v_2^{f-2}v_1[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]$$
$$-2!v_2^2v_1[\sum_{m=0}^{d-4}\binom{2+m}{2}v_2^m v_1^{(d-4)-m}]-\sum_{f=5}^{d-1}a_{k-1,f-2}(f-2)!v_2^{f-2}v_1[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2}v_2^m v_1^{(d-f)-m}]$$
$$+a_{k,d}\times(d-1)!v_2^{d-1}+a_{k,d-1}\times(d-2)!v_2^{d-1}-a_{k-1,d-1}\times(d-2)(d-2)!v_2^{d-2}v_1$$
$$-a_{k-1,d-2}\times(d-2)!v_2^{d-2}v_1.(4.1.3)$$

It remains to prove the equations (4.1.2) = (4.1.3).

To this end, we compare the ten terms $(4.1.2.i), 1\leq i\leq 10$, of equation (4.1.2) with those twelve terms $(4.1.3.j), 1\leq j\leq 12$, of equation (4.1.3). By direct calculations, the following eight terms are equal: (4.1.2.1) = (4.1.3.1), (4.1.2.2) = (4.1.3.3), (4.1.2.3) = (4.1.3.2), (4.1.2.5) = (4.1.3.5), (4.1.2.6) = (4.1.3.7), (4.1.2.7) = (4.1.3.6), (4.1.2.9) = (4.1.3.9), (4.1.2.10) = (4.1.3.11).

For the rest, it remains to check term by term, using direct calculations that

$(4.1.2.4) = (4.1.3.4) + (4.1.3.10)$ and $(4.1.2.8) = (4.1.3.8) + (4.1.3.12)$:

$$(4.1.2.4) = \sum_{f=3}^{d-1} a_{k,f}(f-2)!v_2^{f-1}[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2} m v_2^m v_1^{(d-f)-m}]$$

$$= \sum_{f=3}^{d-2} a_{k,f}(f-1)!v_2^{f}[\sum_{m=1}^{d-f}\frac{(f-2+m)!}{(f-1)!(m-1)!} v_2^{m-1} v_1^{[d-(f+1)]-(m-1)}] + a_{k,d-1}(d-2)!v_2^{d-1}.(4.1.2.4')$$

Now, the crucial step is to show that $(4.1.2.4') = (4.1.3.4) + (4.1.3.10)$: Let $g = m-1$, then

$$(4.1.2.4') = \sum_{f=3}^{d-2} a_{k,f}(f-1)!v_2^{f}[\sum_{g=0}^{d-(f+1)}\binom{f-1+g}{f-1} v_2^g v_1^{[d-(f+1)]-g}] + a_{k,d-1}(d-2)!v_2^{d-1}$$

$$= \sum_{f=4}^{d-1} a_{k,f-1}(f-2)!v_2^{f-1}[\sum_{m=0}^{d-f}\binom{f-2+m}{f-2} v_2^m v_1^{(d-f)-m}] + a_{k,d-1}(d-2)!v_2^{d-1}$$

$= (4.1.3.4) + (4.1.3.10)$.

Similarly, we also have $(4.1.2.8) = (4.1.3.8) + (4.1.3.12)$.

This completes the proof of the equality $(4.1.2) = (4.1.3)$.

To prove the equality $(4.1.1')$, it can easily be shown that $(4.1.1')$ holds for $d = 1, 2, 3$, by definitions.

Using the same technique as in the case of equation (4.1.1), we can obtain that $(4.1.1')$ holds for $4 \le d \le k+1$, proving the equation $(4.1.1')$ and completing the proof of Theorem 4.1.1.

**Example 4.1.2** To illustrate our result, we give an example for $n = 7$: $V_{\{2;1,6\}} = L_7 U_7$, where

$$V_{\{2;1,6\}} = \begin{bmatrix} 1 & v_1 & v_1^2 & v_1^3 & v_1^4 & v_1^5 & v_1^6 \\ 1 & v_2 & v_2^2 & v_2^3 & v_2^4 & v_2^5 & v_2^6 \\ 0 & v_2 & 2v_2^2 & 3v_2^3 & 4v_2^4 & 5v_2^5 & 6v_2^6 \\ 0 & v_2 & 4v_2^2 & 9v_2^3 & 16v_2^4 & 25v_2^5 & 36v_2^6 \\ 0 & v_2 & 8v_2^2 & 27v_2^3 & 64v_2^4 & 125v_2^5 & 216v_2^6 \\ 0 & v_2 & 16v_2^2 & 81v_2^3 & 256v_2^4 & 625v_2^5 & 1296v_2^6 \\ 0 & v_2 & 32v_2^2 & 243v_2^3 & 1024v_2^4 & 3125v_2^5 & 7776v_2^6 \end{bmatrix},$$

$$U_7=\begin{bmatrix} 1 & v_1 & v_1^2 & v_1^3 & v_1^4 & v_1^5 & v_1^6 \\ 0 & v_2-v_1 & (v_2-v_1)(v_2+v_1) & (v_2-v_1)(v_2^2+v_2v_1+v_1^2) & (v_2-v_1)(v_2^3+v_2^2v_1+v_2v_1^2+v_1^3) & (v_2-v_1)(v_2^4+v_2^3v_1+v_2^2v_1^2+v_2v_1^3+v_1^4) & (v_2-v_1)(v_2^5+v_2^4v_1+v_2^3v_1^2+v_2^2v_1^3+v_2v_1^4+v_1^5) \\ 0 & 0 & v_2(v_2-v_1) & v_2(v_2-v_1)(2v_2+v_1) & v_2(v_2-v_1)(3v_2^2+2v_2v_1+v_1^2) & v_2(v_2-v_1)(4v_2^3+3v_2^2v_1+2v_2v_1^2+v_1^3) & v_2(v_2-v_1)(5v_2^4+4v_2^3v_1+3v_2^2v_1^2+2v_2v_1^3+v_1^4) \\ 0 & 0 & 0 & 2v_2^2(v_2-v_1) & 2v_2^2(v_2-v_1)(3v_2+v_1) & 2v_2^2(v_2-v_1)(6v_2^2+3v_2v_1+v_1^2) & 2v_2^2(v_2-v_1)(10v_2^3+6v_2^2v_1+3v_2v_1^2+v_1^3) \\ 0 & 0 & 0 & 0 & 6v_2^3(v_2-v_1) & 6v_2^3(v_2-v_1)(4v_2+v_1) & 6v_2^3(v_2-v_1)(10v_2^2+4v_2v_1+v_1^2) \\ 0 & 0 & 0 & 0 & 0 & 24v_2^4(v_2-v_1) & 24v_2^4(v_2-v_1)(5v_2+v_1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 120v_2^5(v_2-v_1) \end{bmatrix},$$

and

$$L_7=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{v_2}{v_2-v_1} & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{v_2}{v_2-v_1} & \frac{3v_2-v_1}{v_2-v_1} & 1 & 0 & 0 & 0 \\ 0 & \frac{v_2}{v_2-v_1} & \frac{7v_2-v_1}{v_2-v_1} & \frac{6v_2-3v_1}{v_2-v_1} & 1 & 0 & 0 \\ 0 & \frac{v_2}{v_2-v_1} & \frac{15v_2-v_1}{v_2-v_1} & \frac{25v_2-7v_1}{v_2-v_1} & \frac{10v_2-6v_1}{v_2-v_1} & 1 & 0 \\ 0 & \frac{v_2}{v_2-v_1} & \frac{31v_2-v_1}{v_2-v_1} & \frac{90v_2-15v_1}{v_2-v_1} & \frac{65v_2-25v_1}{v_2-v_1} & \frac{15v_2-10v_1}{v_2-v_1} & 1 \end{bmatrix}.$$

**Example 4.1.3** Let $n=10$, then $V_{\{10;1,9\}}=L_{10}U_{10}$, where

$$V_{\{10;1,9\}}=\begin{bmatrix} 1 & v_1 & v_1^2 & v_1^3 & v_1^4 & v_1^5 & v_1^6 \\ 1 & v_2 & v_2^2 & v_2^3 & v_2^4 & v_2^5 & v_2^6 \\ 0 & v_2 & 2v_2^2 & 3v_2^3 & 4v_2^4 & 5v_2^5 & 6v_2^6 \\ 0 & v_2 & 4v_2^2 & 9v_2^3 & 16v_2^4 & 25v_2^5 & 36v_2^6 \\ 0 & v_2 & 8v_2^2 & 27v_2^3 & 64v_2^4 & 125v_2^5 & 216v_2^6 \\ 0 & v_2 & 16v_2^2 & 81v_2^3 & 256v_2^4 & 625v_2^5 & 1296v_2^6 \\ 0 & v_2 & 32v_2^2 & 243v_2^3 & 1024v_2^4 & 3125v_2^5 & 7776v_2^6 \\ 0 & v_2 & 64v_2^2 & 729v_2^3 & 4096v_2^4 & 15625v_2^5 & 46656v_2^6 \\ 0 & v_2 & 128v_2^2 & 2187v_2^3 & 16384v_2^4 & 78125v_2^5 & 279936v_2^6 \\ 0 & v_2 & 256v_2^2 & 6561v_2^3 & 65536v_2^4 & 390625v_2^5 & 1679616v_2^6 \end{bmatrix}$$

$$
\left.\begin{matrix}
v_1^7 & v_1^8 & v_1^9 \\
v_2^7 & v_2^8 & v_2^9 \\
7v_2^7 & 8v_2^8 & 9v_2^9 \\
49v_2^7 & 64v_2^8 & 81v_2^9 \\
343v_2^7 & 512v_2^8 & 729v_2^9 \\
2401v_2^7 & 4096v_2^8 & 6561v_2^9 \\
16807v_2^7 & 32768v_2^8 & 59049v_2^9 \\
117649v_2^7 & 262144v_2^8 & 531441v_2^9 \\
823543v_2^7 & 2097152v_2^8 & 4782969v_2^9 \\
5764801v_2^7 & 16777216v_2^8 & 43046721v_2^9
\end{matrix}\right],
$$

$$
U_{10}=\left[\begin{matrix}
1 & v_1 & v_1^2 & v_1^3 & v_1^4 & v_1^5 \\
0 & v_2-v_1 & (v_2-v_1)(v_2+v_1) & (v_2-v_1)(v_2^2+v_2v_1+v_1^2) & (v_2-v_1)(v_2^3+v_2^2v_1+v_2v_1^2+v_1^3) & (v_2-v_1)(v_2^4+v_2^3v_1+v_2^2v_1^2+v_2v_1^3+v_1^4) \\
0 & 0 & v_2(v_2-v_1) & v_2(v_2-v_1)(2v_2+v_1) & v_2(v_2-v_1)(3v_2^2+2v_2v_1+v_1^2) & v_2(v_2-v_1)(4v_2^3+3v_2^2v_1+2v_2v_1^2+v_1^3) \\
0 & 0 & 0 & 2v_2^2(v_2-v_1) & 2v_2^2(v_2-v_1)(3v_2+v_1) & 2v_2^2(v_2-v_1)(6v_2^2+3v_2v_1+v_1^2) \\
0 & 0 & 0 & 0 & 6v_2^3(v_2-v_1) & 6v_2^3(v_2-v_1)(4v_2+v_1) \\
0 & 0 & 0 & 0 & 0 & 24v_2^4(v_2-v_1) \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{matrix}\right.
$$

$$
\begin{matrix}
v_1^6 & v_1^7 \\
(v_2-v_1)(v_2^5+v_2^4v_1+v_2^3v_1^2+v_2^2v_1^3+v_2v_1^4+v_1^5) & (v_2-v_1)(v_2^6+v_2^5v_1+v_2^4v_1^2+v_2^3v_1^3+v_2^2v_1^4+v_2v_1^5+v_1^6) \\
v_2(v_2-v_1)(5v_2^4+4v_2^3v_1+3v_2^2v_1^2+2v_2v_1^3+v_1^4) & v_2(v_2-v_1)(6v_2^5+5v_2^4v_1+4v_2^3v_1^2+3v_2^2v_1^3+2v_2v_1^4+v_1^5) \\
2v_2^2(v_2-v_1)(10v_2^3+6v_2^2v_1+3v_2v_1^2+v_1^3) & 2v_2^2(v_2-v_1)(15v_2^4+10v_2^3v_1+6v_2^2v_1^2+3v_2v_1^3+v_1^4) \\
6v_2^3(v_2-v_1)(10v_2^2+4v_2v_1+v_1^2) & 6v_2^3(v_2-v_1)(20v_2^3+10v_2^2v_1+4v_2v_1^2+v_1^3) \\
24v_2^4(v_2-v_1)(5v_2+v_1) & 24v_2^4(v_2-v_1)(15v_2^2+5v_2v_1+v_1^2) \\
120v_2^5(v_2-v_1) & 120v_2^5(v_2-v_1)(6v_2+v_1) \\
0 & 720v_2^6(v_2-v_1) \\
0 & 0 \\
0 & 0
\end{matrix}
$$

$$
\begin{matrix}
v_1^8 & v_1^9 \\
(v_2-v_1)(v_2^7+v_2^6v_1+v_2^5v_1^2+v_2^4v_1^3+v_2^3v_1^4+v_2^2v_1^5+v_2v_1^6+v_1^7) & (v_2-v_1)(v_2^8+v_2^7v_1+v_2^6v_1^2+v_2^5v_1^3+v_2^4v_1^4+v_2^3v_1^5+v_2^2v_1^6+v_2v_1^7+v_1^8) \\
v_2(v_2-v_1)(7v_2^6+6v_2^5v_1+5v_2^4v_1^2+4v_2^3v_1^3+3v_2^2v_1^4+2v_2v_1^5+v_1^6) & v_2(v_2-v_1)(8v_2^7+7v_2^6v_1+6v_2^5v_1^2+5v_2^4v_1^3+4v_2^3v_1^4+3v_2^2v_1^5+2v_2v_1^6+v_1^7) \\
2v_2^2(v_2-v_1)(21v_2^5+15v_2^4v_1+10v_2^3v_1^2+6v_2^2v_1^3+3v_2v_1^4+v_1^5) & 2v_2^2(v_2-v_1)(28v_2^6+21v_2^5v_1+15v_2^4v_1^2+10v_2^3v_1^3+6v_2^2v_1^4+3v_2v_1^5+v_1^6) \\
6v_2^3(v_2-v_1)(35v_2^4+20v_2^3v_1+10v_2^2v_1^2+4v_2v_1^3+v_1^4) & 6v_2^3(v_2-v_1)(56v_2^5+35v_2^4v_1+20v_2^3v_1^2+10v_2^2v_1^3+4v_2v_1^4+v_1^5) \\
24v_2^4(v_2-v_1)(35v_2^3+15v_2^2v_1+5v_2v_1^2+v_1^3) & 24v_2^4(v_2-v_1)(70v_2^4+35v_2^3v_1+15v_2^2v_1^2+5v_2v_1^3+v_1^4) \\
120v_2^5(v_2-v_1)(21v_2^2+6v_2v_1+v_1^2) & 120v_2^5(v_2-v_1)(56v_2^3+21v_2^2v_1+6v_2v_1^2+v_1^3) \\
720v_2^6(v_2-v_1)(7v_2+v_1) & 720v_2^6(v_2-v_1)(28v_2^2+7v_2v_1+v_1^2) \\
5040v_2^7(v_2-v_1) & 5040v_2^7(v_2-v_1)(8v_2+v_1) \\
0 & 40320v_2^7(v_2-v_1)
\end{matrix}
\Bigg],
$$

and

$$
L_{10} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{v_2}{v_2-v_1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{v_2}{v_2-v_1} & \frac{3v_2-v_1}{v_2-v_1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{v_2}{v_2-v_1} & \frac{7v_2-v_1}{v_2-v_1} & \frac{6v_2-3v_1}{v_2-v_1} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{v_2}{v_2-v_1} & \frac{15v_2-v_1}{v_2-v_1} & \frac{25v_2-7v_1}{v_2-v_1} & \frac{10v_2-6v_1}{v_2-v_1} & 1 & 0 & 0 & 0 & 0 \\
0 & \frac{v_2}{v_2-v_1} & \frac{31v_2-v_1}{v_2-v_1} & \frac{90v_2-15v_1}{v_2-v_1} & \frac{65v_2-25v_1}{v_2-v_1} & \frac{15v_2-10v_1}{v_2-v_1} & 1 & 0 & 0 & 0 \\
0 & \frac{v_2}{v_2-v_1} & \frac{63v_2-v_1}{v_2-v_1} & \frac{301v_2-31v_1}{v_2-v_1} & \frac{350v_2-90v_1}{v_2-v_1} & \frac{140v_2-65v_1}{v_2-v_1} & \frac{21v_2-15v_1}{v_2-v_1} & 1 & 0 & 0 \\
0 & \frac{v_2}{v_2-v_1} & \frac{127v_2-v_1}{v_2-v_1} & \frac{966v_2-63v_1}{v_2-v_1} & \frac{1701v_2-301v_1}{v_2-v_1} & \frac{1050v_2-350v_1}{v_2-v_1} & \frac{266v_2-140v_1}{v_2-v_1} & \frac{28v_2-21v_1}{v_2-v_1} & 1 & 0 \\
0 & \frac{v_2}{v_2-v_1} & \frac{255v_2-v_1}{v_2-v_1} & \frac{3025v_2-127v_1}{v_2-v_1} & \frac{7770v_2-966v_1}{v_2-v_1} & \frac{6951v_2-1701v_1}{v_2-v_1} & \frac{2646v_2-1050v_1}{v_2-v_1} & \frac{462v_2-266v_1}{v_2-v_1} & \frac{36v_2-28v_1}{v_2-v_1} & 1
\end{bmatrix}.
$$

## 4.2 Factorization of $V_{\{2;1,n-1\}}$ into 1-Banded(Bidiagonal) Matrices

Motivated by previous results by several authors (see [4], [15], [21]), we now formulate our second main result on the 1-banded factorizations of the special generalized Vandermonde matrix $V_{\{2;1,n-1\}}$.

**Theorem 4.2.1** *$V_{\{2;1,n-1\}}$ can be factorized into $n-2$ 1-lower banded matrices and $n-1$ 1-upper banded matrices such that*

$$for\ all\ n \geq 3,\ V_{\{2;1,n-1\}} = L_n^{(1)} L_n^{(2)} \cdots L_n^{(n-2)} U_n^{(n-1)} U_n^{(n-2)} \cdots U_n^{(1)},$$

*where for all $1 \leq l \leq n-3$,*

$$L_n^{(l)}(i,j) = \begin{cases} 1, & i = j; \\ j-(n-l-1), & i = j+1, n-l \leq j \leq n-1; \\ 0, & otherwise. \end{cases}$$

$$L_n^{(n-2)}(i,j) = \begin{cases} 1, & i = j; \\ 1, & i = 2, j = 1; \\ \frac{(j-1)v_2}{v_2-v_1}, & i = j+1, j \geq 2; \\ 0 & otherwise. \end{cases}$$

*and*

$$U_2^{(1)} = \begin{bmatrix} 1 & v_1 \\ 0 & v_2 - v_1 \end{bmatrix},$$

$$for\ all\ 1 \leq l \leq n-2,\ U_n^{(l)} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & U_{n-1}^{(l)} \end{bmatrix},$$

*where* $\mathbf{0}$ *denotes the appropriate zero row vector,*

$$U_n^{(n-1)}(i,j) = \begin{cases} 1, & i=j=1; \\ v_1, & i=1, j=2; \\ v_2 - v_1, & i=j=2; \\ (j-2)v_2, & i=j \geq 3; \\ v_2, & i=j-1, j \geq 3; \\ 0, & \textit{otherwise.} \end{cases}$$

*Hence*

$$L_n = L_n^{(1)} L_n^{(2)} \cdots L_n^{(n-2)},$$
$$U_n = U_n^{(n-1)} U_n^{(n-2)} \cdots U_n^{(1)}.$$

Before proving Theorem 4.2.1, we need the following lemma:

**Lemma 4.2.2** *For* $n \geq 4$*,*

$$L_n^{(1)} L_n^{(2)} \cdots L_n^{(n-3)} = \widetilde{L}_n,$$

*where*

$$\widetilde{L}_n(i,j) = \begin{cases} 1, & i=j; \\ a_{i-1,j-1}, & i > j \geq 3; \\ 0, & \textit{otherwise.} \end{cases}$$

*where* $a_{i,j}$ *are defined as in Theorem 4.1.1.*

**Proof.** We use mathematical induction on $n$.

(1) The case $n = 4$ :

$$L_4^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{3,2} & 1 \end{bmatrix} = \widetilde{L}_4.$$

(2) The case $k \Rightarrow k+1$ with $k > 4$ : Assume $L_k^{(1)} L_k^{(2)} \cdots L_k^{(k-3)} = \widetilde{L}_k$ holds, we want to prove $L_{k+1}^{(1)} L_{k+1}^{(2)} \cdots L_{k+1}^{(k-2)} = \widetilde{L}_{k+1}$. Observe that

$$\widetilde{L}'_{k+1} := L_{k+1}^{(1)} L_{k+1}^{(2)} \cdots L_{k+1}^{(k-2)} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & \widetilde{L}_k \end{bmatrix} L_{k+1}^{(k-2)},$$

since the matrices $L_n^{(l)}, 1 \le l \le n-3$, in Theorem 4.2.1 are all of the form $\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & M \end{bmatrix}$.
To show $\widetilde{L}'_{k+1} = \widetilde{L}_{k+1}$, we calculate for all $i > j \ge 3$,

$$\begin{aligned}\widetilde{L}'_{k+1}(i,j) &= \widetilde{L}_k(i-1,j-1)L_{k+1}^{(k-2)}(j,j) + \widetilde{L}_k(i-1,j)L_{k+1}^{(k-2)}(j+1,j) \\ &= a_{i-2,j-2} \times 1 + a_{i-2,j-1} \times (j-2) = a_{i-1,j-1},\end{aligned}$$

i.e.

$$\widetilde{L}'_{k+1}(i,j) = \begin{cases} 1, & i = j; \\ a_{i-1,j-1}, & i > j \ge 3; \\ 0, & otherwise. \end{cases}$$

yielding $\widetilde{L}'_{k+1} = \widetilde{L}_{k+1}$.
Thus by induction, we complete the proof of this lemma.

Now we are in a position to prove Theorem 4.2.1.

**Proof of Theorem 4.2.1.** First, we show that $L_n = L_n^{(1)} L_n^{(2)} \cdots L_n^{(n-3)} L_n^{(n-2)}$.
Let $L'_n = L_n^{(1)} L_n^{(2)} \cdots L_n^{(n-3)} L_n^{(n-2)} = \widetilde{L}_n L_n^{(n-2)}$(by Lemma 4.2.2), then for all $i \ge 3$,

$$L'_n(i,2) = \widetilde{L}_n(i,2)L_n^{(n-2)}(2,2) + \widetilde{L}_n(i,3)L_n^{(n-2)}(3,2) = \frac{v_2}{v_2 - v_1},$$

and for all $i \ge 3,\ j \ge 2,\ i > j$,

$$L'_n(i,j) = \widetilde{L}_n(i,j)L_n^{(n-2)}(j,j) + \widetilde{L}_n(i,j+1)L_n^{(n-2)}(j+1,j) = \frac{a_{i,j}v_2 - a_{i-1,j-1}v_1}{v_2 - v_1},$$

so

$$L'_n(i,j) = \begin{cases} 0, & i < j; \\ 1, & i = j; \\ 1, & i = 2, j = 1; \\ 0, & j = 1, i \ge 3; \\ \frac{a_{i,j}v_2 - a_{i-1,j-1}v_1}{v_2 - v_1}, & i \ge 3, j \ge 2, i > j. \end{cases}$$

thus $L'_n = L_n$.
Next, we claim that $U_n = U_n^{(n-1)} U_n^{(n-2)} \cdots U_n^{(1)}$; we use mathematical induction on $n$.
(1) The case $n = 2$ :

$$U_2^{(1)} = \begin{bmatrix} 1 & v_1 \\ 0 & v_2 - v_1 \end{bmatrix} = U_2.$$

(2) The case $k \Rightarrow k+1$ with $k > 4$ : Assume $U_k = U_k^{(k-1)} U_k^{(k-2)} \cdots U_k^{(1)}$ holds, we claim that $U_{k+1} = U_{k+1}^{(k)} U_{k+1}^{(k-1)} \cdots U_{k+1}^{(1)}$.
Observe that

$$U'_{k+1} := U_{k+1}^{(k)} U_{k+1}^{(k-1)} \cdots U_{k+1}^{(1)} = U_{(k+1)}^{(k)} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & U_k^{(k-1)} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & U_k^{(k-2)} \end{bmatrix} \cdots \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & U_k^{(1)} \end{bmatrix}$$

$$= U_{(k+1)}^{(k)} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & U_k \end{bmatrix},$$

by definition and inductive hypothesis. To prove $U'_{k+1} = U_{k+1}$, we calculate for all $k+1 \geq j \geq i \geq 3$,

$$U'_{k+1}(i,j) = U_{k+1}^{(k)}(i,i) U_k(i-1, j-1) + U_{k+1}^{(k)}(i, i+1) U_k(i, j-1)$$

$$= (i-2) v_2 (i-3)! v_2^{i-3} (v_2 - v_1) [\sum_{m=0}^{j-i} \tbinom{i-3+m}{i-3} v_2^m v_1^{(j-i)-m}]$$

$$+ v_2 (i-2)! v_2^{i-2} (v_2 - v_1) [\sum_{m=0}^{j-i-1} \tbinom{i-2+m}{i-2} v_2^m v_1^{(j-i-1)-m}]$$

$$= (i-2)! v_2^{i-2} (v_2 - v_1) [\sum_{m=0}^{j-i} \tbinom{i-3+m}{i-3} v_2^m v_1^{(j-i)-m} + \sum_{g=1}^{j-i} \tbinom{i-3+g}{i-2} v_2^g v_1^{(j-i)-g}]$$

$$= (i-2)! v_2^{i-2} (v_2 - v_1) [v_1^{j-i} + \sum_{m=1}^{j-i} \tbinom{i-3+m}{m} (\frac{i-2+m}{i-2}) v_2^m v_1^{(j-i)-m}]$$

$$= (i-2)! v_2^{i-2} (v_2 - v_1) [\sum_{m=0}^{j-i} \tbinom{i-2+m}{i-2} v_2^m v_1^{(j-i)-m}],$$

hence

$$U'_{k+1}(i,j) = \begin{cases} 0, & i > j; \\ 1, & i = j = 1; \\ v_1^{j-1}, & i = 1, j \geq 2; \\ (i-2)! v_2^{i-2} (v_2 - v_1) [\sum_{m=0}^{j-i} \tbinom{i-2+m}{i-2} v_2^m v_1^{(j-i)-m}], & j \geq i \geq 2. \end{cases}$$

thus $U'_{k+1} = U_{k+1}$.
So by induction, we conclude that $U_n = U_n^{(n-1)} U_n^{(n-2)} \cdots U_n^{(1)}$, and the proof of Theorem 4.2.1. is completed.

**Example 4.2.3** For $n = 5$, we have $V_{\{2;1,4\}} = L_5 U_5$, then $L_5 = L_5^{(1)} L_5^{(2)} L_5^{(3)}$, where

$$L_5^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$L_5^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix},$$

$$L_5^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & \frac{v_2}{v_2-v_1} & 1 & 0 & 0 \\ 0 & 0 & \frac{2v_2}{v_2-v_1} & 1 & 0 \\ 0 & 0 & 0 & \frac{3v_2}{v_2-v_1} & 1 \end{bmatrix}.$$

And $U_5 = U_5^{(4)} U_5^{(3)} U_5^{(2)} U_5^{(1)}$, where

$$U_5^{(4)} = \begin{bmatrix} 1 & v_1 & 0 & 0 & 0 \\ 0 & v_2 - v_1 & v_2 & 0 & 0 \\ 0 & 0 & v_2 & v_2 & 0 \\ 0 & 0 & 0 & 2v_2 & v_2 \\ 0 & 0 & 0 & 0 & 3v_2 \end{bmatrix},$$

$$U_5^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & v_1 & 0 & 0 \\ 0 & 0 & v_2 - v_1 & v_2 & 0 \\ 0 & 0 & 0 & v_2 & v_2 \\ 0 & 0 & 0 & 0 & 2v_2 \end{bmatrix},$$

$$U_5^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & v_1 & 0 \\ 0 & 0 & 0 & v_2 - v_1 & v_2 \\ 0 & 0 & 0 & 0 & v_2 \end{bmatrix},$$

$$U_5^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & v_1 \\ 0 & 0 & 0 & 0 & v_2 - v_1 \end{bmatrix}.$$

**Example 4.2.4** For $n = 10$, we have $V_{\{10;1,9\}} = L_{10}U_{10}$, then

$L_{10} = L_{10}^{(1)} L_{10}^{(2)} L_{10}^{(3)} L_{10}^{(4)} L_{10}^{(5)} L_{10}^{(6)} L_{10}^{(7)} L_{10}^{(8)}$, where

$$L_{10}^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$L_{10}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix},$$

$$L_{10}^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \end{bmatrix},$$

$$L_{10}^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 \end{bmatrix},$$

$$L_{10}^{(5)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 \end{bmatrix},$$

$$L_{10}^{(6)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 1 \end{bmatrix},$$

$$L_{10}^{(7)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 \end{bmatrix},$$

$$L_{10}^{(8)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{v_2}{v_2-v_1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2v_2}{v_2-v_1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3v_2}{v_2-v_1} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4v_2}{v_2-v_1} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5v_2}{v_2-v_1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{6v_2}{v_2-v_1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7v_2}{v_2-v_1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{8v_2}{v_2-v_1} & 1 \end{bmatrix}.$$

And $U_{10} = U_{10}^{(9)} U_{10}^{(8)} U_{10}^{(7)} U_{10}^{(6)} U_{10}^{(5)} U_{10}^{(4)} U_{10}^{(3)} U_{10}^{(2)} U_{10}^{(1)}$, where

$$U_{10}^{(9)} = \begin{bmatrix} 1 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_2 - v_1 & v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_2 & v_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2v_2 & v_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3v_2 & v_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4v_2 & v_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5v_2 & v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6v_2 & v_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7v_2 & v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8v_2 \end{bmatrix},$$

$$U_{10}^{(8)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_2 - v_1 & v_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2 & v_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2v_2 & v_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3v_2 & 0v_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4v_2 & v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5v_2 & v_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6v_2 & v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7v_2 \end{bmatrix},$$

$$U_{10}^{(7)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & v_2 - v_1 & v_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_2 & v_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2v_2 & v_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3v_2 & v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4v_2 & v_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5v_2 & v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6v_2 \end{bmatrix},$$

$$U_{10}^{(6)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & v_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_2 - v_1 & v_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & v_2 & v_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2v_2 & v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3v_2 & v_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4v_2 & v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5v_2 \end{bmatrix},$$

$$U_{10}^{(5)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & v_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & v_2 - v_1 & v_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v_2 & v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2v_2 & v_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3v_2 & v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4v_2 \end{bmatrix},$$

$$U_{10}^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & v_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v_2 - v_1 & v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_2 & v_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2v_2 & v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3v_2 \end{bmatrix},$$

$$U_{10}^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & v_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_2 - v_1 & v_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_2 & v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2v_2 \end{bmatrix},$$

$$U_{10}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & v_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_2 - v_1 & v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_2 \end{bmatrix},$$

$$U_{10}^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & v_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_2 - v_1 \end{bmatrix}.$$

## 4.3 Application to the Closed-form Formula of $V^{-1}_{\{2;1,n-1\}}$

As a first application, the calculation of the determinant of $V_{\{2;1,n-1\}}$ is obvious, by Theorem 2.1: It is equal to the product of the entries of the main diagonal of $U_n$ which is equal to the product of the products of the main diagonals of $U_n^{(l)}, 1 \leq l \leq n-1$, and this coincides with the corresponding formula (2.2.1). Furthermore, using our result of 1-banded factorizations we can get, as an application of Theorem 4.2.1, the closed-form formula of the inverse of $V_{\{2;1,n-1\}}$. This is feasible, since the explicit formula of the inverse of a tridiagonal matrix had been calculated in the literature (see e.g. a recent result [5], P.713, by M. El-Mikkawy and A. Karawia). For convenience's sake we reproduce the result on the inversion of a general tridiagonal matrix in the following: Let $T = (t_{ij})_{1\leq i,j\leq n}$ be a general tridiagonal matrix, i.e.

$$T = \begin{bmatrix} d_1 & a_1 & 0 & \cdots & \cdots & 0 \\ b_2 & d_2 & a_2 & \cdots & \cdots & 0 \\ 0 & b_3 & d_3 & a_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & b_{n-1} & d_{n-1} & a_{n-1} \\ 0 & 0 & \cdots & 0 & b_n & d_n \end{bmatrix}$$

in which $t_{ij} = 0$ for $|i-j| \geq 2$, the inverse matrix $T^{-1} = (u_{ij})_{1\leq i,j\leq n}$ of the matrix $T$ is given by:

$$u_{11} = (d_1 - \frac{b_2 a_1 \beta_3}{\beta_2})^{-1},$$

$$u_{nn} = (d_n - \frac{b_n a_{n-1} \alpha_{n-2}}{\alpha_{n-1}})^{-1},$$

$$u_{ii} = (d_i - \frac{b_i a_{i-1} \alpha_{i-2}}{\alpha_{i-1}} - \frac{b_{i+1} a_i \beta_{i+2}}{\beta_{i+1}})^{-1}, i = 2, 3, \cdots, n-1,$$

$$u_{ij} = \begin{cases} (-1)^{j-i} (\prod_{k=1}^{j-i} a_{j-k}) \frac{\alpha_{i-1}}{\alpha_{j-1}} u_{jj}, & i < j; \\ (-1)^{i-j} (\prod_{k=1}^{i-j} b_{j+k}) \frac{\beta_{i+1}}{\beta_{j+1}} u_{jj}, & i > j. \end{cases}$$

where

$$\alpha_i = \begin{cases} 1, & i = 0; \\ d_1, & i = 1; \\ d_i \alpha_{i-1} - b_i a_{i-1} \alpha_{i-2}, & i = 2, 3, \cdots, n. \end{cases}$$

and

$$\beta_i = \begin{cases} 1, & i = n+1; \\ d_n, & i = n; \\ d_i\beta_{i+1} - b_{i+1}a_i\beta_{i+2}, & i = n-1, n-2, \cdots, 1. \end{cases}$$

Now let us state our result in the following where the proof is based on direct calculations and omitted.

**Theorem 4.3.1** *For all $n \geq 3$,*

$$V_{\{2;1,n-1\}}^{-1} = (U_n^{(1)})^{-1}(U_n^{(2)})^{-1} \cdots (U_n^{(n-1)})^{-1}(L_n^{(n-2)})^{-1}(L_n^{(n-3)})^{-1} \cdots (L_n^{(1)})^{-1},$$

where

$$(U_2^{(1)})^{-1} = \begin{bmatrix} 1 & -\frac{v_1}{v_2-v_1} \\ 0 & \frac{1}{v_2-v_1} \end{bmatrix},$$

$$for\ all\ 1 \leq l \leq n-2,\ (U_n^{(l)})^{-1} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0^T} & (U_{n-1}^{(l)})^{-1} \end{bmatrix},$$

where $\mathbf{0}$ denotes the appropriate zero row vector,

$$(U_n^{(n-1)})^{-1}(i,j) = \begin{cases} 0, & i > j; \\ 1, & i = j = 1; \\ (-1)^{j-1}\frac{v_1}{(j-2)!(v_2-v_1)}, & i = 1, j \geq 2; \\ (-1)^{j-2}\frac{1}{(j-2)!(v_2-v_1)}, & i = 2, j \geq 2; \\ (-1)^{j-i}\frac{(i-3)!}{(j-2)!v_2}, & i \geq 3, i \leq j. \end{cases}$$

and

$$(L_n^{(n-2)})^{-1}(i,j) = \begin{cases} 0, & i < j; \\ 1, & i = j; \\ -1, & i = 2, j = 1; \\ (-1)^{i-1}(i-2)!\frac{v_2^{i-2}}{(v_2-v_1)^{i-2}}, & i \geq 3, j = 1; \\ (-1)^{i-j}\frac{(i-2)!}{(j-2)!}\frac{v_2^{i-j}}{(v_2-v_1)^{i-j}}, & i > j \geq 2, i \geq 3. \end{cases}$$

for all $1 \leq l \leq n-3$,

$$(L_n^{(l)})^{-1}(i,j) = \begin{cases} 0, & i < j; \\ 1, & i = j; \\ 0, & j < n-l; \\ (-1)^{i-j}\frac{(i-n+l)!}{(j-n+l)!}, & j \geq n-l. \end{cases}$$

**Example 4.3.2** For $n = 5$, in Example 4.2.3 in Section 2, we have $V_{\{2;1,4\}}^{-1} = U_5^{-1}L_5^{-1}$, then $U_5^{-1} = (U_5^{(1)})^{-1}(U_5^{(2)})^{-1}(U_5^{(3)})^{-1}(U_5^{(4)})^{-1}$ and $L_5^{-1} = (L_5^{(3)})^{-1}(L_5^{(2)})^{-1}(L_5^{(1)})^{-1}$. By Theorem 4.3.1, we obtain

$$(U_5^{(1)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{v_1}{v_2-v_1} \\ 0 & 0 & 0 & 0 & \frac{1}{v_2-v_1} \end{bmatrix},$$

$$(U_5^{(2)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{v_1}{v_2-v_1} & \frac{v_1}{v_2-v_1} \\ 0 & 0 & 0 & \frac{1}{v_2-v_1} & -\frac{1}{v_2-v_1} \\ 0 & 0 & 0 & 0 & \frac{1}{v_2} \end{bmatrix},$$

$$(U_5^{(3)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{v_1}{v_2-v_1} & \frac{v_1}{v_2-v_1} & -\frac{v_1}{2(v_2-v_1)} \\ 0 & 0 & \frac{1}{v_2-v_1} & -\frac{1}{v_2-v_1} & \frac{1}{2(v_2-v_1)} \\ 0 & 0 & 0 & \frac{1}{v_2} & -\frac{1}{2v_2} \\ 0 & 0 & 0 & 0 & \frac{1}{2v_2} \end{bmatrix},$$

$$(U_5^{(4)})^{-1} = \begin{bmatrix} 1 & -\frac{v_1}{v_2-v_1} & \frac{v_1}{v_2-v_1} & -\frac{v_1}{2(v_2-v_1)} & \frac{v_1}{6(v_2-v_1)} \\ 0 & \frac{1}{v_2-v_1} & -\frac{1}{v_2-v_1} & \frac{1}{2(v_2-v_1)} & -\frac{1}{6(v_2-v_1)} \\ 0 & 0 & \frac{1}{v_2} & -\frac{1}{2v_2} & \frac{1}{6v_2} \\ 0 & 0 & 0 & \frac{1}{2v_2} & -\frac{1}{6v_2} \\ 0 & 0 & 0 & 0 & \frac{1}{3v_2} \end{bmatrix},$$

$$(L_5^{(3)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ \frac{v_2}{v_2-v_1} & -\frac{v_2}{v_2-v_1} & 1 & 0 & 0 \\ -\frac{2v_2^2}{(v_2-v_1)^2} & \frac{2v_2^2}{(v_2-v_1)^2} & -\frac{2v_2}{v_2-v_1} & 1 & 0 \\ \frac{6v_2^3}{(v_2-v_1)^3} & -\frac{6v_2^3}{(v_2-v_1)^3} & \frac{6v_2^2}{(v_2-v_1)^2} & -\frac{3v_2}{v_2-v_1} & 1 \end{bmatrix},$$

$$(L_5^{(2)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -2 & 1 \end{bmatrix},$$

$$(L_5^{(1)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Furthermore, we give another example for $V_{\{10;1,9\}}^{-1} = U_{10}^{-1}L_{10}^{-1}$, then

$$U_{10}^{-1} = (U_{10}^{(1)})^{-1}(U_{10}^{(2)})^{-1}(U_{10}^{(3)})^{-1}(U_{10}^{(4)})^{-1}(U_{10}^{(5)})^{-1}(U_{10}^{(6)})^{-1}(U_{10}^{(7)})^{-1}(U_{10}^{(8)})^{-1}(U_{10}^{(9)})^{-1}$$

and

$$L_{10}^{-1} = (L_{10}^{(8)})^{-1}(L_{10}^{(7)})^{-1}(L_{10}^{(6)})^{-1}(L_{10}^{(5)})^{-1}(L_{10}^{(4)})^{-1}(L_{10}^{(3)})^{-1}(L_{10}^{(2)})^{-1}(L_{10}^{(1)})^{-1}.$$

By Theorem 4.3.1, we obtain

$$(U_{10}^{(1)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{v_1}{v_2-v_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v_2-v_1} \end{bmatrix},$$

$$(U_{10}^{(2)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{v_1}{v_2-v_1} & \frac{v_1}{v_2-v_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v_2-v_1} & -\frac{1}{v_2-v_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v_2} \end{bmatrix},$$

$$(U_{10}^{(3)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{v_1}{v_2-v_1} & \frac{v_1}{v_2-v_1} & -\frac{v_1}{2(v_2-v_1)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v_2-v_1} & -\frac{1}{v_2-v_1} & \frac{1}{2(v_2-v_1)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v_2} & -\frac{1}{2v_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2v_2} \end{bmatrix},$$

$$(U_{10}^{(4)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{v_1}{v_2-v_1} & \frac{v_1}{v_2-v_1} & -\frac{v_1}{2(v_2-v_1)} & \frac{v_1}{6(v_2-v_1)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v_2-v_1} & -\frac{1}{v_2-v_1} & \frac{1}{2(v_2-v_1)} & -\frac{1}{6(v_2-v_1)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v_2} & -\frac{1}{2v_2} & \frac{1}{6v_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2v_2} & -\frac{1}{6v_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3v_2} \end{bmatrix},$$

$$
(U_{10}^{(5)})^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -\frac{v_1}{v_2-v_1} & \frac{v_1}{v_2-v_1} & -\frac{v_1}{2(v_2-v_1)} & \frac{v_1}{6(v_2-v_1)} & -\frac{v_1}{24(v_2-v_1)} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{v_2-v_1} & -\frac{1}{v_2-v_1} & \frac{1}{2(v_2-v_1)} & -\frac{1}{6(v_2-v_1)} & \frac{1}{24(v_2-v_1)} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v_2} & -\frac{1}{2v_2} & \frac{1}{6v_2} & -\frac{1}{24v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2v_2} & -\frac{1}{6v_2} & \frac{1}{24v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3v_2} & -\frac{1}{12v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4v_2}
\end{bmatrix},
$$

$$
(U_{10}^{(6)})^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -\frac{v_1}{v_2-v_1} & \frac{v_1}{v_2-v_1} & -\frac{v_1}{2(v_2-v_1)} & \frac{v_1}{6(v_2-v_1)} & -\frac{v_1}{24(v_2-v_1)} & \frac{v_1}{120(v_2-v_1)} \\
0 & 0 & 0 & 0 & \frac{1}{v_2-v_1} & -\frac{1}{v_2-v_1} & \frac{1}{2(v_2-v_1)} & -\frac{1}{6(v_2-v_1)} & \frac{1}{24(v_2-v_1)} & -\frac{1}{120(v_2-v_1)} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{v_2} & -\frac{1}{2v_2} & \frac{1}{6v_2} & -\frac{1}{24v_2} & \frac{1}{120v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2v_2} & -\frac{1}{6v_2} & \frac{1}{24v_2} & -\frac{1}{120v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3v_2} & -\frac{1}{12v_2} & \frac{1}{60v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4v_2} & -\frac{1}{20v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5v_2}
\end{bmatrix},
$$

$$
(U_{10}^{(7)})^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -\frac{v_1}{v_2-v_1} & \frac{v_1}{v_2-v_1} & -\frac{v_1}{2(v_2-v_1)} & \frac{v_1}{6(v_2-v_1)} & -\frac{v_1}{24(v_2-v_1)} & \frac{v_1}{120(v_2-v_1)} & -\frac{v_1}{720(v_2-v_1)} \\
0 & 0 & 0 & \frac{1}{v_2-v_1} & -\frac{1}{v_2-v_1} & \frac{1}{2(v_2-v_1)} & -\frac{1}{6(v_2-v_1)} & \frac{1}{24(v_2-v_1)} & -\frac{1}{120(v_2-v_1)} & \frac{1}{720(v_2-v_1)} \\
0 & 0 & 0 & 0 & \frac{1}{v_2} & -\frac{1}{2v_2} & \frac{1}{6v_2} & -\frac{1}{24v_2} & \frac{1}{120v_2} & -\frac{1}{720v_2} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2v_2} & -\frac{1}{6v_2} & \frac{1}{24v_2} & -\frac{1}{120v_2} & \frac{1}{720v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3v_2} & -\frac{1}{12v_2} & \frac{1}{60v_2} & -\frac{1}{360v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4v_2} & -\frac{1}{20v_2} & \frac{1}{120v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5v_2} & -\frac{1}{30v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6v_2}
\end{bmatrix},
$$

$$
(U_{10}^{(8)})^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -\frac{v_1}{v_2-v_1} & \frac{v_1}{v_2-v_1} & -\frac{v_1}{2(v_2-v_1)} & \frac{v_1}{6(v_2-v_1)} & -\frac{v_1}{24(v_2-v_1)} & \frac{v_1}{120(v_2-v_1)} & -\frac{v_1}{720(v_2-v_1)} & \frac{v_1}{5040(v_2-v_1)} \\
0 & 0 & \frac{1}{v_2-v_1} & -\frac{1}{v_2-v_1} & \frac{1}{2(v_2-v_1)} & -\frac{1}{6(v_2-v_1)} & \frac{1}{24(v_2-v_1)} & -\frac{1}{120(v_2-v_1)} & \frac{1}{720(v_2-v_1)} & -\frac{1}{5040(v_2-v_1)} \\
0 & 0 & 0 & \frac{1}{v_2} & -\frac{1}{2v_2} & \frac{1}{6v_2} & -\frac{1}{24v_2} & \frac{1}{120v_2} & -\frac{1}{720v_2} & \frac{1}{5040v_2} \\
0 & 0 & 0 & 0 & \frac{1}{2v_2} & -\frac{1}{6v_2} & \frac{1}{24v_2} & -\frac{1}{120v_2} & \frac{1}{720v_2} & -\frac{1}{5040v_2} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3v_2} & -\frac{1}{12v_2} & \frac{1}{60v_2} & -\frac{1}{360v_2} & \frac{1}{2520v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4v_2} & -\frac{1}{20v_2} & \frac{1}{120v_2} & -\frac{1}{840v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5v_2} & -\frac{1}{30v_2} & \frac{1}{210v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6v_2} & -\frac{1}{42v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7v_2}
\end{bmatrix},
$$

$$
(U_{10}^{(9)})^{-1} = \begin{bmatrix}
1 & -\frac{v_1}{v_2-v_1} & \frac{v_1}{v_2-v_1} & -\frac{v_1}{2(v_2-v_1)} & \frac{v_1}{6(v_2-v_1)} & -\frac{v_1}{24(v_2-v_1)} & \frac{v_1}{120(v_2-v_1)} & -\frac{v_1}{720(v_2-v_1)} & \frac{v_1}{5040(v_2-v_1)} & -\frac{v_1}{40320(v_2-v_1)} \\
0 & \frac{1}{v_2-v_1} & -\frac{1}{v_2-v_1} & \frac{1}{2(v_2-v_1)} & -\frac{1}{6(v_2-v_1)} & \frac{1}{24(v_2-v_1)} & -\frac{1}{120(v_2-v_1)} & \frac{1}{720(v_2-v_1)} & -\frac{1}{5040(v_2-v_1)} & \frac{1}{40320(v_2-v_1)} \\
0 & 0 & \frac{1}{v_2} & -\frac{1}{2v_2} & \frac{1}{6v_2} & -\frac{1}{24v_2} & \frac{1}{120v_2} & -\frac{1}{720v_2} & \frac{1}{5040v_2} & -\frac{1}{40320v_2} \\
0 & 0 & 0 & \frac{1}{2v_2} & -\frac{1}{6v_2} & \frac{1}{24v_2} & -\frac{1}{120v_2} & \frac{1}{720v_2} & -\frac{1}{5040v_2} & \frac{1}{40320v_2} \\
0 & 0 & 0 & 0 & \frac{1}{3v_2} & -\frac{1}{12v_2} & \frac{1}{60v_2} & -\frac{1}{360v_2} & \frac{1}{2520v_2} & -\frac{1}{20160v_2} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4v_2} & -\frac{1}{20v_2} & \frac{1}{120v_2} & -\frac{1}{840v_2} & \frac{1}{6720v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5v_2} & -\frac{1}{30v_2} & \frac{1}{210v_2} & -\frac{1}{1680v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6v_2} & -\frac{1}{42v_2} & \frac{1}{336v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7v_2} & -\frac{1}{56v_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8v_2}
\end{bmatrix},
$$

$$
(L_{10}^{(8)})^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{v_2}{v_2-v_1} & -\frac{v_2}{v_2-v_1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2v_2^2}{(v_2-v_1)^2} & \frac{2v_2^2}{(v_2-v_1)^2} & -\frac{2v_2}{v_2-v_1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{6v_2^3}{(v_2-v_1)^3} & -\frac{6v_2^3}{(v_2-v_1)^3} & \frac{6v_2^2}{(v_2-v_1)^2} & -\frac{3v_2}{v_2-v_1} & 1 & 0 & 0 & 0 & 0 & 0 \\
-\frac{24v_2^4}{(v_2-v_1)^4} & \frac{24v_2^4}{(v_2-v_1)^4} & -\frac{24v_2^3}{(v_2-v_1)^3} & \frac{12v_2^2}{(v_2-v_1)^2} & -\frac{4v_2}{v_2-v_1} & 1 & 0 & 0 & 0 & 0 \\
\frac{120v_2^5}{(v_2-v_1)^5} & -\frac{120v_2^5}{(v_2-v_1)^5} & \frac{120v_2^4}{(v_2-v_1)^4} & -\frac{60v_2^3}{(v_2-v_1)^3} & \frac{20v_2^2}{(v_2-v_1)^2} & -\frac{5v_2}{v_2-v_1} & 1 & 0 & 0 & 0 \\
-\frac{720v_2^6}{(v_2-v_1)^6} & \frac{720v_2^6}{(v_2-v_1)^6} & -\frac{720v_2^5}{(v_2-v_1)^5} & \frac{360v_2^4}{(v_2-v_1)^4} & -\frac{120v_2^3}{(v_2-v_1)^3} & \frac{30v_2^2}{(v_2-v_1)^2} & -\frac{6v_2}{v_2-v_1} & 1 & 0 & 0 \\
\frac{5040v_2^7}{(v_2-v_1)^7} & -\frac{5040v_2^7}{(v_2-v_1)^7} & \frac{5040v_2^6}{(v_2-v_1)^6} & -\frac{2520v_2^5}{(v_2-v_1)^5} & \frac{840v_2^4}{(v_2-v_1)^4} & -\frac{210v_2^3}{(v_2-v_1)^3} & \frac{42v_2^2}{(v_2-v_1)^2} & -\frac{7v_2}{v_2-v_1} & 1 & 0 \\
-\frac{40320v_2^8}{(v_2-v_1)^8} & \frac{40320v_2^8}{(v_2-v_1)^8} & -\frac{40320v_2^7}{(v_2-v_1)^7} & \frac{20160v_2^6}{(v_2-v_1)^6} & -\frac{6720v_2^5}{(v_2-v_1)^5} & \frac{1680v_2^4}{(v_2-v_1)^4} & -\frac{336v_2^3}{(v_2-v_1)^3} & \frac{56v_2^2}{(v_2-v_1)^2} & -\frac{8v_2}{v_2-v_1} & 1
\end{bmatrix},
$$

$$
(L_{10}^{(7)})^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -6 & 6 & -3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 24 & -24 & 12 & -4 & 1 & 0 & 0 & 0 \\
0 & 0 & -120 & 120 & -60 & 20 & -5 & 1 & 0 & 0 \\
0 & 0 & 720 & -720 & 360 & -120 & 30 & -6 & 1 & 0 \\
0 & 0 & -5040 & 5040 & -2520 & 840 & -210 & 42 & -7 & 1
\end{bmatrix},
$$

$$(L_{10}^{(6)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & 6 & -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 24 & -24 & 12 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -120 & 120 & -60 & 20 & -5 & 1 & 0 \\ 0 & 0 & 0 & 720 & -720 & 360 & -120 & 30 & -6 & 1 \end{bmatrix},$$

$$(L_{10}^{(5)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & 6 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 24 & -24 & 12 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & -120 & 120 & -60 & 20 & -5 & 1 \end{bmatrix},$$

$$(L_{10}^{(4)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & 6 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 24 & -24 & 12 & -4 & 1 \end{bmatrix},$$

$$(L_{10}^{(3)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 & 6 & -3 & 1 \end{bmatrix},$$

$$(L_{10}^{(2)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 1 \end{bmatrix},$$

$$(L_{10}^{(1)})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

In conclusion, this example showed that our Theorem 4.3.1 provided us with the explicit inversion formula for $V_{\{2;1,n-1\}}$.