Chapter 5

An Inductive Computation of Totally Positive Generalized Vandermonde Matrices Avoiding Schur Functions

An LU factorization is to express a matrix as a product of a lower triangular matrix L and an upper triangular matrix U. If L is a lower triangular matrix with unit main diagonal and U is an upper triangular matrix, the LU factorization is unique. In this chapter we will study the LU factorizations of the $n \times n$ generalized Vandermonde matrix $G_{\{n;a_1,a_2,\cdots,a_n\}}$, introduced in Chapter 1.

Let

$$G_{\{n;a_1,a_2,\cdots,a_n\}} = \begin{bmatrix} x_1^{a_1} & x_1^{a_2} & \cdots & x_1^{a_n} \\ x_2^{a_1} & x_2^{a_2} & \cdots & x_2^{a_n} \\ \vdots & \vdots & \vdots & \vdots \\ x_n^{a_1} & x_n^{a_2} & \cdots & x_n^{a_n} \end{bmatrix}$$

be a generalized Vandermonde matrix and is called a totally positive(TP) generalized Vandermonde matrix if $0 \le a_1 < a_2 < \cdots < a_n$ are integers and $0 < x_1 < x_2 < \cdots < x_n$. In connection with Schur functions we define the partition λ associated with $(a_1, a_2, \cdots, a_n)$ as the nonincreasing sequence of nonnegative integers

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) = (a_n - (n-1), a_{n-1} - (n-2), \cdots, a_1),$$

and get

$$G_{\{n;a_1,a_2,\cdots,a_n\}} = [x_i^{j-1+\lambda_{n-j+1}}]_{1\le i,j\le n},$$

and Schur function associated to λ is defined as

$$s_\lambda(x_1,x_2,\cdots,x_n) = \frac{\det G_{\{n;a_1,a_2,\cdots,a_n\}}}{\det G_{\{n;0,1,\cdots,n-1\}}}.$$

In a recent paper [4], J. Demmel and P. Koev performed accurate and efficient matrix computations of $G_n := G_{\{n;a_1,a_2,\cdots,a_n\}}$: Their results, among others, are an explicit bidiagonal decompositions of G_n^{-1} and an LDU decomposition of G_n which involves Schur functions. In what follows the submatrix of A consisting of rows i through j and columns k through l will be denoted as $A(i:j,k:l)$ or, equivalently, $A_{i;j,k:l}$. Similarly, $x_{1:n}$ will mean the vector $(x_1,x_2,\cdots,x_n)$. Now we give the results of Demmel and Koev explicitly as follows:

1. Bidiagonal decomposition of G_n^{-1}. G_n^{-1} can be factored as

$$G_n^{-1} = U_n^{(1)}U_n^{(2)}\cdots U_n^{(n-1)}D_n^{-1}L_n^{(n-1)}L_n^{(n-2)}\cdots L_n^{(1)}$$

where $L_n^{(k)}$ and $U_n^{(k)}$, $k=1,2,\cdots,n$, are unit lower and upper bidiagonal matrices, respectively, such that $L_n^{(k)}(i+1,i) = U_n^{(k)}(i,i+1) = 0$ for $i<k$, and D_n is diagonal. Also for $i\ge k$,

$$L_n^{(k)}(i+1,i) = -\frac{s_{(\lambda_{n-k+1:n})}(x_{i-k+2:i+1})}{s_{(\lambda_{n-k+2:n})}(x_{i-k+2:i})}\times\frac{s_{(\lambda_{n-k+2:n})}(x_{i-k+1:i-1})}{s_{(\lambda_{n-k+1:n})}(x_{i-k+1:i})}\times\prod_{j=i-k+1}^{i-1}\frac{x_{i+1}-x_{j+1}}{x_i-x_j},$$

$$U_n^{(k)}(i,i+1) = -x_k\times\frac{s_{(\lambda_{n-i:n-i+k-1})}(x_{1:k})}{s_{(\lambda_{n-i+1:n-i+k-1})}(x_{1:k-1})}\times\frac{s_{(\lambda_{n-i+2:n-i+k})}(x_{1:k-1})}{s_{(\lambda_{n-i+1:n-i+k})}(x_{1:k})},$$

and

$$D_n^{-1}(i,i) = \frac{s_{(\lambda_{n-i+2:n})}(x_{1:i-1})}{s_{(\lambda_{n-i+1:i})}(x_{1:i})}\times\prod_{j=1}^{i-1}\frac{1}{x_i-x_j}.$$

2. The inverse of G_n. The explicit formula for the entries of G_n^{-1} is

$$G_n^{-1}(l,k) = (-1)^{n-l}\frac{s_{\lambda^{(k,l)}}(x_1,\cdots,\hat{x}_k,\cdots,x_n)}{s_\lambda(x_1,\cdots,x_n)}\times\prod_{i=1,i\ne k}^{n}\frac{1}{x_i-x_j},$$

where $\lambda^{(k,l)} = (\lambda_1+1,\lambda_2+1,\cdots,\lambda_{n-l}+1,\lambda_{n-l+2},\cdots,\lambda_n)$. The "hats" indicate the omission of the symbols they cover.

3. The LDU decomposition of G_n. Let $G_n = L_n D_n U_n$, where L_n and U_n are unit lower and upper triangular matrices, respectively. Then

$$D_n(i,i) = \frac{s_{(\lambda_{n-i+1},\cdots,\lambda_p)}(x_{1:i})}{s_{(\lambda_{n-i+2},\cdots,\lambda_p)}(x_{1:i-1})} \times \prod_{j=1}^{i-1}(x_i - x_j);$$

$$U_n(i,j) = \frac{s_{(\lambda_{n-j+1}+j-i,\lambda_{n-i+2},\cdots,\lambda_p)}(x_{1:i})}{s_{(\lambda_{n-i+1},\cdots,\lambda_p)}(x_{1:i})}, i < j;$$

$$L_n(i,j) = \frac{s_{(\lambda_{n-j+1},\cdots,\lambda_p)}(x_{[1:j-1,i]})}{s_{(\lambda_{n-j+1},\cdots,\lambda_p)}(x_{1:j})} \times \prod_{k=1}^{j-1}\frac{x_i - x_k}{x_j - x_k}, i > j.$$

In a previous paper [11], using only mathematical induction, we succeeded to provide the unique LU factorization of a special case $V_{\{2;1,n-1\}}$ of generalized Vandermonde matrices , and this prompts us to do the same thing with G_n. Our main result presents an explicit LU factorization without involving Schur functions. As by-products, we calculate the determinant, the inverse of G_n and the Schur functions $s_\lambda(x_1, x_2, \cdots, x_n)$.

5.1 The LU Factorization of G_n

Now we give the explicit form of the LU factorization of G_n.

Theorem 5.1.1 *G_n can be factorized as $G_n = L_n U_n$, where $L_n = [L_n(i,j)]$ is a lower triangular matrix with unit main diagonal and $U_n = [U_n(i,j)]$ is an upper triangular matrix, whose entries are defined as follows:*

$$L_n(i,j) = \begin{cases} 1, & i = j; \\ 0, & i < j; \\ (\frac{x_i}{x_1})^{a_1}, & j = 1, i \geq 2; \\ (\frac{x_i}{x_j})^{a_1} \frac{S_{\{x_j \to x_i\}}(A_j)}{A_j}, & i \geq j+1, j \geq 2. \end{cases}$$

and

$$U_n(i,j) = \begin{cases} x_1^{a_j}, & i = 1; \\ 0, & i > j; \\ x_i^{a_1} B_i, & i = j \geq 2; \\ x_i^{a_1} S^{\{a_i \to a_j\}}(B_i), & j \geq i+1, i \geq 2. \end{cases}$$

where

$$A_2 = B_2 = x_2^{a_2-a_1} - x_1^{a_2-a_1};$$

$$A_k = \{S_{\{x_{k-1}\to x_k\}}^{\{a_{k-1}\to a_k\}}(A_{k-1})\}A_{k-1} - S_{\{x_{k-1}\to x_k\}}(A_{k-1})S^{\{a_{k-1}\to a_k\}}(A_{k-1}), k \geq 3;$$

$$B_k = S_{\{x_{k-1}\to x_k\}}^{\{a_{k-1}\to a_k\}}(B_{k-1}) - \{S^{\{a_{k-1}\to a_k\}}(B_{k-1})\}(\frac{S_{\{x_{k-1}\to x_k\}}(A_{k-1})}{A_{k-1}}), k \geq 3;$$

$$S_{\{x_{k-1}\to x_k\}}(A_{k-1}) := x_k \textit{ substitutes for } x_{k-1} \textit{ in } A_{k-1};$$

$$S^{\{a_{k-1}\to a_k\}}(A_{k-1}) := a_k \textit{ substitutes for } a_{k-1} \textit{ in } A_{k-1}.$$

Proof. We use mathematical induction on n, the size of G_n.

(1) The case $n = 2$:

$$L_2U_2 = \begin{bmatrix} 1 & 0 \\ (\frac{x_2}{x_1})^{a_1} & 1 \end{bmatrix} \begin{bmatrix} x_1^{a_1} & x_1^{a_2} \\ 0 & x_2^{a_1}(x_2^{a_2-a_1} - x_1^{a_2-a_1}) \end{bmatrix} = \begin{bmatrix} x_1^{a_1} & x_1^{a_2} \\ x_2^{a_1} & x_2^{a_2} \end{bmatrix} = G_2.$$

(2) The case $k \Rightarrow k+1$ with $k > 1$: Assume $G_k = L_kU_k$ holds, we want to prove $G_{k+1} = L_{k+1}L_{k+1}$. By inductive hypothesis, we know that

$$G_k(k,l) = \sum_{m=1}^{l} L_k(k,m)U_k(m,l), 1 \leq l \leq k, \tag{5.1.1}$$

$$G_k(l,k) = \sum_{m=1}^{l} L_k(l,m)U_k(m,k), 1 \leq l \leq k. \tag{5.1.2}$$

It's sufficient to show that

$$G_{k+1}(k+1,l) = \sum_{m=1}^{k+1} L_{k+1}(k+1,m)U_{k+1}(m,l), 1 \leq l \leq k,$$

$$G_{k+1}(l,k+1) = \sum_{m=1}^{k+1} L_{k+1}(l,m)U_{k+1}(m,k+1), 1 \leq l \leq k+1.$$

At first, we find that

$$\sum_{m=1}^{k+1} L_{k+1}(k+1,m)U_{k+1}(m,1) = L_{k+1}(k+1,1)U_{k+1}(1,1) = (\frac{x_{k+1}}{x_1})^{a_1} \times x_1^{a_1} = x_1^{a_1} = G_{k+1}(k+1,1),$$

and by (5.1.1), we can get

$$\sum_{m=1}^{k} L_k(k,m)U_k(m,l) = \sum_{m=1}^{l} L_k(k,m)U_k(m,l)$$

$= (\frac{x_k}{x_1})^{a_1} \times x_1^{a_l} + \sum_{m=2}^{l} (\frac{x_k}{x_m})^{a_1} \frac{S_{\{x_m \to x_k\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m \to a_l\}}(B_m)$
$= G_k(k,l) = x_k^{a_l},\ 2 \le l \le k-1,$

so

$\sum_{m=1}^{k+1} L_{k+1}(k+1,m) U_{k+1}(m,l) = \sum_{m=1}^{l} L_{k+1}(k+1,m) U_{k+1}(m,l)$
$= (\frac{x_{k+1}}{x_1})^{a_1} \times x_1^{a_l} + \sum_{m=2}^{l} (\frac{x_k}{x_m})^{a_1} \frac{S_{\{x_m \to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m \to a_l\}}(B_m)$
$= x_{k+1}^{a_l} = G_{k+1}(k+1,l),\ 2 \le l \le k-1.$

Besides, we have

$$\sum_{m=1}^{k+1} L_{k+1}(1,m) U_{k+1}(m,k+1) = L_{k+1}(1,1) U_{k+1}(1,k+1) = 1 \times x_1^{a_{k+1}} = x_1^{a_{k+1}} = G_{k+1}(1,k+1),$$

and by (5.1.2), we can get

$\sum_{m=1}^{k} L_k(l,m) U_k(m,k) = \sum_{m=1}^{l} L_k(l,m) U_k(m,k)$
$= (\frac{x_l}{x_1})^{a_1} \times x_1^{a_k} + \sum_{m=2}^{l} (\frac{x_l}{x_m})^{a_1} \frac{S_{\{x_m \to x_l\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m \to a_k\}}(B_m)$
$= G_k(l,k) = x_l^{a_k},\ 2 \le l \le k-1,$

so

$\sum_{m=1}^{k+1} L_{k+1}(l,m) U_{k+1}(m,k+1) = \sum_{m=1}^{l} L_{k+1}(l,m) U_{k+1}(m,k+1)$
$= (\frac{x_l}{x_1})^{a_1} \times x_1^{a_{k+1}} + \sum_{m=2}^{l} (\frac{x_l}{x_m})^{a_1} \frac{S_{\{x_m \to x_l\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m \to a_{k+1}\}}(B_m)$
$= x_l^{a_{k+1}} = G_{k+1}(l,k+1),\ 2 \le l \le k-1;$

let $l = k$ in (5.1.2), we establish the following equation:

$x_k^{a_k} = \sum_{m=1}^{k} L_k(k,m) U_k(m,k)$
$= (\frac{x_k}{x_1})^{a_1} \times x_1^{a_k} + \sum_{m=2}^{k-1} (\frac{x_k}{x_m})^{a_1} \frac{S_{\{x_m \to x_k\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m \to a_k\}}(B_m) + 1 \times x_k^{a_1} B_k,$ (5.1.3)

so from above, we conclude that

$\sum_{m=1}^{k+1} L_{k+1}(k,m) U_{k+1}(m,k) = \sum_{m=1}^{k} L_{k+1}(k,m) U_{k+1}(m,k)$
$= (\frac{x_k}{x_1})^{a_1} \times x_1^{a_{k+1}} + \sum_{m=2}^{k-1} (\frac{x_k}{x_m})^{a_1} \frac{S_{\{x_m \to x_k\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m \to a_{k+1}\}}(B_m)$
$+ 1 \times x_k^{a_1} \times S^{\{a_k \to a_{k+1}\}} B_k$
$= x_k^{a_{k+1}} = G_{k+1}(k,k+1),$

on the other hand, we assume
$x_{k+1}^{a_k} \neq (\frac{x_{k+1}}{x_1})^{a_1} \times x_1^{a_k} + \sum_{m=2}^{k-1}(\frac{x_{k+1}}{x_m})^{a_1}\frac{S_{\{x_m\to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1}S^{\{a_m\to a_k\}}(B_m)$
$+ x_{k+1}^{a_1}(\frac{S_{\{x_k\to x_{k+1}\}}(A_k)}{A_k}) \times B_k,$
then
$S_{\{x_{k+1}\to x_k\}}(x_{k+1}^{a_k})$
$\neq S_{\{x_{k+1}\to x_k\}}((\frac{x_{k+1}}{x_1})^{a_1} \times x_1^{a_k} + \sum_{m=2}^{k-1}(\frac{x_{k+1}}{x_m})^{a_1}\frac{S_{\{x_m\to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1}S^{\{a_m\to a_k\}}(B_m))$
$+ S_{\{x_{k+1}\to x_k\}}(x_{k+1}^{a_1}(\frac{S_{\{x_k\to x_{k+1}\}}(A_k)}{A_k}) \times B_k),$
so

$$x_k^{a_k} \neq (\frac{x_k}{x_1})^{a_1} \times x_1^{a_k} + \sum_{m=2}^{k-1}(\frac{x_k}{x_m})^{a_1}\frac{S_{\{x_m\to x_k\}}(A_m)}{A_m} \times x_m^{a_1}S^{\{a_m\to a_k\}}(B_m) + 1 \times x_k^{a_1}B_k,$$

contradicting to (5.1.3), hence
$\sum_{m=1}^{k+1} L_{k+1}(k+1,m)U_{k+1}(m,k) = \sum_{m=1}^{k} L_{k+1}(k+1,m)U_{k+1}(m,k)$
$= (\frac{x_{k+1}}{x_1})^{a_1} \times x_1^{a_k} + \sum_{m=2}^{k-1}(\frac{x_{k+1}}{x_m})^{a_1}\frac{S_{\{x_m\to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1}S^{\{a_m\to a_k\}}(B_m)$
$+ (\frac{x_{k+1}}{x_k})^{a_1}(\frac{S_{\{x_k\to x_{k+1}\}}(A_k)}{A_k}) \times x_k^{a_1}B_k$
$= x_{k+1}^{a_k} = G_{k+1}(k+1,k),$
and by (5.1.3),
$\sum_{m=1}^{k+1} L_{k+1}(k,m)U_{k+1}(m,k+1)$
$= (\frac{x_{k+1}}{x_1})^{a_1} \times x_1^{a_{k+1}} + \sum_{m=2}^{k}(\frac{x_{k+1}}{x_m})^{a_1}\frac{S_{\{x_m\to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1}S^{\{a_m\to a_{k+1}\}}(B_m) + 1 \times x_{k+1}^{a_1}B_{k+1}$
$= (\frac{x_{k+1}}{x_1})^{a_1} \times x_1^{a_{k+1}} + \sum_{m=2}^{k-1}(\frac{x_{k+1}}{x_m})^{a_1}\frac{S_{\{x_m\to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1}S^{\{a_m\to a_{k+1}\}}(B_m)$
$+ (\frac{x_{k+1}}{x_k})^{a_1}\frac{S_{\{x_k\to x_{k+1}\}}(A_k)}{A_k} \times x_k^{a_1}S^{\{a_k\to a_{k+1}\}}(B_k) + x_{k+1}^{a_1}B_{k+1}$
$= (\frac{x_{k+1}}{x_1})^{a_1} \times x_1^{a_{k+1}} + \sum_{m=2}^{k-1}(\frac{x_{k+1}}{x_m})^{a_1}\frac{S_{\{x_m\to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1}S^{\{a_m\to a_{k+1}\}}(B_m)$
$+ x_{k+1}^{a_1}\frac{S_{\{x_k\to x_{k+1}\}}(A_k)}{A_k} \times S^{\{a_k\to a_{k+1}\}}(B_k)$
$+ x_{k+1}^{a_1} \times (S_{\{x_k\to x_{k+1}\}}^{\{a_k\to a_{k+1}\}}(B_k) - S^{\{a_k\to a_{k+1}\}}(B_k)(\frac{S_{\{x_k\to x_{k+1}\}}(A_k)}{(A_k)}))$

$$= (\frac{x_{k+1}}{x_1})^{a_1} \times x_1^{a_{k+1}} + \sum_{m=2}^{k-1} (\frac{x_{k+1}}{x_m})^{a_1} \frac{S_{\{x_m \to x_{k+1}\}}(A_m)}{A_m} \times x_m^{a_1} S^{\{a_m \to a_{k+1}\}}(B_m)$$
$$+ x_{k+1}^{a_1} \times S^{\{a_k \to a_{k+1}\}}_{\{x_k \to x_{k+1}\}}(B_k)$$
$$= x_{k+1}^{a_{k+1}} = G_{k+1}(k+1, k+1).$$

So far we complete the proof.

Example 5.1.2 To illustrate our result, we give an example for $n = 4$: $G_4 = L_4 U_4$, where

$$G_4 = \begin{bmatrix} {x_1}^{a_1} & {x_1}^{a_2} & {x_1}^{a_3} & {x_1}^{a_4} \\ {x_2}^{a_1} & {x_2}^{a_2} & {x_2}^{a_3} & {x_2}^{a_4} \\ {x_3}^{a_1} & {x_3}^{a_2} & {x_3}^{a_3} & {x_3}^{a_4} \\ {x_4}^{a_1} & {x_4}^{a_2} & {x_4}^{a_3} & {x_4}^{a_4} \end{bmatrix},$$

$$L_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ (\frac{x_2}{x_1})^{a_1} & 1 & 0 & 0 \\ (\frac{x_3}{x_1})^{a_1} & (\frac{x_3}{x_2})^{a_1}(\frac{{x_3}^{a_2-a_1}-{x_1}^{a_2-a_1}}{{x_2}^{a_2-a_1}-{x_1}^{a_2-a_1}}) & 1 & 0 \\ (\frac{x_4}{x_1})^{a_1} & (\frac{x_4}{x_2})^{a_1}(\frac{{x_4}^{a_2-a_1}-{x_1}^{a_2-a_1}}{{x_2}^{a_2-a_1}-{x_1}^{a_2-a_1}}) & L_4(4,3) & 1 \end{bmatrix}$$

where

$$L_4(4,3)=(\frac{x_4}{x_3})^{a_1}[\frac{({x_4}^{a_3-a_1}-{x_1}^{a_3-a_1})({x_2}^{a_2-a_1}-{x_1}^{a_2-a_1})-({x_4}^{a_2-a_1}-{x_1}^{a_2-a_1})({x_2}^{a_3-a_1}-{x_1}^{a_3-a_1})}{({x_3}^{a_3-a_1}-{x_1}^{a_3-a_1})({x_2}^{a_2-a_1}-{x_1}^{a_2-a_1})-({x_3}^{a_2-a_1}-{x_1}^{a_2-a_1})({x_2}^{a_3-a_1}-{x_1}^{a_3-a_1})}],$$

and

$$U_4 = \begin{bmatrix} x_1^{a_1} & x_1^{a_2} & x_1^{a_3} & x_1^{a_4} \\ 0 & x_2^{a_1}(x_2^{a_2-a_1} - x_1^{a_2-a_1}) & x_2^{a_1}(x_2^{a_3-a_1} - x_1^{a_3-a_1}) & x_2^{a_1}(x_2^{a_4-a_1}-x_1^{a_4-a_1}) \\ 0 & 0 & x_3^{a_1}((x_3^{a_3-a_1} - x_1^{a_3-a_1}) - \frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}) & x_3^{a_1}((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}) \\ 0 & 0 & 0 & \begin{array}{c}\{x_4^{a_1}((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}) \\ \times[\frac{({x_4}^{a_3-a_1}-{x_1}^{a_3-a_1})({x_2}^{a_2-a_1}-{x_1}^{a_2-a_1})-({x_4}^{a_2-a_1}-{x_1}^{a_2-a_1})({x_2}^{a_3-a_1}-{x_1}^{a_3-a_1})}{({x_3}^{a_3-a_1}-{x_1}^{a_3-a_1})({x_2}^{a_2-a_1}-{x_1}^{a_2-a_1})-({x_3}^{a_2-a_1}-{x_1}^{a_2-a_1})({x_2}^{a_3-a_1}-{x_1}^{a_3-a_1})}])\}\end{array} \end{bmatrix}.$$

The following three examples are programs running by Mathematica, the results also help us to show that the LU decomposition is correct.

Example 5.1.3 Let $n = 3$, $x_1 = x, x_2 = y, x_3 = z$ and $a_1 = 0, a_2 = 2, a_3 = 4$. Then the program

$FullSimplify[\{\{1,0,0\},\{1,1,0\},\{1,(z^2-x^2)/(y^2-x^2),1\}\}.\{\{1,x^2,x^4\},\{0,(y^2-x^2),(y^4-x^4)\},\{0,0,((z^4-x^4)-((z^2-x^2)(y^4-x^4))/(y^2-x^2))\}\}]$

yields the result

$$\begin{bmatrix} 1 & x^2 & x^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{bmatrix}.$$

Example 5.1.4 Let $n = 4$, $x_1 = x, x_2 = y, x_3 = z, x_4 = w$ and $a_1 = 0, a_2 = 2, a_3 = 4, a_4 = 6$. Then the program

$FullSimplify[\{\{1,0,0,0\},\{1,1,0,0\},\{1,(z^2-x^2)/(y^2-x^2),1,0\},\{1,(w^2-x^2)/(y^2-x^2),((w^4-x^4)(y^2-x^2)-(w^2-x^2)(y^4-x^4))/((z^4-x^4)(y^2-x^2)-(z^2-x^2)(y^4-x^4)),1\}\}.\{\{1,x^2,x^4,x^6\},\{0,(y^2-x^2),(y^4-x^4),(y^6-x^6)\},\{0,0,((z^4-x^4)-((z^2-x^2)(y^4-x^4))/(y^2-x^2)),((z^6-x^6)-((z^2-x^2)(y^6-x^6))/(y^2-x^2))\},\{0,0,0,((w^6-x^6)-((w^2-x^2)(y^6-x^6))/(y^2-x^2))-((z^6-x^6)-((z^2-x^2)(y^6-x^6))/(y^2-x^2))((w^4-x^4)(y^2-x^2)-(w^2-x^2)(y^4-x^4))/((z^4-x^4)(y^2-x^2)-(z^2-x^2)(y^4-x^4))\}\}]$

produces the result

$$\begin{bmatrix} 1 & x^2 & x^4 & x^6 \\ 1 & y^2 & y^4 & y^6 \\ 1 & z^2 & z^4 & z^6 \\ 1 & w^2 & w^4 & w^6 \end{bmatrix}.$$

Example 5.1.5 Let $n = 5$, $x_1 = x, x_2 = y, x_3 = z, x_4 = w, x_5 = v$ and $a_1 = 0, a_2 = 2, a_3 = 4, a_4 = 6, a_5 = 8$. Then the program

$FullSimplify[\{\{1,0,0,0,0\},\{1,1,0,0,0\},\{1,(z^2-x^2)/(y^2-x^2),1,0,0\},\{1,(w^2-x^2)/(y^2-x^2),((w^4-x^4)(y^2-x^2)-(w^2-x^2)(y^4-x^4))/((z^4-x^4)(y^2-x^2)-(z^2-x^2)(y^4-x^4)),1,0\},\{1,(v^2-x^2)/(y^2-x^2),((v^4-x^4)(y^2-x^2)-(v^2-x^2)(y^4-x^4))/((z^4-x^4)(y^2-x^2)-(z^2-x^2)(y^4-x^4)),(((v^6-x^6)(y^2-x^2)-(v^2-x^2)(y^6-x^6))((z^4-x^4)(y^2-x^2)-(z^2-x^2)(y^4-x^4))-((v^4-x^4)(y^2-x^2)-(v^2-x^2)(y^4-x^4))((z^6-x^6)(y^2-x^2)-(z^2-x^2)(y^6-x^6)))/(((w^6-x^6)(y^2-x^2)-(w^2-x^2)(y^6-x^6))((z^4-x^4)(y^2-x^2)-(z^2-x^2)(y^4-x^4))-((w^4-x^4)(y^2-x^2)-(w^2-x^2)(y^4-x^4))((z^6-x^6)(y^2-x^2)-$

$(z^2 - x^2)(y^6 - x^6))), 1\}\}.\{\{1, x^2, x^4, x^6, x^8\}, \{0, (y^2 - x^2), (y^4 - x^4), (y^6 - x^6), (y^8 -$
$x^8)\}, \{0, 0, ((z^4 - x^4) - ((z^2 - x^2)(y^4 - x^4))/(y^2 - x^2)), ((z^6 - x^6) - ((z^2 - x^2)(y^6 -$
$x^6))/(y^2-x^2)), ((z^8-x^8)-((z^2-x^2)(y^8-x^8))/(y^2-x^2))\}, \{0, 0, 0, ((w^6-x^6)-((w^2-$
$x^2)(y^6 - x^6))/(y^2 - x^2)) - ((z^6 - x^6) - ((z^2 - x^2)(y^6 - x^6))/(y^2 - x^2))((w^4 - x^4)(y^2 -$
$x^2) - (w^2 - x^2)(y^4 - x^4))/((z^4 - x^4)(y^2 - x^2) - (z^2 - x^2)(y^4 - x^4)), ((w^8 - x^8) - ((w^2 -$
$x^2)(y^8 - x^8))/(y^2 - x^2)) - ((z^8 - x^8) - ((z^2 - x^2)(y^8 - x^8))/(y^2 - x^2))((w^4 - x^4)(y^2 -$
$x^2) - (w^2 - x^2)(y^4 - x^4))/((z^4 - x^4)(y^2 - x^2) - (z^2 - x^2)(y^4 - x^4))\}, \{0, 0, 0, 0, (((v^8 -$
$x^8)-((v^2-x^2)(y^8-x^8))/(y^2-x^2))-((z^8-x^8)-((z^2-x^2)(y^8-x^8))/(y^2-x^2))((v^4-$
$x^4)(y^2 - x^2) - (v^2 - x^2)(y^4 - x^4))/((z^4 - x^4)(y^2 - x^2) - (z^2 - x^2)(y^4 - x^4))) - (((w^8 -$
$x^8)-((w^2-x^2)(y^8-x^8))/(y^2-x^2))-((z^8-x^8)-((z^2-x^2)(y^8-x^8))/(y^2-x^2))((w^4-$
$x^4)(y^2 - x^2) - (w^2 - x^2)(y^4 - x^4))/((z^4 - x^4)(y^2 - x^2) - (z^2 - x^2)(y^4 - x^4)))(((v^6 -$
$x^6)(y^2 - x^2) - (v^2 - x^2)(y^6 - x^6))((z^4 - x^4)(y^2 - x^2) - (z^2 - x^2)(y^4 - x^4)) - ((v^4 -$
$x^4)(y^2 - x^2) - (v^2 - x^2)(y^4 - x^4))((z^6 - x^6)(y^2 - x^2) - (z^2 - x^2)(y^6 - x^6)))/(((w^6 -$
$x^6)(y^2 - x^2) - (w^2 - x^2)(y^6 - x^6))((z^4 - x^4)(y^2 - x^2) - (z^2 - x^2)(y^4 - x^4)) - ((w^4 -$
$x^4)(y^2 - x^2) - (w^2 - x^2)(y^4 - x^4))((z^6 - x^6)(y^2 - x^2) - (z^2 - x^2)(y^6 - x^6)))\}\}]$

yields the result

$$\begin{bmatrix} 1 & x^2 & x^4 & x^6 & x^8 \\ 1 & y^2 & y^4 & y^6 & y^8 \\ 1 & z^2 & z^4 & z^6 & z^8 \\ 1 & w^2 & w^4 & w^6 & w^8 \\ 1 & v^2 & v^4 & v^6 & v^8 \end{bmatrix}.$$

5.2 The Determinant of G_n

We know that the determinant of a triangular matrix is the products of the entries of the main diagonal. According to this fact, we establish the following theorem.

Theorem 5.2.1 *The determinant of G_n is as following:*

$$\det G_n = \prod_{1 \le i \le n} U_n(i,i) = x_1^{a_1} \times \prod_{2 \le i \le n} x_i^{a_1} B_i.$$

Proof. A natural consequence by Theorem 5.1.1.

Example 5.2.2 Considering D_1 in [8], i.e.

$$D_1 = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^3 & x_2^3 & x_3^3 \end{bmatrix},$$

then

$$\begin{aligned}
D_1^t &= \det G_{\{3;0,2,3\}} \\
&= x_1^0 x_2^0 (x_2^{2-0} - x_1^{2-0}) x_3^0 ((x_3^{3-0} - x_1^{3-0}) - \frac{(x_3^{2-0} - x_1^{2-0})(x_2^{3-0} - x_1^{3-0})}{(x_2^{2-0} - x_1^{2-0})}) \\
&= ((x_3^3 - x_1^3)(x_2^2 - x_1^2) - (x_3^2 - x_1^2)(x_2^3 - x_1^3)) \\
&= ((x_3 - x_1)(x_3^2 + x_3x_1 + x_1^2)(x_2 - x_1)(x_2 + x_1) - (x_3 - x_1)(x_3 + x_1)(x_2 - x_1)(x_2^2 + x_2x_1 + x_1^2)) \\
&= (x_3 - x_1)(x_2 - x_1)((x_3^2 + x_3x_1 + x_1^2)(x_2 + x_1) - (x_3 + x_1)(x_2^2 + x_2x_1 + x_1^2)) \\
&= (x_3 - x_1)(x_2 - x_1)((x_3^2x_2 + x_3x_2x_1 + x_2x_1^2 + x_3^2x_1 + x_3x_1^2 + x_1^3) - (x_3x_2^2 + x_3x_2x_1 + x_3x_1^2 + x_2^2x_1 + x_2x_1^2 + x_1^3)) \\
&= (x_3 - x_1)(x_2 - x_1)(x_3^2x_2 + x_3^2x_1 - x_3x_2^2 - x_2^2x_1) \\
&= (x_3 - x_1)(x_2 - x_1)((x_3^2x_2 - x_3x_2^2) + (x_3^2x_1 - x_2^2x_1)) \\
&= (x_3 - x_1)(x_2 - x_1)(x_3x_2(x_3 - x_2) + x_1(x_3 - x_2)(x_3 + x_2)) \\
&= (x_3 - x_1)(x_2 - x_1)(x_3 - x_2)(x_3x_2 + x_3x_1 + x_2x_1) \\
&= (x_3x_2 + x_3x_1 + x_2x_1) \det V_3.
\end{aligned}$$

Example 5.2.3 Considering D_2 in [8], i.e.

$$D_2 = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^3 & x_2^3 & x_3^3 \end{bmatrix},$$

then

$$\begin{aligned}
D_2^t &= \det G_{\{3;0,1,3\}} \\
&= x_1^0 x_2^0 (x_2^{1-0} - x_1^{1-0}) x_3^0 ((x_3^{3-0} - x_1^{3-0}) - \frac{(x_3^{1-0} - x_1^{1-0})(x_2^{3-0} - x_1^{3-0})}{(x_2^{1-0} - x_1^{1-0})}) \\
&= (x_2 - x_1)(\frac{(x_3^3 - x_1^3)(x_2 - x_1) - (x_3 - x_1)(x_2^3 - x_1^3)}{(x_2 - x_1)}) \\
&= (x_3^3 - x_1^3)(x_2 - x_1) - (x_3 - x_1)(x_2^3 - x_1^3) \\
&= (x_3 - x_1)(x_3^2 + x_3x_1 + x_1^2)(x_2 - x_1) - (x_3 - x_1)(x_2 - x_1)(x_2^2 + x_2x_1 + x_1^2)
\end{aligned}$$

$$= (x_3 - x_1)(x_2 - x_1)((x_3^2 + x_3x_1 + x_1^2) - (x_2^2 + x_2x_1 + x_1^2))$$

$$= (x_3 - x_1)(x_2 - x_1)(x_3^2 + x_3x_1 - x_2^2 - x_2x_1)$$

$$= (x_3 - x_1)(x_2 - x_1)((x_3^2 - x_2^2) + (x_3x_1 - x_2x_1))$$

$$= (x_3 - x_1)(x_2 - x_1)((x_3 - x_2)(x_3 + x_2) + x_1(x_3 - x_2))$$

$$= (x_3 - x_1)(x_2 - x_1)(x_3 - x_2)(x_3 + x_2 + x_1)$$

$$= (x_3 + x_2 + x_1) \det V_3.$$

Example 5.2.4 Considering D_3 in [8], i.e.

$$D_3 = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1^3 & x_2^3 & x_3^3 \\ x_1^4 & x_2^4 & x_3^4 \end{bmatrix},$$

then

$$D_3^t = \det G_{\{3;0,3,4\}}$$

$$= x_1^0 x_2^0 (x_2^{3-0} - x_1^{3-0}) x_3^0 ((x_3^{4-0} - x_1^{4-0}) - \frac{(x_3^{3-0} - x_1^{3-0})(x_2^{4-0} - x_1^{4-0})}{(x_2^{3-0} - x_1^{3-0})})$$

$$= (x_2^3 - x_1^3)(\frac{(x_3^4 - x_1^4)(x_2^3 - x_1^3) - (x_3^3 - x_1^3)(x_2^4 - x_1^4)}{(x_2^3 - x_1^3)})$$

$$= (x_3^4 - x_1^4)(x_2^3 - x_1^3) - (x_3^3 - x_1^3)(x_2^4 - x_1^4)$$

$$= (x_3-x_1)(x_3+x_1)(x_3^2+x_1^2)(x_2-x_1)(x_2^2+x_2x_1+x_1^2) - (x_3-x_1)(x_3^2+x_3x_1+x_1^2)(x_2-x_1)(x_2+x_1)(x_2^2+x_1^2)$$

$$= (x_3-x_1)(x_2-x_1)((x_3+x_1)(x_3^2+x_1^2)(x_2^2+x_2x_1+x_1^2) - (x_3^2+x_3x_1+x_1^2)(x_2+x_1)(x_2^2+x_1^2))$$

$$= (x_3-x_1)(x_2-x_1)((x_3^3+x_3x_1^2+x_3^2x_1+x_1^3)(x_2^2+x_2x_1+x_1^2) - (x_3^2+x_3x_1+x_1^2)(x_2^3+x_2x_1^2+x_2^2x_1+x_1^3))$$

$$= (x_3-x_1)(x_2-x_1)((x_3^3x_2^2+x_3^3x_2x_1+x_3^3x_1^2+x_3x_2^2x_1^2+x_3x_2x_1^3+x_3x_1^4+x_3^2x_2^2x_1+x_3^2x_2x_1^2+x_3^2x_1^3+x_2^2x_1^3$$

$$+x_2x_1^4+x_1^5) - (x_3^2x_2^3+x_3x_2^3x_1+x_2^3x_1^2+x_3^2x_2x_1^2+x_3x_2x_1^3+x_2x_1^4+x_3^2x_2^2x_1+x_3x_2^2x_1^2+x_2^2x_1^3+x_3^2x_1^3+x_3x_1^4+x_1^5))$$

$$= (x_3 - x_1)(x_2 - x_1)((x_3^3x_2^2 - x_3^2x_2^3) + (x_3^3x_2x_1 - x_3x_2^3x_1) + (x_3^3x_1^2 - x_2^3x_1^2))$$

$$= (x_3-x_1)(x_2-x_1)(x_3^2x_2^2(x_3-x_2)+x_3x_2x_1(x_3-x_2)(x_3+x_2)+x_1^2(x_3-x_2)(x_3^2+x_3x_2+x_2^2))$$

$$= (x_3 - x_1)(x_2 - x_1)(x_3 - x_2)(x_3^2x_2^2 + x_3^2x_1^2 + x_2^2x_1^2 + x_3^2x_2x_1 + x_3x_2^2x_1 + x_3x_2x_1^2)$$

$$= (x_3^2x_2^2 + x_3^2x_1^2 + x_2^2x_1^2 + x_3^2x_2x_1 + x_3x_2^2x_1 + x_3x_2x_1^2) \det V_3.$$

Example 5.2.5 Considering Theorem 1 in [8], let $n = 4$, i.e.

$$D_4 = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 \end{bmatrix},$$

then

$$\begin{aligned}
D_4^t &= \det G_{\{4;0,2,3,4\}} \\
&= (x_2^2 - x_1^2) \times ((x_3^3 - x_1^3) - \frac{(x_3^2 - x_1^2)(x_2^3 - x_1^3)}{(x_2^2 - x_1^2)}) \\
&\times((x_4^4-x_1^4)-\frac{(x_4^2-x_1^2)(x_2^4-x_1^4)}{(x_2^2-x_1^2)}-((x_3^4-x_1^4)-\frac{(x_3^2-x_1^2)(x_2^4-x_1^4)}{(x_2^2-x_1^2)})\times(\frac{(x_4^3-x_1^3)(x_2^2-x_1^2)-(x_4^2-x_1^2)(x_2^3-x_1^3)}{(x_3^3-x_1^3)(x_2^2-x_1^2)-(x_3^2-x_1^2)(x_2^3-x_1^3)}) \\
&= ((x_3^3 - x_1^3)(x_2^2 - x_1^2) - (x_3^2 - x_1^2)(x_2^3 - x_1^3)) \\
&\times((\frac{(x_4^4-x_1^4)(x_2^2-x_1^2)-(x_4^2-x_1^2)(x_2^4-x_1^4)}{(x_2^2-x_1^2)})-(\frac{(x_3^4-x_1^4)(x_2^2-x_1^2)-(x_3^2-x_1^2)(x_2^4-x_1^4)}{(x_2^2-x_1^2)})\times(\frac{(x_4^3-x_1^3)(x_2^2-x_1^2)-(x_4^2-x_1^2)(x_2^3-x_1^3)}{(x_3^3-x_1^3)(x_2^2-x_1^2)-(x_3^2-x_1^2)(x_2^3-x_1^3)})) \\
&= ((x_3^3 - x_1^3)(x_2^2 - x_1^2) - (x_3^2 - x_1^2)(x_2^3 - x_1^3)) \times ((x_4^4 - x_1^4) - (x_4^2 - x_1^2)(x_2^2 + x_1^2)) \\
&-((x_3^4 - x_1^4) - (x_3^2 - x_1^2)(x_2^2 + x_1^2)) \times ((x_4^3 - x_1^3)(x_2^2 - x_1^2) - (x_4^2 - x_1^2)(x_2^3 - x_1^3)) \\
&= ((x_3-x_1)(x_3^2+x_3x_1+x_1^2)(x_2+x_1)(x_2-x_1)-(x_3-x_1)(x_3+x_1)(x_2-x_1)(x_2^2+x_2x_1+x_1^2)) \\
&\times((x_4 - x_1)(x_4 + x_1)(x_4^2 + x_1^2) - (x_4 + x_1)(x_4 - x_1)(x_2^2 + x_1^2)) \\
&-((x_3 - x_1)(x_3 + x_1)(x_3^2 + x_1^2) - (x_3 - x_1)(x_3 + x_1)(x_2^2 + x_1^2)) \\
&\times((x_4-x_1)(x_4^2+x_4x_1+x_1^2)(x_2+x_1)(x_2-x_1)-(x_4+x_1)(x_4-x_1)(x_2-x_1)(x_2^2+x_2x_1+x_1^2)) \\
&= (x_3 - x_1)(x_2 - x_1)(x_4 - x_1) \\
&\times\{((x_3^2+x_3x_1+x_1^2)(x_2+x_1)-(x_3+x_1)(x_2^2+x_2x_1+x_1^2))\times((x_4+x_1)(x_4^2+x_1^2)-(x_4+x_1)(x_2^2+x_1^2)) \\
&-((x_3+x_1)(x_3^2+x_1^2)-(x_3+x_1)(x_2^2+x_1^2))\times((x_4^2+x_4x_1+x_1^2)(x_2+x_1)-(x_4+x_1)(x_2^2+x_2x_1+x_1^2))\} \\
&= (x_3 - x_1)(x_2 - x_1)(x_4 - x_1) \\
&\times\{((x_3^2+x_3x_1+x_1^2)(x_2+x_1)-(x_3+x_1)(x_2^2+x_2x_1+x_1^2))\times((x_4+x_1)(x_4+x_2)(x_4-x_2)) \\
&-((x_3+x_1)(x_3+x_2)((x_3-x_2)\times((x_4^2+x_4x_1+x_1^2)(x_2+x_1)-(x_4+x_1)(x_2^2+x_2x_1+x_1^2)))\} \\
&= (x_3 - x_1)(x_2 - x_1)(x_4 - x_1) \\
&\times\{((x_3^2x_2 + x_3^2x_1) - (x_3x_2^2 + x_2^2x_1)) \times ((x_4 + x_1)(x_4 + x_2)(x_4 - x_2)) \\
&-((x_3 + x_1)(x_3 + x_2)(x_3 - x_2)((x_4^2x_2 + x_4^2x_1) - (x_4x_2^2 + x_2^2x_1))\}
\end{aligned}$$

$$= (x_3 - x_1)(x_2 - x_1)(x_4 - x_1)$$

$$\times\{(x_3x_2(x_3 - x_2) + x_1(x_3 + x_2)(x_3 - x_2)) \times ((x_4 + x_1)(x_4 + x_2)(x_4 - x_2))$$

$$-((x_3 + x_1)(x_3 + x_2)(x_3 - x_2)) \times (x_4x_2(x_4 - x_2) + x_1(x_4 + x_2)(x_4 - x_2))\}$$

$$= (x_3 - x_1)(x_2 - x_1)(x_4 - x_1)(x_3 - x_2)(x_4 - x_2)$$

$$\times((x_3x_2 + x_1x_3 + x_1x_2)(x_4 + x_1)(x_4 + x_2) - (x_3 + x_1)(x_3 + x_2)(x_4x_2 + x_4x_1 + x_2x_1))$$

$$= (x_3 - x_1)(x_2 - x_1)(x_4 - x_1)(x_3 - x_2)(x_4 - x_2)$$

$$\times((x_3x_2+x_1x_3+x_1x_2)(x_4^2+x_4x_2+x_4x_1+x_2x_1)-(x_3^2+x_3x_2+x_3x_1+x_2x_1)(x_4x_2+x_4x_1+x_2x_1))$$

$$= (x_3 - x_1)(x_2 - x_1)(x_4 - x_1)(x_3 - x_2)(x_4 - x_2)$$

$$\times((x_4^2x_3x_2 + x_4^2x_3x_1 + x_4^2x_2x_1) - (x_4x_3^2x_2 + x_4x_3^2x_1 + x_3^2x_2x_1))$$

$$= (x_3 - x_1)(x_2 - x_1)(x_4 - x_1)(x_3 - x_2)(x_4 - x_2)$$

$$\times((x_4 - x_3)(x_4x_3x_2 + +(x_4 - x_3)(x_4x_3x_1) + (x_4 - x_3)(x_4x_2x_1 + x_3x_2x_1))$$

$$= (x_3-x_1)(x_2-x_1)(x_4-x_1)(x_3-x_2)(x_4-x_2)(x_4-x_3)(x_4x_3x_2+x_4x_3x_1+x_4x_2x_1+x_3x_2x_1)$$

$$= (x_1x_2x_3x_4)(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}) \times \det V_4.$$

5.3 The Inverse of G_n

Let A be an $n \times n$ matrix, and A_{ij} be the matrix obtained from A by deleting the i-th row and the j-th column. Then we have the formula ([9], p. 177)

$$A^{-1} = transpose\ of\ (\frac{(-1)^{i+j}\det(A_{ij})}{\det(A)}).$$

Because

$$(G_n)_{ji} = \begin{bmatrix} x_1^{a_1} & x_1^{a_2} & \cdots & x_1^{a_{i-1}} & x_1^{a_{i+1}} & \cdots & x_1^{a_n} \\ x_2^{a_1} & x_2^{a_2} & \cdots & x_2^{a_{i-1}} & x_2^{a_{i+1}} & \cdots & x_2^{a_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{j-1}^{a_1} & x_{j-1}^{a_2} & \cdots & x_{j-1}^{a_{i-1}} & x_{j-1}^{a_{i+1}} & \cdots & x_{j-1}^{a_n} \\ x_{j+1}^{a_1} & x_{j+1}^{a_2} & \cdots & x_{j+1}^{a_{i-1}} & x_{j+1}^{a_{i+1}} & \cdots & x_{j+1}^{a_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n^{a_1} & x_n^{a_2} & \cdots & x_n^{a_{i-1}} & x_n^{a_{i+1}} & \cdots & x_n^{a_n} \end{bmatrix}$$

is still a totally positive generalized Vandermonde matrix, $\det(G_n)_{ji}$ can be computed by Theorem 5.2.1. Basing on the fact and the above formula, we establish the following theorem:

Theorem 5.3.1 *The entry of the inverse of G_n is*

$$G_n^{-1}(i,j) = \frac{(-1)^{j+i}\det((G_n)_{ji})}{\det(G_n)}.$$

Example 5.3.2 Let $n = 3$, i.e.

$$G_3 = \begin{bmatrix} x_1^{a_1} & x_1^{a_2} & x_1^{a_3} \\ x_2^{a_1} & x_2^{a_2} & x_2^{a_3} \\ x_3^{a_1} & x_3^{a_2} & x_3^{a_3} \end{bmatrix},$$

then

$$G_3^{-1}(1,1) = \frac{(-1)^{1+1}\det((G_3)_{11})}{\det(G_3)}$$

$$= \frac{(-1)^{1+1}\det\begin{bmatrix} x_2^{a_2} & x_2^{a_3} \\ x_3^{a_2} & x_3^{a_3} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$= \frac{(x_2^{a_2}x_3^{a_3}-x_2^{a_3}x_3^{a_2})}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))},$$

$$G_3^{-1}(1,2) = \frac{(-1)^{2+1}\det((G_3)_{21})}{\det(G_3)}$$

$$= \frac{(-1)^{2+1}\det\begin{bmatrix} x_1^{a_2} & x_1^{a_3} \\ x_3^{a_2} & x_3^{a_3} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$= \frac{-(x_1^{a_2}x_3^{a_3}-x_1^{a_3}x_3^{a_2})}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))},$$

$$G_3^{-1}(1,3) = \frac{(-1)^{3+1}\det((G_3)_{31})}{\det(G_3)}$$

$$= \frac{(-1)^{3+1}\det\begin{bmatrix} x_1^{a_2} & x_1^{a_3} \\ x_2^{a_2} & x_2^{a_3} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$= \frac{(x_1^{a_2}x_2^{a_3}-x_1^{a_3}x_2^{a_2})}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))},$$

$$G_3^{-1}(2,1) = \frac{(-1)^{1+2}\det((G_3)_{12})}{\det(G_3)}$$

$$= \frac{(-1)^{1+2}\det\begin{bmatrix} x_2^{a_1} & x_2^{a_3} \\ x_3^{a_1} & x_3^{a_3} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$= \frac{-(x_2^{a_1}x_3^{a_3}-x_2^{a_3}x_3^{a_1})}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))},$$

$$G_3^{-1}(2,2) = \frac{(-1)^{2+2}\det((G_3)_{22})}{\det(G_3)}$$

$$= \frac{(-1)^{2+2}\det\begin{bmatrix} x_1^{a_1} & x_1^{a_3} \\ x_3^{a_1} & x_3^{a_3} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$= \frac{(x_1^{a_1}x_3^{a_3}-x_1^{a_3}x_3^{a_1})}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))},$$

$$G_3^{-1}(2,3) = \frac{(-1)^{3+2}\det((G_3)_{32})}{\det(G_3)}$$

$$= \frac{(-1)^{3+2}\det\begin{bmatrix} x_1^{a_1} & x_1^{a_3} \\ x_2^{a_1} & x_2^{a_3} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$= \frac{-(x_1^{a_1}x_2^{a_3}-x_1^{a_3}x_2^{a_1})}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))},$$

$$G_3^{-1}(3,1) = \frac{(-1)^{1+3}\det((G_3)_{13})}{\det(G_3)}$$

$$= \frac{(-1)^{1+3}\det\begin{bmatrix} x_2^{a_1} & x_2^{a_2} \\ x_3^{a_1} & x_3^{a_2} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$= \frac{(x_2^{a_1}x_3^{a_2}-x_2^{a_2}x_3^{a_1})}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))},$$

$$G_3^{-1}(3,2) = \frac{(-1)^{2+3}\det((G_3)_{23})}{\det(G_3)}$$

$$= \frac{(-1)^{2+3}\det\begin{bmatrix} x_1^{a_1} & x_1^{a_2} \\ x_3^{a_1} & x_3^{a_2} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$= \frac{-(x_1^{a_1}x_3^{a_2} - x_1^{a_2}x_3^{a_1})}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1}))},$$

$$G_3^{-1}(3,3) = \frac{(-1)^{3+3}\det((G_3)_{33})}{\det(G_3)}$$

$$= \frac{(-1)^{3+3}\det\begin{bmatrix} x_1^{a_1} & x_1^{a_2} \\ x_2^{a_1} & x_2^{a_2} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1}))}$$

$$= \frac{(x_1^{a_1}x_2^{a_2} - x_1^{a_2}x_2^{a_1})}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1}))},$$

hence

$$G_3^{-1} = \frac{1}{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1}))}$$

$$\times \begin{bmatrix} (x_2^{a_2}x_3^{a_3} - x_2^{a_3}x_3^{a_2}) & -(x_1^{a_2}x_3^{a_3} - x_1^{a_3}x_3^{a_2}) & (x_1^{a_2}x_2^{a_3} - x_1^{a_3}x_2^{a_2}) \\ -(x_2^{a_1}x_3^{a_3} - x_2^{a_3}x_3^{a_1}) & (x_1^{a_1}x_3^{a_3} - x_1^{a_3}x_3^{a_1}) & -(x_1^{a_1}x_2^{a_3} - x_1^{a_3}x_2^{a_1}) \\ (x_2^{a_1}x_3^{a_2} - x_2^{a_2}x_3^{a_1}) & -(x_1^{a_1}x_3^{a_2} - x_1^{a_2}x_3^{a_1}) & (x_1^{a_1}x_2^{a_2} - x_1^{a_2}x_2^{a_1}) \end{bmatrix}.$$

Example 5.3.3 Let $n = 4$, i.e.

$$G_4 = \begin{bmatrix} x_1^{a_1} & x_1^{a_2} & x_1^{a_3} & x_1^{a_4} \\ x_2^{a_1} & x_2^{a_2} & x_2^{a_3} & x_2^{a_4} \\ x_3^{a_1} & x_3^{a_2} & x_3^{a_3} & x_3^{a_4} \\ x_4^{a_1} & x_4^{a_2} & x_4^{a_3} & x_4^{a_4} \end{bmatrix},$$

then

$$G_4^{-1}(1,1) = \frac{(-1)^{1+1}\det((G_4)_{11})}{\det(G_4)}$$

$$= \frac{(-1)^{1+1}\det\begin{bmatrix} x_2^{a_2} & x_2^{a_3} & x_2^{a_4} \\ x_3^{a_2} & x_3^{a_3} & x_3^{a_4} \\ x_4^{a_2} & x_4^{a_3} & x_4^{a_4} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1} - x_1^{a_4-a_1}) - \frac{(x_4^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_4-a_1} - x_1^{a_4-a_1})}{(x_2^{a_2-a_1} - x_1^{a_2-a_1})} - ((x_3^{a_4-a_1} - x_1^{a_4-a_1}) - \frac{(x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_4-a_1} - x_1^{a_4-a_1})}{(x_2^{a_2-a_1} - x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1} - x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1} - x_1{}^{a_2-a_1}) - (x_4{}^{a_2-a_1} - x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1} - x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1} - x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1} - x_1{}^{a_2-a_1}) - (x_3{}^{a_2-a_1} - x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1} - x_1{}^{a_3-a_1})}])}$$

$$= \frac{x_2^{a_2}x_3^{a_2}x_4^{a_2}((x_4^{a_4-a_2} - x_2^{a_4-a_2})(x_3^{a_3-a_2} - x_2^{a_3-a_2}) - (x_4^{a_3-a_2} - x_2^{a_3-a_2})(x_3^{a_4-a_2} - x_2^{a_4-a_2}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1} - x_1^{a_3-a_1})(x_2^{a_2-a_1} - x_1^{a_2-a_1}) - (x_3^{a_2-a_1} - x_1^{a_2-a_1})(x_2^{a_3-a_1} - x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])},$$

$$G_4^{-1}(1,2) = \frac{(-1)^{2+1}\det((G_4)_{21})}{\det(G_4)}$$

$$= \frac{-\det\begin{bmatrix} x_1^{a_2} & x_1^{a_3} & x_1^{a_4} \\ x_3^{a_2} & x_3^{a_3} & x_3^{a_4} \\ x_4^{a_2} & x_4^{a_3} & x_4^{a_4} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])}$$

$$= \frac{-x_1^{a_2}x_3^{a_2}x_4^{a_2}((x_4^{a_4-a_2}-x_1^{a_4-a_2})(x_3^{a_3-a_2}-x_1^{a_3-a_2})-(x_4^{a_3-a_2}-x_1^{a_3-a_2})(x_3^{a_4-a_2}-x_1^{a_4-a_2}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])},$$

$$G_4^{-1}(1,3) = \frac{(-1)^{3+1}\det((G_4)_{31})}{\det(G_4)}$$

$$= \frac{\det\begin{bmatrix} x_1^{a_2} & x_1^{a_3} & x_1^{a_4} \\ x_2^{a_2} & x_2^{a_3} & x_2^{a_4} \\ x_4^{a_2} & x_4^{a_3} & x_4^{a_4} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])}$$

$$= \frac{x_1^{a_2}x_2^{a_2}x_4^{a_2}((x_4^{a_4-a_2}-x_1^{a_4-a_2})(x_2^{a_3-a_2}-x_1^{a_3-a_2})-(x_4^{a_3-a_2}-x_1^{a_3-a_2})(x_2^{a_4-a_2}-x_1^{a_4-a_2}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])},$$

$$G_4^{-1}(1,4) = \frac{(-1)^{4+1}\det((G_4)_{41})}{\det(G_4)}$$

$$= \frac{-\det\begin{bmatrix} x_1^{a_2} & x_1^{a_3} & x_1^{a_4} \\ x_2^{a_2} & x_2^{a_3} & x_2^{a_4} \\ x_3^{a_2} & x_3^{a_3} & x_3^{a_4} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])}$$

$$= \frac{-x_1^{a_2}x_2^{a_2}x_3^{a_2}((x_3^{a_4-a_2}-x_1^{a_4-a_2})(x_2^{a_3-a_2}-x_1^{a_3-a_2})-(x_3^{a_3-a_2}-x_1^{a_3-a_2})(x_2^{a_4-a_2}-x_1^{a_4-a_2}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])},$$

$$G_4^{-1}(2,1) = \frac{(-1)^{1+2}\det((G_4)_{12})}{\det(G_4)}$$

$$= \frac{-\det\begin{bmatrix} x_2^{a_1} & x_2^{a_3} & x_2^{a_4} \\ x_3^{a_1} & x_3^{a_3} & x_3^{a_4} \\ x_4^{a_1} & x_4^{a_3} & x_4^{a_4} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])}$$

$$= \frac{-x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_4^{a_4-a_1}-x_2^{a_4-a_1})(x_3^{a_3-a_1}-x_2^{a_3-a_1})-(x_4^{a_3-a_1}-x_2^{a_3-a_1})(x_3^{a_4-a_1}-x_2^{a_4-a_1}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])},$$

$$G_4^{-1}(2,2) = \frac{(-1)^{2+2}\det((G_4)_{22})}{\det(G_4)}$$

$$= \frac{\det \begin{bmatrix} x_1^{a_1} & x_1^{a_3} & x_1^{a_4} \\ x_3^{a_1} & x_3^{a_3} & x_3^{a_4} \\ x_4^{a_1} & x_4^{a_3} & x_4^{a_4} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])}$$

$$= \frac{x_1^{a_1}x_3^{a_1}x_4^{a_1}((x_4^{a_4-a_1}-x_1^{a_4-a_1})(x_3^{a_3-a_1}-x_1^{a_3-a_1})-(x_4^{a_3-a_1}-x_1^{a_3-a_1})(x_3^{a_4-a_1}-x_1^{a_4-a_1}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])},$$

$$G_4^{-1}(2,3) = \frac{(-1)^{3+2}\det((G_4)_{32})}{\det(G_4)}$$

$$= \frac{-\det \begin{bmatrix} x_1^{a_1} & x_1^{a_3} & x_1^{a_4} \\ x_2^{a_1} & x_2^{a_3} & x_2^{a_4} \\ x_4^{a_1} & x_4^{a_3} & x_4^{a_4} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])}$$

$$= \frac{-x_1^{a_1}x_2^{a_1}x_4^{a_1}((x_4^{a_4-a_1}-x_1^{a_4-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1})-(x_4^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])},$$

$$G_4^{-1}(2,4) = \frac{(-1)^{4+2}\det((G_4)_{42})}{\det(G_4)}$$

$$= \frac{\det\begin{bmatrix} x_1^{a_1} & x_1^{a_3} & x_1^{a_4} \\ x_2^{a_1} & x_2^{a_3} & x_2^{a_4} \\ x_3^{a_1} & x_3^{a_3} & x_3^{a_4} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])}$$

$$= \frac{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_4-a_1}-x_1^{a_4-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1})-(x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])},$$

$$G_4^{-1}(3,1) = \frac{(-1)^{1+3}\det((G_4)_{13})}{\det(G_4)}$$

$$= \frac{\det\begin{bmatrix} x_2^{a_1} & x_2^{a_2} & x_2^{a_4} \\ x_3^{a_1} & x_3^{a_2} & x_3^{a_4} \\ x_4^{a_1} & x_4^{a_2} & x_4^{a_4} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])}$$

$$= \frac{x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_4^{a_4-a_1}-x_2^{a_4-a_1})(x_3^{a_2-a_1}-x_2^{a_2-a_1})-(x_4^{a_2-a_1}-x_2^{a_2-a_1})(x_3^{a_4-a_1}-x_2^{a_4-a_1}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])},$$

$$G_4^{-1}(3,2) = \frac{(-1)^{2+3}\det((G_4)_{23})}{\det(G_4)}$$

$$= \frac{-\det\begin{bmatrix} x_1^{a_1} & x_1^{a_2} & x_1^{a_4} \\ x_3^{a_1} & x_3^{a_2} & x_3^{a_4} \\ x_4^{a_1} & x_4^{a_2} & x_4^{a_4} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])}$$

$$= \frac{-x_1^{a_1}x_3^{a_1}x_4^{a_1}((x_4^{a_4-a_1}-x_1^{a_4-a_1})(x_3^{a_2-a_1}-x_1^{a_2-a_1})-(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_3^{a_4-a_1}-x_1^{a_4-a_1}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])},$$

$$G_4^{-1}(3,3) = \frac{(-1)^{3+3}\det((G_4)_{33})}{\det(G_4)}$$

$$= \frac{\det\begin{bmatrix} x_1^{a_1} & x_1^{a_2} & x_1^{a_4} \\ x_2^{a_1} & x_2^{a_2} & x_2^{a_4} \\ x_4^{a_1} & x_4^{a_2} & x_4^{a_4} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])}$$

$$= \frac{x_1^{a_1}x_2^{a_1}x_4^{a_1}((x_4^{a_4-a_1}-x_1^{a_4-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1})}{(x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1})}])},$$

$$G_4^{-1}(3,4) = \frac{(-1)^{4+3}\det((G_4)_{43})}{\det(G_4)}$$

$$= \frac{-\det\begin{bmatrix} x_1^{a_1} & x_1^{a_2} & x_1^{a_4} \\ x_2^{a_1} & x_2^{a_2} & x_2^{a_4} \\ x_3^{a_1} & x_3^{a_2} & x_3^{a_4} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1})}{(x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1})}])}$$

$$= \frac{-x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_4-a_1}-x_1^{a_4-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1})}{(x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1})}])},$$

$$G_4^{-1}(4,1) = \frac{(-1)^{1+4}\det((G_4)_{14})}{\det(G_4)}$$

$$= \frac{-\det\begin{bmatrix} x_2^{a_1} & x_2^{a_2} & x_2^{a_3} \\ x_3^{a_1} & x_3^{a_2} & x_3^{a_3} \\ x_4^{a_1} & x_4^{a_2} & x_4^{a_3} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1})}{(x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1})}])}$$

$$= \frac{-x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_4^{a_3-a_1}-x_2^{a_3-a_1})(x_3^{a_2-a_1}-x_2^{a_2-a_1})-(x_4^{a_2-a_1}-x_2^{a_2-a_1})(x_3^{a_3-a_1}-x_2^{a_3-a_1}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])},$$

$$G_4^{-1}(4,2) = \frac{(-1)^{2+4}\det((G_4)_{24})}{\det(G_4)}$$

$$= \frac{\det\begin{bmatrix} x_1^{a_1} & x_1^{a_2} & x_1^{a_3} \\ x_3^{a_1} & x_3^{a_2} & x_3^{a_3} \\ x_4^{a_1} & x_4^{a_2} & x_4^{a_3} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])}$$

$$= \frac{x_1^{a_1}x_3^{a_1}x_4^{a_1}((x_4^{a_3-a_1}-x_1^{a_3-a_1})(x_3^{a_2-a_1}-x_1^{a_2-a_1})-(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_3^{a_3-a_1}-x_1^{a_3-a_1}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])},$$

$$G_4^{-1}(4,3) = \frac{(-1)^{3+4}\det((G_4)_{34})}{\det(G_4)}$$

$$= \frac{-\det\begin{bmatrix} x_1^{a_1} & x_1^{a_2} & x_1^{a_3} \\ x_2^{a_1} & x_2^{a_2} & x_2^{a_3} \\ x_4^{a_1} & x_4^{a_2} & x_4^{a_3} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])}$$

$$= \frac{-x_1^{a_1}x_2^{a_1}x_4^{a_1}((x_4^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])},$$

$$G_4^{-1}(4,4) = \frac{(-1)^{4+4}\det((G_4)_{44})}{\det(G_4)}$$

$$= \frac{\det\begin{bmatrix} x_1^{a_1} & x_1^{a_2} & x_1^{a_3} \\ x_2^{a_1} & x_2^{a_2} & x_2^{a_3} \\ x_3^{a_1} & x_3^{a_2} & x_3^{a_3} \end{bmatrix}}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])}$$

$$= \frac{x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])},$$

hence

$$G_4^{-1} = \frac{1}{x_1^{a_1}x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))}$$

$$\times \frac{1}{((x_4^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})}-((x_3^{a_4-a_1}-x_1^{a_4-a_1})-\frac{(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1})}{(x_2^{a_2-a_1}-x_1^{a_2-a_1})})}$$

$$\frac{1}{\times[\frac{(x_4{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_4{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}{(x_3{}^{a_3-a_1}-x_1{}^{a_3-a_1})(x_2{}^{a_2-a_1}-x_1{}^{a_2-a_1})-(x_3{}^{a_2-a_1}-x_1{}^{a_2-a_1})(x_2{}^{a_3-a_1}-x_1{}^{a_3-a_1})}])}$$

$$
\times\left[\begin{array}{l}
x_2^{a_2}x_3^{a_2}x_4^{a_2}((x_4^{a_4-a_2}-x_2^{a_4-a_2})(x_3^{a_3-a_2}-x_2^{a_3-a_2})-(x_4^{a_3-a_2}-x_2^{a_3-a_2})(x_3^{a_4-a_2}-x_2^{a_4-a_2}))\\
-x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_4^{a_4-a_1}-x_2^{a_4-a_1})(x_3^{a_3-a_1}-x_2^{a_3-a_1})-(x_4^{a_3-a_1}-x_2^{a_3-a_1})(x_3^{a_4-a_1}-x_2^{a_4-a_1}))\\
x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_4^{a_4-a_1}-x_2^{a_4-a_1})(x_3^{a_2-a_1}-x_2^{a_2-a_1})-(x_4^{a_2-a_1}-x_2^{a_2-a_1})(x_3^{a_4-a_1}-x_2^{a_4-a_1}))\\
-x_2^{a_1}x_3^{a_1}x_4^{a_1}((x_4^{a_3-a_1}-x_2^{a_3-a_1})(x_3^{a_2-a_1}-x_2^{a_2-a_1})-(x_4^{a_2-a_1}-x_2^{a_2-a_1})(x_3^{a_3-a_1}-x_2^{a_3-a_1}))\\
\\
-x_1^{a_2}x_3^{a_2}x_4^{a_2}((x_4^{a_4-a_2}-x_1^{a_4-a_2})(x_3^{a_3-a_2}-x_1^{a_3-a_2})-(x_4^{a_3-a_2}-x_1^{a_3-a_2})(x_3^{a_4-a_2}-x_1^{a_4-a_2}))\\
x_1^{a_1}x_3^{a_1}x_4^{a_1}((x_4^{a_4-a_1}-x_1^{a_4-a_1})(x_3^{a_3-a_1}-x_1^{a_3-a_1})-(x_4^{a_3-a_1}-x_1^{a_3-a_1})(x_3^{a_4-a_1}-x_1^{a_4-a_1}))\\
-x_1^{a_1}x_3^{a_1}x_4^{a_1}((x_4^{a_4-a_1}-x_1^{a_4-a_1})(x_3^{a_2-a_1}-x_1^{a_2-a_1})-(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_3^{a_4-a_1}-x_1^{a_4-a_1}))\\
x_1^{a_1}x_3^{a_1}x_4^{a_1}((x_4^{a_3-a_1}-x_1^{a_3-a_1})(x_3^{a_2-a_1}-x_1^{a_2-a_1})-(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_3^{a_3-a_1}-x_1^{a_3-a_1}))\\
\\
x_1^{a_2}x_2^{a_2}x_4^{a_2}((x_4^{a_4-a_2}-x_1^{a_4-a_2})(x_2^{a_3-a_2}-x_1^{a_3-a_2})-(x_4^{a_3-a_2}-x_1^{a_3-a_2})(x_2^{a_4-a_2}-x_1^{a_4-a_2}))\\
-x_1^{a_1}x_2^{a_1}x_4^{a_1}((x_4^{a_4-a_1}-x_1^{a_4-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1})-(x_4^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1}))\\
x_1^{a_1}x_2^{a_1}x_4^{a_1}((x_4^{a_4-a_1}-x_1^{a_4-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1}))\\
-x_1^{a_1}x_2^{a_1}x_4^{a_1}((x_4^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_4^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))\\
\\
-x_1^{a_2}x_2^{a_2}x_3^{a_2}((x_3^{a_4-a_2}-x_1^{a_4-a_2})(x_2^{a_3-a_2}-x_1^{a_3-a_2})-(x_3^{a_3-a_2}-x_1^{a_3-a_2})(x_2^{a_4-a_2}-x_1^{a_4-a_2}))\\
x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_4-a_1}-x_1^{a_4-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1})-(x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1}))\\
-x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_4-a_1}-x_1^{a_4-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_4-a_1}-x_1^{a_4-a_1}))\\
x_1^{a_1}x_2^{a_1}x_3^{a_1}((x_3^{a_3-a_1}-x_1^{a_3-a_1})(x_2^{a_2-a_1}-x_1^{a_2-a_1})-(x_3^{a_2-a_1}-x_1^{a_2-a_1})(x_2^{a_3-a_1}-x_1^{a_3-a_1}))
\end{array}\right].
$$

5.4 The Calculation of Schur Function

Following from the definition of Schur function:

$$
\det G_n = s_\lambda(x_1, x_2, \cdots, x_n) \times \det V_n = s_\lambda(x_1, x_2, \cdots, x_n) \times \prod_{1\le i<j\le n}(x_j - x_i) \quad (5.4.1)
$$

is the product of the Schur function $s_\lambda(x_1, x_2, \cdots, x_n)$ and $\det V_n = \prod_{1\le i<j\le n}(x_j - x_i)$. Obviously, by Theorem 5.2.1 and (5.4.1), we can get the following theorem:

Theorem 5.4.1 *The Schur function $s_\lambda(x_1, x_2, \cdots, x_n)$ can be expressed as*

$$
s_\lambda(x_1, x_2, \cdots, x_n) = \frac{x_1^{a_1} \times \prod_{2\le i\le n} x_i^{a_1} B_i}{\prod_{1\le i<j\le n}(x_j - x_i)}
$$

where B_i are defined in Theorem 5.1.1.

Example 5.4.2 Let $\lambda = (1) = (1, 0, 0, 0)$ and $n = 4$, then $(a_1, a_2, a_3, a_4) = (0, 1, 2, 4)$ and

$$x_1^{a_1} \times \prod_{2 \le i \le 4} x_i^{a_1} B_i$$

$$= (x_2 - x_1)((x_3^2 - x_1^2) - \frac{(x_3 - x_1)(x_2^2 - x_1^2)}{(x_2 - x_1)})((x_4^4 - x_1^4) - \frac{(x_4 - x_1)(x_2^4 - x_1^4)}{(x_2 - x_1)} - ((x_3^4 - x_1^4) - \frac{(x_3 - x_1)(x_2^4 - x_1^4)}{(x_2 - x_1)})(\frac{(x_4^2 - x_1^2)(x_2 - x_1) - (x_4 - x_1)(x_2^2 - x_1^2)}{(x_3^2 - x_1^2)(x_2 - x_1) - (x_3 - x_1)(x_2^2 - x_1^2)}))$$

$$= (x_1 + x_2 + x_3 + x_4) \times \prod_{1 \le i < j \le 4} (x_j - x_i)$$

so by Theorem 5.4.1, we can get

$$s_{(1)}(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3 + x_4.$$

Example 5.4.3 Let $\lambda = (7, 5, 3, 1)$ and $n = 4$, then $(a_1, a_2, a_3, a_4) = (1, 4, 7, 10)$ and

$$\det G_{\{4;1,4,7,10\}} = x_1^{a_1} \times \prod_{2 \le i \le 4} x_i^{a_1} B_i$$

$$= x_1 x_2 (x_2^3 - x_1^3) x_3 ((x_3^6 - x_1^6) - \frac{(x_3^3 - x_1^3)(x_2^6 - x_1^6)}{(x_2^3 - x_1^3)})$$

$$\times x_4((x_4^9 - x_1^9) - \frac{(x_4^3 - x_1^3)(x_2^9 - x_1^9)}{(x_2^3 - x_1^3)} - ((x_3^9 - x_1^9) - \frac{(x_3^3 - x_1^3)(x_2^9 - x_1^9)}{(x_2^3 - x_1^3)})(\frac{(x_4^6 - x_1^6)(x_2^3 - x_1^3) - (x_4^3 - x_1^3)(x_2^6 - x_1^6)}{(x_3^6 - x_1^6)(x_2^3 - x_1^3) - (x_3^3 - x_1^3)(x_2^6 - x_1^6)}))$$

$$= x_1 x_2 x_3 x_4 (x_4^2 + x_4 x_3 + x_3^2)(x_4^2 + x_4 x_2 + x_2^2)(x_4^2 + x_4 x_1 + x_1^2)(x_3^2 + x_3 x_2 + x_2^2)(x_3^2 + x_3 x_1 + x_1^2)(x_2^2 + x_2 x_1 + x_1^2) \times \prod_{1 \le i < j \le 4} (x_j - x_i),$$

so by Theorem 5.4.1, we can get

$$s_{(7,5,3,1)}(x_1, x_2, x_3, x_4)$$

$$= x_1 x_2 x_3 x_4 (x_4^2 + x_4 x_3 + x_3^2)(x_4^2 + x_4 x_2 + x_2^2)(x_4^2 + x_4 x_1 + x_1^2)(x_3^2 + x_3 x_2 + x_2^2)(x_3^2 + x_3 x_1 + x_1^2)(x_2^2 + x_2 x_1 + x_1^2).$$

As we can see, there are $3^6 = 729$ semistandard $(7, 5, 3, 1)$ tableaux, it seems not easy to write out all of the semistandard $(7, 5, 3, 1)$ tableaux.

Example 5.4.4 Let $\lambda = (5, 2, 1)$ and $n = 4$, then $(a_1, a_2, a_3, a_4) = (0, 2, 4, 8)$ and

$$\det G_{\{4;0,2,4,8\}} = x_1^{a_1} \times \prod_{2 \le i \le 4} x_i^{a_1} B_i$$

$$= (x_2^2 - x_1^2)((x_3^4 - x_1^4) - \frac{(x_3^2 - x_1^2)(x_2^4 - x_1^4)}{(x_2^2 - x_1^2)})$$

$$\times ((x_4^8 - x_1^8) - \frac{(x_4^2 - x_1^2)(x_2^8 - x_1^8)}{(x_2^2 - x_1^2)} - ((x_3^8 - x_1^8) - \frac{(x_3^2 - x_1^2)(x_2^8 - x_1^8)}{(x_2^2 - x_1^2)})(\frac{(x_4^4 - x_1^4)(x_2^2 - x_1^2) - (x_4^2 - x_1^2)(x_2^4 - x_1^4)}{(x_3^4 - x_1^4)(x_2^2 - x_1^2) - (x_3^2 - x_1^2)(x_2^4 - x_1^4)}))$$

$$= [(x_3^4 - x_1^4)(x_2^2 - x_1^2) - (x_3^2 - x_1^2)(x_2^4 - x_1^4)][(x_4^8 - x_1^8) - (x_4^2 + x_1^2)(x_2^2 + x_1^2)(x_2^4 + x_1^4)]$$

$$-[(x_3^8-x_1^8)-(x_3^2-x_1^2)(x_2^2+x_1^2)(x_2^4+x_1^4)][(x_4^4-x_1^4)(x_2^2-x_1^2)-(x_4^2-x_1^2)(x_2^4-x_1^4)]$$

$$=(x_4-x_3)(x_4-x_2)(x_4-x_1)(x_3-x_2)(x_3-x_1)(x_2-x_1)(x_4+x_3)(x_4+x_2)(x_4+x_1)(x_3+x_2)(x_3+x_1)(x_2+x_1)(x_4^2+x_3^2+x_2^2+x_1^2)$$

$$=(x_4+x_3)(x_4+x_2)(x_4+x_1)(x_3+x_2)(x_3+x_1)(x_2+x_1)(x_4^2+x_3^2+x_2^2+x_1^2)\times\prod_{1\le i<j\le 4}(x_j-x_i),$$

so by Theorem 5.4.1, we can get

$$s_{(5,2,1)}(x_1,x_2,x_3,x_4)$$

$$=(x_4+x_3)(x_4+x_2)(x_4+x_1)(x_3+x_2)(x_3+x_1)(x_2+x_1)(x_4^2+x_3^2+x_2^2+x_1^2).$$

As we can see, there are $2^6\times 4=256$ semistandard $(5,2,1)$ tableaux (see Appendix).

As is known, the finitely many Schur functions of degree m in n variables form a basis of the space of homogeneous symmetric functions of degree m in n variables (see [14], p.2). Below we give an example to illustrate this.

Example 5.4.5 Consider $n=4$ and $|\lambda|=6$, by using Ferrers diagrams, we find that there are two partitions of 6 into precisely four parts. i.e.

$$s_{(3,1,1,1)}(x_1,x_2,x_3,x_4)=\frac{\det\begin{bmatrix}x_1 & x_1^2 & x_1^3 & x_1^6\\ x_2 & x_2^2 & x_2^3 & x_2^6\\ x_3 & x_3^2 & x_3^3 & x_3^6\\ x_4 & x_4^2 & x_4^3 & x_4^6\end{bmatrix}}{\prod_{1\le i<j\le 4}(x_j-x_i)},$$

where

$$\det\begin{bmatrix}x_1 & x_1^2 & x_1^3 & x_1^6\\ x_2 & x_2^2 & x_2^3 & x_2^6\\ x_3 & x_3^2 & x_3^3 & x_3^6\\ x_4 & x_4^2 & x_4^3 & x_4^6\end{bmatrix}=x_1x_2(x_2-x_1)x_3((x_3^2-x_1^2)-\frac{(x_3-x_1)(x_2^2-x_1^2)}{(x_2-x_1)})$$

$$\times x_4((x_4^5-x_1^5)-\frac{(x_4-x_1)(x_2^5-x_1^5)}{(x_2-x_1)}-((x_3^5-x_1^5)-\frac{(x_3-x_1)(x_2^5-x_1^5)}{(x_2-x_1)})(\frac{(x_4^2-x_1^2)(x_2-x_1)-(x_4-x_1)(x_2^2-x_1^2)}{(x_3^2-x_1^2)(x_2-x_1)-(x_3-x_1)(x_2^2-x_1^2)}))$$

$$=x_1x_2x_3x_4((x_3^2-x_1^2)(x_2-x_1)-(x_3-x_1)(x_2^2-x_1^2))$$

$$\times(\frac{(x_4^5-x_1^5)(x_2-x_1)-(x_4-x_1)(x_2^5-x_1^5)}{(x_2-x_1)}-(\frac{(x_3^5-x_1^5)(x_2-x_1)-(x_3-x_1)(x_2^5-x_1^5)}{(x_2-x_1)})(\frac{(x_4^2-x_1^2)(x_2-x_1)-(x_4-x_1)(x_2^2-x_1^2)}{(x_3^2-x_1^2)(x_2-x_1)-(x_3-x_1)(x_2^2-x_1^2)}))$$

$$=x_1x_2x_3x_4((x_3^2-x_1^2)-(x_3-x_1)(x_2+x_1))((x_4^5-x_1^5)(x_2-x_1)-(x_4-x_1)(x_2^5-x_1^5))$$

$$-x_1x_2x_3x_4(x_3^5-x_1^5)(x_2-x_1)-(x_3-x_1)(x_2^5-x_1^5))((x_4^2-x_1^2)-(x_4-x_1)(x_2+x_1))$$

$$= x_1x_2x_3x_4((x_3-x_1)(x_3-x_2)(x_4-x_1)(x_2-x_1)$$

$$\times(x_4^4+x_4^3x_1+x_4^2x_1^2+x_4x_1^3+x_1^4-x_2^4-x_2^3x_1-x_2^2x_1^2-x_2x_1^3-x_1^4))$$

$$-x_1x_2x_3x_4((x_4-x_1)(x_4-x_2)(x_3-x_1)(x_2-x_1)$$

$$\times(x_3^4+x_3^3x_1+x_3^2x_1^2+x_3x_1^3+x_1^4-x_2^4-x_2^3x_1-x_2^2x_1^2-x_2x_1^3-x_1^4))$$

$$= x_1x_2x_3x_4((x_3-x_1)(x_3-x_2)(x_4-x_1)(x_2-x_1)$$

$$\times((x_4^4-x_2^4)+(x_4^3x_1-x_2^3x_1)+(x_4^2x_1^2-x_2^2x_1^2)+(x_4x_1^3-x_2x_1^3)))$$

$$-x_1x_2x_3x_4((x_4-x_1)(x_4-x_2)(x_3-x_1)(x_2-x_1)$$

$$\times((x_3^4-x_2^4)+(x_3^3x_1-x_2^3x_1)+(x_3^2x_1^2-x_2^2x_1^2)+(x_3x_1^3-x_2x_1^3)))$$

$$= x_1x_2x_3x_4((x_3-x_1)(x_3-x_2)(x_4-x_1)(x_2-x_1)$$

$$\times((x_4-x_2)(x_4^3+x_4^2x_2+x_4x_2^2+x_2^3)+(x_4-x_2)x_1(x_4^2+x_4x_2+x_2^2)+(x_4-x_2)x_1^2(x_4+x_2)+x_1^3(x_4-x_2))$$

$$-x_1x_2x_3x_4((x_4-x_1)(x_4-x_2)(x_3-x_1)(x_2-x_1)(x_3-x_2)$$

$$\times(x_3^3+x_3^2x_2+x_3x_2^2+x_2^3+x_3^2x_1+x_3x_2x_1+x_2^2x_1+x_3x_1^2+x_2x_1^2+x_1^3)$$

$$= x_1x_2x_3x_4((x_4-x_1)(x_4-x_2)(x_3-x_1)(x_2-x_1)(x_3-x_2)$$

$$\times((x_4^3-x_3^3)+(x_4^2x_2-x_3^2x_2)+(x_4x_2^2-x_3x_2^2)+x_4^2x_1-x_3^2x_1)+(x_4x_2x_1-x_3x_2x_1)+(x_4x_1^2-x_3x_1^2))$$

$$= x_1x_2x_3x_4((x_4-x_1)(x_4-x_2)(x_3-x_1)(x_2-x_1)(x_3-x_2)$$

$$\times((x_4-x_3)(x_4^2+x_4x_3+x_3^2)+(x_4-x_3)(x_4x_2+x_3x_2)+(x_4-x_3)x_2^2+(x_4-x_3)(x_4x_1+x_3x_1)+(x_4-x_3)x_2x_1+(x_4-x_3)x_1^2)$$

$$= x_1x_2x_3x_4((x_4-x_1)(x_4-x_2)(x_3-x_1)(x_2-x_1)(x_3-x_2)(x_4-x_3)$$

$$\times(x_4^2+x_4x_3+x_3^2+x_4x_2+x_3x_2+x_2^2+x_4x_1+x_3x_1+x_1^2)$$

$$= x_1x_2x_3x_4(x_1^2+x_2^2+x_3^2+x_4^2+x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)\prod_{1\le i<j\le 4}(x_j-x_i),$$

hence

$$s_{(3,1,1,1)}(x_1,x_2,x_3,x_4) = x_1x_2x_3x_4(x_1^2+x_2^2+x_3^2+x_4^2+x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4);$$

and

$$s_{(2,2,1,1)}(x_1,x_2,x_3,x_4) = \frac{\det\begin{bmatrix} x_1 & x_1^2 & x_1^4 & x_1^5 \\ x_2 & x_2^2 & x_2^4 & x_2^5 \\ x_3 & x_3^2 & x_3^4 & x_3^5 \\ x_4 & x_4^2 & x_4^4 & x_4^5 \end{bmatrix}}{\prod_{1\le i<j\le 4}(x_j-x_i)},$$

where

$$\det \begin{bmatrix} x_1 & x_1^2 & x_1^4 & x_1^5 \\ x_2 & x_2^2 & x_2^4 & x_2^5 \\ x_3 & x_3^2 & x_3^4 & x_3^5 \\ x_4 & x_4^2 & x_4^4 & x_4^5 \end{bmatrix} = x_1x_2(x_2-x_1)x_3((x_3^3-x_1^3)-\frac{(x_3-x_1)(x_2^3-x_1^3)}{(x_2-x_1)})$$

$$\times x_4((x_4^4-x_1^4)-\frac{(x_4-x_1)(x_2^4-x_1^4)}{(x_2-x_1)}-((x_3^4-x_1^4)-\frac{(x_3-x_1)(x_2^4-x_1^4)}{(x_2-x_1)})(\frac{(x_4^3-x_1^3)(x_2-x_1)-(x_4-x_1)(x_2^3-x_1^3)}{(x_3^3-x_1^3)(x_2-x_1)-(x_3-x_1)(x_2^3-x_1^3)}))$$

$$= x_1x_2x_3x_4((x_3^3-x_1^3)-(x_3-x_1)(x_2^3-x_1^3))$$

$$\times((x_4^4-x_1^4)-\frac{(x_4-x_1)(x_2^4-x_1^4)}{(x_2-x_1)}-((x_3^4-x_1^4)-\frac{(x_3-x_1)(x_2^4-x_1^4)}{(x_2-x_1)})(\frac{(x_4^3-x_1^3)(x_2-x_1)-(x_4-x_1)(x_2^3-x_1^3)}{(x_3^3-x_1^3)(x_2-x_1)-(x_3-x_1)(x_2^3-x_1^3)}))$$

$$= x_1x_2x_3x_4((x_3^3-x_1^3)-(x_3-x_1)(x_2^2+x_2x_1+x_1^2))((x_4^4-x_1^4)(x_2-x_1)-(x_4-x_1)(x_2^4-x_1^4))$$

$$-x_1x_2x_3x_4((x_3^4-x_1^4)(x_2-x_1)-(x_3-x_1)(x_2^4-x_1^4))((x_4^3-x_1^3)-(x_4-x_1)(x_2^2+x_2x_1+x_1^2))$$

$$= x_1x_2x_3x_4(x_3-x_1)(x_3^2+x_3x_1+x_1^2-x_2^2-x_2x_1-x_1^2)$$

$$\times((x_4-x_1)(x_2-x_1)(x_4^3+x_4^2x_1+x_4x_1^2+x_1^3-x_2^3-x_2x_1-x_2x_1^2-x_1^3))$$

$$-x_1x_2x_3x_4(x_4-x_1)(x_4^2+x_4x_1+x_1^2-x_2^2-x_2x_1-x_1^2)$$

$$\times(x_3-x_1)(x_2-x_1)(x_3^3+x_3^2x_1+x_3x_1^2+x_1^3-x_2^3-x_2x_1-x_2x_1^2-x_1^3)$$

$$= x_1x_2x_3x_4(x_3-x_1)(x_4-x_1)(x_2-x_1)$$

$$\times((x_3^2-x_2^2)+x_1(x_3-x_2))((x_4^3-x_2^3)+(x_4^2x_1-x_2^2x_1)+(x_4x_1^2-x_2x_1^2))$$

$$-x_1x_2x_3x_4(x_4-x_1)(x_3-x_1)(x_2-x_1)$$

$$\times((x_4-x_2)(x_4+x_2+x_1))((x_3^3-x_2^3)+(x_3^2x_1-x_2^2x_1)+(x_3x_1^2-x_2x_1^2))$$

$$= x_1x_2x_3x_4(x_3-x_1)(x_4-x_1)(x_2-x_1)$$

$$\times(x_3-x_2)(x_3+x_2+x_1)((x_4-x_2)(x_4^2+x_4x_2+x_2^2)+(x_4-x_2)(x_4x_1+x_2x_1)+(x_4-x_2)x_1^2)$$

$$-x_1x_2x_3x_4(x_4-x_1)(x_3-x_1)(x_2-x_1)(x_4-x_2)(x_4+x_2+x_1)$$

$$\times((x_3^3-x_2^3)+(x_3^2x_1-x_2^2x_1)+(x_3x_1^2-x_2x_1^2))$$

$$= x_1x_2x_3x_4(x_3-x_1)(x_4-x_1)(x_2-x_1)(x_3-x_2)(x_4-x_2)$$

$$\times((x_3+x_2+x_1)(x_4^2+x_4x_2+x_2^2+x_4x_1+x_2x_1+x_1^2)-(x_4+x_2+x_1)(x_3^2+x_3x_2+x_2^2+x_3x_1+x_2x_1+x_1^2))$$

$$= x_1x_2x_3x_4(x_3-x_1)(x_4-x_1)(x_2-x_1)(x_3-x_2)(x_4-x_2)$$

$$\times((x_3x_4^2-x_3x_4)+(x_2x_4^2-x_2x_3^2)+(x_1x_2x_4-x_1x_2x_3)+(x_1x_4^2-x_1x_3^2))$$

$$= x_1x_2x_3x_4(x_3-x_1)(x_4-x_1)(x_2-x_1)(x_3-x_2)(x_4-x_2)$$

$$\times(x_3x_4(x_4-x_3)+(x_4-x_3)(x_2x_4+x_2x_3)+(x_4-x_3)x_1x_2+(x_4-x_3)(x_1x_4+x_1x_3))$$

$$=x_1x_2x_3x_4(x_3-x_1)(x_4-x_1)(x_2-x_1)(x_3-x_2)(x_4-x_2)(x_4-x_3)$$

$$\times(x_3x_4+x_2x_4+x_2x_3+x_1x_2+x_1x_4+x_1x_3)$$

$$=x_1x_2x_3x_4(x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)\prod_{1\le i<j\le 4}(x_j-x_i),$$

hence

$$s_{(2,2,1,1)}(x_1,x_2,x_3,x_4)=x_1x_2x_3x_4(x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4).$$

Let

$$\psi(x_1,x_2,x_3,x_4)=x_1^3x_2x_3x_4+x_1x_2^3x_3x_4+x_1x_2x_3^3x_4+x_1x_2x_3x_4^3,$$

it is obviously a symmetric polynomial which is homogeneous of degree 6, and we have

$$\psi(x_1,x_2,x_3,x_4)=s_{(3,1,1,1)}(x_1,x_2,x_3,x_4)-s_{(2,2,1,1)}(x_1,x_2,x_3,x_4).$$

5.5 An Application to Kostka Numbers

As a final application of our explicit formula for Schur functions in Theorem 5.4.1, we present a way to determine Kostka numbers by expanding Schur functions. We illustrate our results in the following three examples 5.5.1, 5.5.2 and 5.5.3.

Example 5.5.1 Let $x_1=a, x_2=b, x_3=c, x_4=d$ in Example 5.4.3, then the program

$Expand[a*b*c*d*(d^2+d*c+c^2)*(d^2+d*b+b^2)(d^2+d*a+a^2)*(c^2+c*b+b^2)*(c^2+c*a+a^2)*(b^2+b*a+a^2)]$

running by Mathematica yields the result

$a^7*b^5*c^3*d+a^6*b^6*c^3*d+a^5*b^7*c^3*d+a^7*b^4*c^4*d+2*a^6*b^5*c^4*d+2*a^5*b^6*c^4*d+a^4*b^7*c^4*d$

$+a^7*b^3*c^5*d+2*a^6*b^4*c^5*d+3*a^5*b^5*c^5*d+2*a^4*b^6*c^5*d+a^3*b^7*c^5*d+a^6*b^3*c^6*d$

$+2*a^5*b^4*c^6*d+2*a^4*b^5*c^6*d+a^3*b^6*c^6*d+a^5*b^3*c^7*d+a^4*b^4*c^7*d+a^3*b^5*c^7*d$

$$+a^7*b^5*c^2*d^2+a^6*b^6*c^2*d^2+a^5*b^7*c^2*d^2+2*a^7*b^4*c^3*d^2+4*a^6*b^5*c^3*d^2$$

$$+4*a^5*b^6*c^3*d^2+2*a^4*b^7*c^3*d^2+2*a^7*b^3*c^4*d^2+5*a^6*b^4*c^4*d^2+7*a^5*b^5*c^4*d^2$$

$$+5*a^4*b^6*c^4*d^2+2*a^3*b^7*c^4*d^2+a^7*b^2*c^5*d^2+4*a^6*b^3*c^5*d^2+7*a^5*b^4*c^5*d^2$$

$$+7*a^4*b^5*c^5*d^2+4*a^3*b^6*c^5*d^2+a^2*b^7*c^5*d^2+a^6*b^2*c^6*d^2+4*a^5*b^3*c^6*d^2$$

$$+5*a^4*b^4*c^6*d^2+4*a^3*b^5*c^6*d^2+a^2*b^6*c^6*d^2+a^5*b^2*c^7*d^2+2*a^4*b^3*c^7*d^2$$

$$+2*a^3*b^4*c^7*d^2+a^2*b^5*c^7*d^2+a^7*b^5*c*d^3+a^6*b^6*c*d^3+a^5*b^7*c*d^3+2*a^7*b^4*c^2*d^3$$

$$+4*a^6*b^5*c^2*d^3+4*a^5*b^6*c^2*d^3+2*a^4*b^7*c^2*d^3+3*a^7*b^3*c^3*d^3+7*a^6*b^4*c^3*d^3$$

$$+10*a^5*b^5*c^3*d^3+7*a^4*b^6*c^3*d^3+3*a^3*b^7*c^3*d^3+2*a^7*b^2*c^4*d^3+7*a^6*b^3*c^4*d^3$$

$$+12*a^5*b^4*c^4*d^3+12*a^4*b^5*c^4*d^3+7*a^3*b^6*c^4*d^3+2*a^2*b^7*c^4*d^3+a^7*b*c^5*d^3$$

$$+4*a^6*b^2*c^5*d^3+10*a^5*b^3*c^5*d^3+12*a^4*b^4*c^5*d^3+10*a^3*b^5*c^5*d^3+4*a^2*b^6*c^5*d^3$$

$$+a*b^7*c^5*d^3+a^6*b*c^6*d^3+4*a^5*b^2*c^6*d^3+7*a^4*b^3*c^6*d^3+7*a^3*b^4*c^6*d^3$$

$$+4*a^2*b^5*c^6*d^3+a*b^6*c^6*d^3+a^5*b*c^7*d^3+2*a^4*b^2*c^7*d^3+3*a^3*b^3*c^7*d^3$$

$$+2*a^2*b^4*c^7*d^3+a*b^5*c^7*d^3+a^7*b^4*c*d^4+2*a^6*b^5*c*d^4+2*a^5*b^6*c*d^4+a^4*b^7*c*d^4$$

$$+2*a^7*b^3*c^2*d^4+5*a^6*b^4*c^2*d^4+7*a^5*b^5*c^2*d^4+5*a^4*b^6*c^2*d^4+2*a^3*b^7*c^2*d^4$$

$$+2*a^7*b^2*c^3*d^4+7*a^6*b^3*c^3*d^4+12*a^5*b^4*c^3*d^4+12*a^4*b^5*c^3*d^4+7*a^3*b^6*c^3*d^4$$

$$+2*a^2*b^7*c^3*d^4+a^7*b*c^4*d^4+5*a^6*b^2*c^4*d^4+12*a^5*b^3*c^4*d^4+15*a^4*b^4*c^4*d^4$$

$$+12*a^3*b^5*c^4*d^4+5*a^2*b^6*c^4*d^4+a*b^7*c^4*d^4+2*a^6*b*c^5*d^4+7*a^5*b^2*c^5*d^4$$

$$+12*a^4*b^3*c^5*d^4+12*a^3*b^4*c^5*d^4+7*a^2*b^5*c^5*d^4+2*a*b^6*c^5*d^4+2*a^5*b*c^6*d^4$$

$$+5*a^4*b^2*c^6*d^4+7*a^3*b^3*c^6*d^4+5*a^2*b^4*c^6*d^4+2*a*b^5*c^6*d^4+a^4*b*c^7*d^4$$

$$+2*a^3*b^2*c^7*d^4+2*a^2*b^3*c^7*d^4+a*b^4*c^7*d^4+a^7*b^3*c*d^5+2*a^6*b^4*c*d^5$$

$$+3*a^5*b^5*c*d^5+2*a^4*b^6*c*d^5+a^3*b^7*c*d^5+a^7*b^2*c^2*d^5+4*a^6*b^3*c^2*d^5$$

$$+7*a^5*b^4*c^2*d^5+7*a^4*b^5*c^2*d^5+4*a^3*b^6*c^2*d^5+a^2*b^7*c^2*d^5+a^7*b*c^3*d^5$$

$$+4*a^6*b^2*c^3*d^5+10*a^5*b^3*c^3*d^5+12*a^4*b^4*c^3*d^5+10*a^3*b^5*c^3*d^5+4*a^2*b^6*c^3*d^5$$

$$+a*b^7*c^3*d^5+2*a^6*b*c^4*d^5+7*a^5*b^2*c^4*d^5+12*a^4*b^3*c^4*d^5+12*a^3*b^4*c^4*d^5$$

$$+7*a^2*b^5*c^4*d^5+2*a*b^6*c^4*d^5+3*a^5*b*c^5*d^5+7*a^4*b^2*c^5*d^5+10*a^3*b^3*c^5*d^5$$

$$+7*a^2*b^4*c^5*d^5+3*a*b^5*c^5*d^5+2*a^4*b*c^6*d^5+4*a^3*b^2*c^6*d^5+4*a^2*b^3*c^6*d^5$$

$$+2\ast a\ast b^4\ast c^6\ast d^5+a^3\ast b\ast c^7\ast d^5+a^2\ast b^2\ast c^7\ast d^5+a\ast b^3\ast c^7\ast d^5+a^6\ast b^3\ast c\ast d^6+2\ast a^5\ast b^4\ast c\ast d^6$$

$$+2\ast a^4\ast b^5\ast c\ast d^6+a^3\ast b^6\ast c\ast d^6+a^6\ast b^2\ast c^2\ast d^6+4\ast a^5\ast b^3\ast c^2\ast d^6+5\ast a^4\ast b^4\ast c^2\ast d^6$$

$$+4\ast a^3\ast b^5\ast c^2\ast d^6+a^2\ast b^6\ast c^2\ast d^6+a^6\ast b\ast c^3\ast d^6+4\ast a^5\ast b^2\ast c^3\ast d^6+7\ast a^4\ast b^3\ast c^3\ast d^6$$

$$+7\ast a^3\ast b^4\ast c^3\ast d^6+4\ast a^2\ast b^5\ast c^3\ast d^6+a\ast b^6\ast c^3\ast d^6+2\ast a^5\ast b\ast c^4\ast d^6+5\ast a^4\ast b^2\ast c^4\ast d^6$$

$$+7\ast a^3\ast b^3\ast c^4\ast d^6+5\ast a^2\ast b^4\ast c^4\ast d^6+2\ast a\ast b^5\ast c^4\ast d^6+2\ast a^4\ast b\ast c^5\ast d^6+4\ast a^3\ast b^2\ast c^5\ast d^6$$

$$+4\ast a^2\ast b^3\ast c^5\ast d^6+2\ast a\ast b^4\ast c^5\ast d^6+a^3\ast b\ast c^6\ast d^6+a^2\ast b^2\ast c^6\ast d^6+a\ast b^3\ast c^6\ast d^6+a^5\ast b^3\ast c\ast d^7$$

$$+a^4\ast b^4\ast c\ast d^7+a^3\ast b^5\ast c\ast d^7+a^5\ast b^2\ast c^2\ast d^7+2\ast a^4\ast b^3\ast c^2\ast d^7+2\ast a^3\ast b^4\ast c^2\ast d^7+a^2\ast b^5\ast c^2\ast d^7$$

$$+a^5\ast b\ast c^3\ast d^7+2\ast a^4\ast b^2\ast c^3\ast d^7+3\ast a^3\ast b^3\ast c^3\ast d^7+2\ast a^2\ast b^4\ast c^3\ast d^7+a\ast b^5\ast c^3\ast d^7+a^4\ast b\ast c^4\ast d^7$$

$$+2\ast a^3\ast b^2\ast c^4\ast d^7+2\ast a^2\ast b^3\ast c^4\ast d^7+a\ast b^4\ast c^4\ast d^7+a^3\ast b\ast c^5\ast d^7+a^2\ast b^2\ast c^5\ast d^7+a\ast b^3\ast c^5\ast d^7.$$

From above, we find $K_{(7,5,3,1),(5,5,4,2)} = 7$, the 7 semi-standard Young tableaux ($SSYT^{7531}$) of weight (5,5,4,2) are as follows:

1	1	1	1	1	2	2
2	2	2	3	3		
3	3	4				
4						

1	1	1	1	1	2	3
2	2	2	2	3		
3	3	4				
4						

1	1	1	1	1	3	3
2	2	2	2	2		
3	3	4				
4						

1	1	1	1	1	2	2
2	2	2	3	4		
3	3	3				
4						

1	1	1	1	1	2	4
2	2	2	2	3		
3	3	3				
4						

1	1	1	1	1	2	3
2	2	2	2	4		
3	3	3				
4						

1	1	1	1	1	3	4
2	2	2	2	2		
3	3	3				
4						

and $K_{(7,5,3,1),(5,5,3,3)} = 10$, the 10 semi-standard Young tableaux ($SSYT^{7531}$) of weight (5,5,3,3) are as follows:

1	1	1	1	1	2	2
2	2	2	3	3		
3	4	4				
4						

1	1	1	1	1	2	3
2	2	2	2	3		
3	4	4				
4						

1	1	1	1	1	3	3
2	2	2	2	2		
3	4	4				
4						

1	1	1	1	1	2	2
2	2	2	4	4		
3	3	3				
4						

1	1	1	1	1	2	4
2	2	2	2	4		
3	3	3				
4						

1	1	1	1	1	2	3
2	2	2	2	4		
3	3	4				
4						

1	1	1	1	1	3	4
2	2	2	2	2		
3	3	4				
4						

1	1	1	1	1	2	2
2	2	2	3	4		
3	3	4				
4						

1	1	1	1	1	2	4
2	2	2	2	3		
3	3	4				
4						

1	1	1	1	1	4	4
2	2	2	2	2		
3	3	3				
4						

and $K_{(7,5,3,1),(5,4,4,3)} = 12$, the 12 semi-standard Young tableaux ($SSYT^{7531}$) of weight (5,4,4,3) are as follows:

1	1	1	1	1	2	3
2	2	2	3	3		
3	4	4				
4						

1	1	1	1	1	2	3
2	2	2	3	4		
3	3	4				
4						

1	1	1	1	1	2	3
2	2	2	4	4		
3	3	3				
4						

1	1	1	1	1	2	4
2	2	2	3	4		
3	3	3				
4						

1	1	1	1	1	2	4
2	2	2	3	3		
3	3	4				
4						

1	1	1	1	1	3	4
2	2	2	2	3		
3	3	4				
4						

1	1	1	1	1	3	4
2	2	2	2	4		
3	3	3				
4						

1	1	1	1	1	3	3
2	2	2	2	3		
3	4	4				
4						

1	1	1	1	1	3	3
2	2	2	2	4		
3	3	4				
4						

1	1	1	1	1	4	4
2	2	2	2	3		
3	3	3				
4						

1	1	1	1	1	2	2
2	2	3	3	3		
3	4	4				
4						

1	1	1	1	1	2	2
2	2	3	3	4		
3	3	4				
4						

Example 5.5.2 Let $x_1 = a, x_2 = b, x_3 = c, x_4 = d$ in Example 5.4.4, then the program

$$Expand[(d+c)\ast(d+b)\ast(d+a)\ast(c+b)\ast(c+a)\ast(b+a)\ast(d^2+c^2+b^2+a^2)]$$

running by Mathematica yields the result

$$a^5\ast b^2\ast c+a^4\ast b^3\ast c+a^3\ast b^4\ast c+a^2\ast b^5\ast c+a^5\ast b\ast c^2+2\ast a^4\ast b^2\ast c^2+2\ast a^3\ast b^3\ast c^2+2\ast a^2\ast b^4\ast c^2$$

$$+a\ast b^5\ast c^2+a^4\ast b\ast c^3+2\ast a^3\ast b^2\ast c^3+2\ast a^2\ast b^3\ast c^3+a\ast b^4\ast c^3+a^3\ast b\ast c^4+2\ast a^2\ast b^2\ast c^4+a\ast b^3\ast c^4$$

$$+a^2\ast b\ast c^5+a\ast b^2\ast c^5+a^5\ast b^2\ast d+a^4\ast b^3\ast d+a^3\ast b^4\ast d+a^2\ast b^5\ast d+2\ast a^5\ast b\ast c\ast d+4\ast a^4\ast b^2\ast c\ast d$$

$$+4\ast a^3\ast b^3\ast c\ast d+4\ast a^2\ast b^4\ast c\ast d+2\ast a\ast b^5\ast c\ast d+a^5\ast c^2\ast d+4\ast a^4\ast b\ast c^2\ast d+6\ast a^3\ast b^2\ast c^2\ast d+6\ast a^2\ast b^3\ast c^2\ast d$$

$$+4\ast a\ast b^4\ast c^2\ast d+b^5\ast c^2\ast d+a^4\ast c^3\ast d+4\ast a^3\ast b\ast c^3\ast d+6\ast a^2\ast b^2\ast c^3\ast d+4\ast a\ast b^3\ast c^3\ast d+b^4\ast c^3\ast d$$

$$+a^3\ast c^4\ast d+4\ast a^2\ast b\ast c^4\ast d+4\ast a\ast b^2\ast c^4\ast d+b^3\ast c^4\ast d+a^2\ast c^5\ast d+2\ast a\ast b\ast c^5\ast d+b^2\ast c^5\ast d+a^5\ast b\ast d^2$$

$$+2*a^4*b^2*d^2+2*a^3*b^3*d^2+2*a^2*b^4*d^2+a*b^5*d^2+a^5*c*d^2+4*a^4*b*c*d^2+6*a^3*b^2*c*d^2$$

$$+6*a^2*b^3*c*d^2+4*a*b^4*c*d^2+b^5*c*d^2+2*a^4*c^2*d^2+6*a^3*b*c^2*d^2+8*a^2*b^2*c^2*d^2$$

$$+6*a*b^3*c^2*d^2+2*b^4*c^2*d^2+2*a^3*c^3*d^2+6*a^2*b*c^3*d^2+6*a*b^2*c^3*d^2+2*b^3*c^3*d^2$$

$$+2*a^2*c^4*d^2+4*a*b*c^4*d^2+2*b^2*c^4*d^2+a*c^5*d^2+b*c^5*d^2+a^4*b*d^3+2*a^3*b^2*d^3$$

$$+2*a^2*b^3*d^3+a*b^4*d^3+a^4*c*d^3+4*a^3*b*c*d^3+6*a^2*b^2*c*d^3+4*a*b^3*c*d^3+b^4*c*d^3$$

$$+2*a^3*c^2*d^3+6*a^2*b*c^2*d^3+6*a*b^2*c^2*d^3+2*b^3*c^2*d^3+2*a^2*c^3*d^3+4*a*b*c^3*d^3$$

$$+2*b^2*c^3*d^3+a*c^4*d^3+b*c^4*d^3+a^3*b*d^4+2*a^2*b^2*d^4+a*b^3*d^4+a^3*c*d^4+4*a^2*b*c*d^4$$

$$+4*a*b^2*c*d^4+b^3*c*d^4+2*a^2*c^2*d^4+4*a*b*c^2*d^4+2*b^2*c^2*d^4+a*c^3*d^4+b*c^3*d^4+a^2*b*d^5$$

$$+a*b^2*d^5+a^2*c*d^5+2*a*b*c*d^5+b^2*c*d^5+a*c^2*d^5+b*c^2*d^5.$$

From above, we find $K_{(5,2,1),(3,2,2,1)} = 6$, the 6 semi-standard Young tableaux ($SSYT^{521}$) of weight (3,2,2,1) are as follows:

1	1	1	2	4
2	3			
3				

1	1	1	2	3
2	3			
4				

1	1	1	2	3
2	4			
3				

1	1	1	3	4
2	2			
3				

1	1	1	3	3
2	2			
4				

1	1	1	2	2
3	3			
4				

and $K_{(5,2,1),(3,3,1,1)} = 4$, the 4 semi-standard Young tableaux ($SSYT^{521}$) of weight (3,3,1,1) are as follows:

1	1	1	2	4
2	2			
3				

1	1	1	2	2
2	4			
3				

1	1	1	2	3
2	2			
4				

1	1	1	2	2
2	3			
4				

Example 5.5.3 Let $\lambda = (3,1)$ and $n = 4$, then $(a_1, a_2, a_3, a_4) = (0, 1, 3, 6)$ and

$$\det G_{\{4;0,1,3,6\}} = x_1^{a_1} \times \prod_{2 \le i \le 4} x_i^{a_1} B_i$$

$$= (x_2 - x_1)((x_3^3 - x_1^3) - \frac{(x_3 - x_1)(x_2^3 - x_1^3)}{(x_2 - x_1)})$$

$$\times((x_4^6 - x_1^6) - \frac{(x_4 - x_1)(x_2^6 - x_1^6)}{(x_2 - x_1)} - ((x_3^6 - x_1^6) - \frac{(x_3 - x_1)(x_2^6 - x_1^6)}{(x_2 - x_1)})(\frac{(x_4^3 - x_1^3)(x_2 - x_1) - (x_4 - x_1)(x_2^3 - x_1^3)}{(x_3^3 - x_1^3)(x_2 - x_1) - (x_3 - x_1)(x_2^3 - x_1^3)}))$$

$$= (x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_1^3 x_3 + 2x_1^2 x_2 x_3 + 2x_1 x_2^2 x_3 + x_2^3 x_3 + x_1^2 x_3^2 + 2x_1 x_2 x_3^2$$

$$+ x_2^2 x_3^2 + x_1 x_3^3 + x_2 x_3^3 + x_1^3 x_4 + 2x_1^2 x_2 x_4 + 2x_1 x_2^2 x_4 + x_2^3 x_4 + 2x_1^2 x_3 x_4 + 3x_1 x_2 x_3 x_4$$

$$+ 2x_2^2 x_3 x_4 + 2x_1 x_3^2 x_4 + 2x_2 x_3 x_4^2 + x_3^2 x_4^2 + x_1 x_4^3 + x_2 x_4^3 + x_3 x_4^3$$

$$+ 2x_1 x_2 x_4^2 + 2x_1 x_3 x_4^2 + 2x_2 x_3^2 x_4 + x_1^2 x_4^2 + x_2^2 x_4^2 + x_3^3 x_4) \times \prod_{1 \le i < j \le 4} (x_j - x_i),$$

so by Theorem 5.4.1, we can get

$$s_{(3,1)}(x_1, x_2, x_3, x_4)$$

$$= x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_1^3 x_3 + 2x_1^2 x_2 x_3 + 2x_1 x_2^2 x_3 + x_2^3 x_3 + x_1^2 x_3^2 + 2x_1 x_2 x_3^2$$

$$+ x_2^2 x_3^2 + x_1 x_3^3 + x_2 x_3^3 + x_1^3 x_4 + 2x_1^2 x_2 x_4 + 2x_1 x_2^2 x_4 + x_2^3 x_4 + 2x_1^2 x_3 x_4 + 3x_1 x_2 x_3 x_4$$

$$+ 2x_2^2 x_3 x_4 + 2x_1 x_3^2 x_4 + 2x_2 x_3 x_4^2 + x_3^2 x_4^2 + x_1 x_4^3 + x_2 x_4^3 + x_3 x_4^3$$

$$+ 2x_1 x_2 x_4^2 + 2x_1 x_3 x_4^2 + 2x_2 x_3^2 x_4 + x_1^2 x_4^2 + x_2^2 x_4^2 + x_3^3 x_4.$$

Hence we obtain the following Kostka numbers:

$$K_{(3,1),(3,1)} = K_{(3,1),(2,2)} = K_{(3,1),(1,3)} = K_{(3,1),(3,0,1)}$$

$$= K_{(3,1),(0,3,1)} = K_{(3,1),(2,0,2)} = K_{(3,1),(0,2,2)} = K_{(3,1),(1,0,3)}$$

$$= K_{(3,1),(0,1,3)} = K_{(3,1),(3,0,0,1)} = K_{(3,1),(0,3,0,1)} = K_{(3,1),(0,0,3,1)}$$

$$= K_{(3,1),(2,0,0,2)} = K_{(3,1),(0,2,0,2)} = K_{(3,1),(0,0,2,2)} = K_{(3,1),(1,0,0,3)}$$

$$= K_{(3,1),(0,1,0,3)} = K_{(3,1),(0,0,1,3)} = 1,$$

$$K_{(3,1),(2,1,1)} = K_{(3,1),(1,2,1)} = K_{(3,1),(1,1,2)} = K_{(3,1),(2,1,0,1)}$$

$$= K_{(3,1),(1,2,0,1)} = K_{(3,1),(2,0,1,1)} = K_{(3,1),(0,2,1,1)} = K_{(3,1),(1,0,2,1)}$$

$$= K_{(3,1),(0,1,2,1)} = K_{(3,1),(1,1,0,2)} = K_{(3,1),(1,0,1,2)} = K_{(3,1),(0,1,1,2)} = 2,$$

and

$$K_{(3,1),(1,1,1,1)} = 3.$$