## 2 A Class of Prime Rational Functions

In this section, we will prove a new class of prime functions given by rational functions. First, we recall the definitions about prime and pseudo-prime functions:

**Definition 2.1** *A meromorphic function of the form* $h(z) = f \circ g(z)$ *is said to have* $f(z)$ *and* $g(z)$ *as left and right factors, respectively, provided that either* $f(z)$ *is non-bilinear and meromorphic and* $g(z)$ *is non-bilinear and entire, or* $f(z)$ *is rational and* $g(z)$ *is meromorphic. In particular, we say that* $f \circ g(z)$ *is a factorization of* $h(z)$.

**Definition 2.2** *A meromorphic function* $h(z)$ *is said to be pseudo-prime if every factorization* $h(z) = f \circ g(z)$ *implies that either* $f(z)$ *is rational or* $g(z)$ *is a polynomial.*

**Definition 2.3** *A meromorphic function* $h(z)$ *is said to be prime if every factorization* $h(z) = f \circ g(z)$ *implies that either* $f(z)$ *is bilinear or* $g(z)$ *is bilinear.*

**Definition 2.4** *A transcendental meromorphic function is a meromorphic function other than a rational function, a transcendental entire function is an entire function other than a polynomial.*

Now, we begin our main work with the following Proposition:

**Proposition 2.5** *Every rational function* $Q(z)$ *is pseudo-prime.*

***Proof***. If not, write $Q(z) = f \circ g(z)$, where $f(z)$ and $g(z)$ are both transcendental. By Picard's Theorem, we can find $z_0 \in \mathbb{C}$ such that the equation $g(z) = g(z_0)$ has infinity many number of roots. So the equation $Q(z) = Q(z_0)$ also has infinity many number of roots which is impossible. Hence, $Q(z)$ is pseudo-prime. ❑

**Proposition 2.6** *The factors of a rational function $Q(z)$ are both rational functions.*

***Proof***. Let $Q(z) = f \circ g(z)$. Since $Q(z)$ is pseudo-prime, we have either $f(z)$ is a rational function or $g(z)$ is a polynomial. If $g(z)$ is a polynomial, suppose $f(z)$ is a transcendental meromorphic function, we can find $z_1 \in \mathbb{C}$ such that the equation $Q(z) = Q(z_1)$ has infinity many number of roots, which is impossible. Hence, $f(z)$ must be a rational function.

If $f(z)$ is a rational function, suppose $g(z)$ is a transcendental meromorphic function, then we can find $z_2 \in \mathbb{C}$ such that the equation $Q(z) = Q(z_2)$ has infinity many number of roots, which is also impossible. Hence, $g(z)$ must be a rational function. ❑

Now, we come to our main work. We are interested in the rational functions of the form

$$Q(z) = \frac{(z-a_1)^{p_1}(z-a_2)^{p_2}\cdots(z-a_n)^{p_n}}{(z-b_1)^{q_1}(z-b_2)^{q_2}\cdots(z-b_m)^{q_m}},$$

where $a_1$, $a_2$, ..., $a_n$, $b_1$, $b_2$, ..., $b_m$ are distinct complex numbers and $p_1$, $p_2$, ..., $p_n$, $q_1$, $q_2$, ..., $q_m$ are distinct prime integers. One may ask the following question: Is $Q(z)$ prime? The answer is yes. In order to prove this general result, we first prove some special cases, so that the general case can be followed. For simplicity, we only consider monic rational functions.

**Theorem 2.7** *If $Q(z) = \dfrac{(z-a)^p}{(z-b)^q}$, where $a$, $b$ are distinct complex numbers and $p$, $q$ are distinct prime integers. Then $Q(z)$ is prime.*

***Proof***. If not, by Proposition 2.6, we may assume $Q(z) = f \circ g(z)$, where $f(z)$, $g(z)$ are both non-bilinear rational functions. Write

$$f(z) = \frac{(z-c_1)^{l_1}(z-c_2)^{l_2}\cdots(z-c_n)^{l_n}}{(z-d_1)^{k_1}(z-d_2)^{k_2}\cdots(z-d_m)^{k_m}},$$

where $c_1, c_2, \ldots, c_n$ are distinct zeros of $f(z)$ with multiplicity $l_1, l_2, \ldots, l_n$, respectively, and $d_1, d_2, \ldots, d_m$ are distinct poles of $f(z)$ with multiplicity $k_1, k_2, \ldots, k_m$, respectively. So

$$\frac{(z-a)^p}{(z-b)^q} = Q(z) = f \circ g(z) = \frac{(g(z)-c_1)^{l_1}(g(z)-c_2)^{l_2}\cdots(g(z)-c_n)^{l_n}}{(g(z)-d_1)^{k_1}(g(z)-d_2)^{k_2}\cdots(g(z)-d_m)^{k_m}},$$

i.e.,

$$\begin{aligned}&(z-a)^p(g(z)-d_1)^{k_1}(g(z)-d_2)^{k_2}\cdots(g(z)-d_m)^{k_m}\\ &= (z-b)^q(g(z)-c_1)^{l_1}(g(z)-c_2)^{l_2}\cdots(g(z)-c_n)^{l_n}.\end{aligned}$$

Note that the zeros of $(g(z)-d_j)^{k_j}$ and $(g(z)-c_i)^{l_i}$ are all distinct for $1 \leq j \leq m$, $1 \leq i \leq n$. Hence, we can immediately get $n = m = 1$. i.e.,

$$(z-a)^p(g(z)-d_1)^{k_1} = (z-b)^q(g(z)-c_1)^{l_1}.$$

Moreover, we may assume

$$g(z) - c_1 = \frac{(z-a)^{r_1}}{\beta(z)}$$

$$g(z) - d_1 = \frac{(z-b)^{r_2}}{\beta(z)},$$

where $\beta(z)$ is a nonzero polynomial and non-vanishing at $a$, $b$. Then

$$(g(z)-c_1)^{l_1} = \frac{(z-a)^{r_1 l_1}}{\beta^{l_1}(z)}$$

$$(g(z)-d_1)^{k_1} = \frac{(z-b)^{r_2 k_1}}{\beta^{k_1}(z)}.$$

Therefore, we will get $r_1 l_1 = p$, $r_2 k_1 = q$, and $l_1 = k_1$. Since $p$, $q$ are distinct prime integers, we have $l_1 = k_1 = 1$, this implies

$$f(z) = \frac{z-c_1}{z-d_1},$$

which contradicts to our hypothesis. Hence, $Q(z)$ is prime. ❑

We immediately get:

**Corollary 2.8** *If* $Q(z) = \dfrac{(z-a)^p}{(z-b)^q}$, *where* $a$, $b$ *are distinct complex numbers and* $p$, $q$ *are relative prime integers. Then* $Q(z)$ *is prime.*

**Theorem 2.9** *If* $Q(z) = \dfrac{(z-a)^p(z-b)^q}{(z-c)^r}$, *where* $a$, $b$, $c$ *are distinct complex numbers and* $p$, $q$, $r$ *are distinct prime integers. Then* $Q(z)$ *is prime.*

***Proof***. If not, by Proposition 2.6, we may assume $Q(z) = f \circ g(z)$, where $f(z)$, $g(z)$ are both non-bilinear rational functions. Write

$$f(z) = \frac{(z-c_1)^{l_1}(z-c_2)^{l_2}\cdots(z-c_n)^{l_n}}{(z-d_1)^{k_1}(z-d_2)^{k_2}\cdots(z-d_m)^{k_m}},$$

where $c_1, c_2, \ldots, c_n$ are distinct zeros of $f(z)$ with multiplicity $l_1, l_2, \ldots, l_n$, respectively, and $d_1, d_2, \ldots, d_m$ are distinct poles of $f(z)$ with multiplicity $k_1, k_2, \ldots, k_m$, respectively. So

$$\frac{(z-a)^p(z-b)^q}{(z-c)^r} = Q(z) = f \circ g(z) = \frac{(g(z)-c_1)^{l_1}(g(z)-c_2)^{l_2}\cdots(g(z)-c_n)^{l_n}}{(g(z)-d_1)^{k_1}(g(z)-d_2)^{k_2}\cdots(g(z)-d_m)^{k_m}},$$

i.e.,

$$\begin{aligned}&(z-a)^p(z-b)^q(g(z)-d_1)^{k_1}(g(z)-d_2)^{k_2}\cdots(g(z)-d_m)^{k_m}\\ &= (z-c)^r(g(z)-c_1)^{l_1}(g(z)-c_2)^{l_2}\cdots(g(z)-c_n)^{l_n}.\end{aligned}$$

Note that the zeros of $(g(z)-d_j)^{k_j}$ and $(g(z)-c_i)^{l_i}$ are all distinct for $1 \le j \le m$, $1 \le i \le n$. Hence, we have two cases:

*Case 1:* n=2, m=1.

In this case,

$$(z-a)^p(z-b)^q(g(z)-d_1)^{k_1} = (z-c)^r(g(z)-c_1)^{l_1}(g(z)-c_2)^{l_2}.$$

We may assume

$$\begin{aligned} g(z)-c_1 &= \frac{(z-a)^{s_1}}{\beta(z)}\\ g(z)-c_2 &= \frac{(z-b)^{s_2}}{\beta(z)}\\ g(z)-d_1 &= \frac{(z-c)^{s_3}}{\beta(z)},\end{aligned}$$

where $\beta(z)$ is a nonzero polynomial and non-vanishing at $a$, $b$, and $c$. Then

$$(g(z)-c_1)^{l_1} = \frac{(z-a)^{s_1 l_1}}{\beta^{l_1}(z)}$$

$$(g(z)-c_2)^{l_2} = \frac{(z-b)^{s_2 l_2}}{\beta^{l_2}(z)}$$
$$(g(z)-d_1)^{k_1} = \frac{(z-c)^{s_3 k_1}}{\beta^{k_1}(z)}.$$

Therefore, we will get $s_1 l_1 = p$, $s_2 l_2 = q$, $s_3 k_1 = r$ and $l_1 + l_2 = k_1$. Clearly, $k_1 = r$, $s_3 = 1$. Moreover, if $s_1 = 1$, then

$$g(z) - c_1 = \frac{z-a}{\beta(z)}$$
$$g(z) - d_1 = \frac{z-c}{\beta(z)},$$

this implies $\beta(z)$ is a constant function, so $g(z)$ is linear which contradicts to our hypothesis. Similarly, we can also get a contradiction if $s_2 = 1$. Hence, we have $s_1 = p$, $s_2 = q$, and $l_1 = 1$, $l_2 = 1$. Since $l_1 + l_2 = k_1$, we will get a contradiction if $r \neq 2$. If $r = 2$, then

$$g(z) - c_1 = \frac{(z-a)^p}{\beta(z)}$$
$$g(z) - c_2 = \frac{(z-b)^q}{\beta(z)}$$
$$g(z) - d_1 = \frac{z-c}{\beta(z)},$$

this implies $p = q$, which is impossible.

*Case 2:* n=1, m=1.

In this case,

$$(z-a)^p (z-b)^q (g(z)-d_1)^{k_1} = (z-c)^r (g(z)-c_1)^{l_1}.$$

We may assume

$$\begin{aligned} g(z) - c_1 &= \frac{(z-a)^{s_1}(z-b)^{s_2}}{\beta(z)} \\ g(z) - d_1 &= \frac{(z-c)^{s_3}}{\beta(z)}, \end{aligned}$$

where $\beta(z)$ is a nonzero polynomial and non-vanishing at $a$, $b$, and $c$. Then

$$\begin{aligned}(g(z)-c_1)^{l_1} &= \frac{(z-a)^{s_1 l_1}(z-b)^{s_2 l_1}}{\beta^{l_1}(z)}\\ (g(z)-d_1)^{k_1} &= \frac{(z-c)^{s_3 k_1}}{\beta^{k_1}(z)}.\end{aligned}$$

Therefore, we will get $s_1 l_1 = p$, $s_2 l_1 = q$, $s_3 k_1 = r$ and $l_1 = k_1$. This implies $l_1 = k_1 = 1$, so $f(z)$ is bilinear, which contradicts to our hypothesis.

Hence, we conclude that $Q(z)$ is prime. ❑

Similarly, we have the following result:

**Corollary 2.10** *If $Q(z) = \dfrac{(z-a)^p}{(z-b)^q(z-c)^r}$, where $a$, $b$, $c$ are distinct complex numbers and $p$, $q$, $r$ are distinct prime integers. Then $Q(z)$ is prime.*

**Theorem 2.11** *If $Q(z) = \dfrac{(z-a)^p(z-b)^q}{(z-c)^r(z-d)^s}$, where $a$, $b$, $c$, $d$ are distinct complex numbers and $p$, $q$, $r$, $s$ are distinct prime integers. Then $Q(z)$ is prime.*

***Proof***. If not, by Proposition 2.6, we may assume $Q(z) = f \circ g(z)$, where $f(z)$, $g(z)$ are both non-bilinear rational functions. Write

$$f(z) = \frac{(z-c_1)^{l_1}(z-c_2)^{l_2}\cdots(z-c_n)^{l_n}}{(z-d_1)^{k_1}(z-d_2)^{k_2}\cdots(z-d_m)^{k_m}},$$

where $c_1, c_2, \ldots, c_n$ are distinct zeros of $f(z)$ with multiplicity $l_1, l_2, \ldots, l_n$, respectively, and $d_1, d_2, \ldots, d_m$ are distinct poles of $f(z)$ with multiplicity $k_1, k_2, \ldots, k_m$, respectively. So

$$\frac{(z-a)^p(z-b)^q}{(z-c)^r(z-d)^s} = Q(z) = f\circ g(z) = \frac{(g(z)-c_1)^{l_1}(g(z)-c_2)^{l_2}\cdots(g(z)-c_n)^{l_n}}{(g(z)-d_1)^{k_1}(g(z)-d_2)^{k_2}\cdots(g(z)-d_m)^{k_m}},$$

i.e.,

$$\begin{aligned}&(z-a)^p(z-b)^q(g(z)-d_1)^{k_1}(g(z)-d_2)^{k_2}\cdots(g(z)-d_m)^{k_m}\\ &= (z-c)^r(z-d)^s(g(z)-c_1)^{l_1}(g(z)-c_2)^{l_2}\cdots(g(z)-c_n)^{l_n}.\end{aligned}$$

Note that the zeros of $(g(z)-d_j)^{k_j}$ and $(g(z)-c_i)^{l_i}$ are all distinct for $1 \le j \le m$, $1 \le i \le n$. Hence, we have four cases:

*Case 1:* n=2, m=2.

In this case,

$$(z-a)^p(z-b)^q(g(z)-d_1)^{k_1}(g(z)-d_2)^{k_2} = (z-c)^r(z-d)^s(g(z)-c_1)^{l_1}(g(z)-c_2)^{l_2}.$$

We may assume

$$g(z)-c_1 = \frac{(z-a)^{t_1}}{\beta(z)}$$
$$g(z)-c_2 = \frac{(z-b)^{t_2}}{\beta(z)}$$
$$g(z)-d_1 = \frac{(z-c)^{t_3}}{\beta(z)}$$
$$g(z)-d_2 = \frac{(z-d)^{t_4}}{\beta(z)},$$

where $\beta(z)$ is a nonzero polynomial and non-vanishing at $a$, $b$, $c$ and $d$. Then

$$(g(z)-c_1)^{l_1} = \frac{(z-a)^{t_1 l_1}}{\beta^{l_1}(z)}$$
$$(g(z)-c_2)^{l_2} = \frac{(z-b)^{t_2 l_2}}{\beta^{l_2}(z)}$$
$$(g(z)-d_1)^{k_1} = \frac{(z-c)^{t_3 k_1}}{\beta^{k_1}(z)}$$
$$(g(z)-d_2)^{k_2} = \frac{(z-d)^{t_4 k_2}}{\beta^{k_2}(z)}.$$

By the same argument as in case 1 of Theorem 2.9, if there are two or more $t_i = 1$, $i = 1,2,3,4$, it will imply $g(z)$ is linear which contracdicts to our hypothesis. Hence, there are at most one $t_i = 1$, $i = 1,2,3,4$. WLOG, we may assume $t_1 = 1$, then $t_2 = q$, $t_3 = r$, and $t_4 = s$. By the same argument as in case 1 of Theorem 2.9, it implies $q = r$ or $r = s$ or $q = s$ which is impossible. Therefore, we must have $t_1 = p$, $t_2 = q$, $t_3 = r$, and $t_4 = s$. i.e.,

$$g(z)-c_1 = \frac{(z-a)^p}{\beta(z)}$$

$$g(z)-c_2 = \frac{(z-b)^q}{\beta(z)}$$

$$g(z) - d_1 = \frac{(z-c)^r}{\beta(z)}$$
$$g(z) - d_2 = \frac{(z-d)^s}{\beta(z)},$$

so

$$c_2 - c_1 = \frac{(z-a)^p - (z-b)^q}{\beta(z)}$$
$$d_2 - d_1 = \frac{(z-c)^r - (z-d)^s}{\beta(z)},$$

and we can get

$$\beta(z) = \frac{(z-a)^p - (z-b)^q}{c_2 - c_1} = \frac{(z-c)^r - (z-d)^s}{d_2 - d_1}.$$

By comparing the highest degree of $z$, we have $\max\{p, q\} = \max\{r, s\}$, which is also impossible.

*Case 2:* n=1, m=2.

In this case,

$$(z-a)^p(z-b)^q(g(z)-d_1)^{k_1}(g(z)-d_2)^{k_2} = (z-c)^r(z-d)^s(g(z)-c_1)^{l_1}.$$

We may assume

$$\begin{aligned} g(z) - c_1 &= \frac{(z-a)^{t_1}(z-b)^{t_2}}{\beta(z)} \\ g(z) - d_1 &= \frac{(z-c)^{t_3}}{\beta(z)} \\ g(z) - d_2 &= \frac{(z-d)^{t_4}}{\beta(z)}, \end{aligned}$$

where $\beta(z)$ is a nonzero polynomial and non-vanishing at $a$, $b$, $c$ and $d$. Then

$$\begin{aligned} (g(z) - c_1)^{l_1} &= \frac{(z-a)^{t_1 l_1}(z-b)^{t_2 l_1}}{\beta^{l_1}(z)} \\ (g(z) - d_1)^{k_1} &= \frac{(z-c)^{t_3 k_1}}{\beta^{k_1}(z)} \\ (g(z) - d_2)^{k_2} &= \frac{(z-d)^{t_4 k_2}}{\beta^{k_2}(z)}. \end{aligned}$$

Therefore, we get $t_1 l_1 = p$, $t_2 l_1 = q$, $t_3 k_1 = r$, $t_4 k_2 = s$ and $l_1 = k_1 + k_2$. This implies $l_1 = 1$, which is impossible.

*Case 3:* n=2, m=1.

This case can be done as in case 2.

*Case 4:* n=1, m=1.

In this case,

$$(z-a)^p(z-b)^q(g(z)-d_1)^{k_1} = (z-c)^r(z-d)^s(g(z)-c_1)^{l_1}.$$

We may assume

$$\begin{aligned} g(z)-c_1 &= \frac{(z-a)^{t_1}(z-b)^{t_2}}{\beta(z)} \\ g(z)-d_1 &= \frac{(z-c)^{t_3}(z-d)^{t_4}}{\beta(z)}, \end{aligned}$$

where $\beta(z)$ is a nonzero polynomial and non-vanishing at $a$, $b$, $c$ and $d$. Then

$$\begin{aligned} (g(z)-c_1)^{l_1} &= \frac{(z-a)^{t_1 l_1}(z-b)^{t_2 l_1}}{\beta^{l_1}(z)} \\ (g(z)-d_1)^{k_1} &= \frac{(z-c)^{t_3 k_1}(z-d)^{t_4 k_1}}{\beta^{k_1}(z)}. \end{aligned}$$

Therefore, we get $t_1 l_1 = p$, $t_2 l_1 = q$, $t_3 k_1 = r$, $t_4 k_1 = s$ and $l_1 = k_1$. This implies $l_1 = k_1 = 1$, so $f(z)$ is bilinear, which contradicts to our hypothesis.

Hence, we conclude that $Q(z)$ is prime. ❑

From these special cases, we can begin to discuss the general result. First, we need some Lemmas:

**Lemma 2.12** [2] *If $h(z)$ is prime, then $h(az+b)$ is also prime, where $a \neq 0$ and $b$ are complex numbers.*

***Proof***. Let $h(az+b) = f \circ g(z)$, then we get

$$h(z) = f \circ g\left(\frac{z-b}{a}\right) = f \circ g_1(z),$$

where $g_1(z) = g((z-b)/a)$. Since $h(z)$ is prime, either $f(z)$ is linear or $g_1(z)$ is linear. If $f(z)$ is linear, then we are done. If $g_1(z)$ is linear, then it is clear that $g(z)$ is also linear. Hence, $h(az+b)$ is prime. ❑

The following Lemma is similar to the so-called Newton's identity[11,14].

**Lemma 2.13** **[11]** *If a polynomial $Q(z)$ is of the form*

$$Q(z) = (1+a_1z)(1+a_2z)\cdots(1+a_nz),$$

*where $a_1$, $a_2$, $\ldots$, $a_n$ are non-zero complex numbers (may not be distinct), then for each $k>1$, we have the identity*

$$ks_k = s_{k-1}t_1 - s_{k-2}t_2 + \cdots + (-1)^{k-1}t_k,$$

*where*

$$s_k(a_1,a_2,\ldots,a_n) = \sum_{1\le i_1<i_2<\cdots<i_k\le n} a_{i_1}a_{i_2}\cdots a_{i_k}$$

*and*

$$t_k = a_1^k + a_2^k + \cdots + a_n^k,$$

*for $1 \le k \le n$.*

***Proof***. Suppose that

$$f(z) = (1+a_1z)(1+a_2z)\cdots(1+a_nz) = 1 + s_1z + s_2z^2 + \cdots + s_nz^n,$$

where

$$s_k(a_1,a_2,\ldots,a_n) = \sum_{i_1<i_2<\cdots<i_k} a_{i_1}a_{i_2}\cdots a_{i_k}, \text{ for } 1\le k\le n.$$

Let

$$s_k^{(i)} = s_k(a_1,a_2,\ldots,\hat{a}_i,\ldots,a_n),$$

where $\hat{a}_i$ denoted the term $a_i$ being deleted. It follows that

$$f(z) = (1+a_iz)\left(1 + s_1^{(i)}z + s_2^{(i)}z^2 + \cdots + s_{n-1}^{(i)}z^{n-1}\right), \text{ for } 1\le i\le n.$$

Then we obtain

$$a_i\left(\frac{f(z)}{1+a_iz}\right)=a_i\left(1+s_1^{(i)}z+s_2^{(i)}z^2+\cdots+s_{n-1}^{(i)}z^{n-1}\right),\ \text{for } 1\le i\le n.$$

Adding these equations for $i=1,2,\ldots,n$, we get

$$\sum_{i=1}^{n}a_i\left(\frac{f(z)}{1+a_iz}\right)=\sum_{i=1}^{n}a_i+\sum_{i=1}^{n}a_is_1^{(i)}z+\cdots+\sum_{i=1}^{n}a_is_{n-1}^{(i)}z^{n-1}.$$

Applying the product rule of differentiation to $f(z)$, it can be seen that the left hand side of the last equation equals to $f'(z)$. Hence, we get

$$\sum_{i=1}^{n}a_i+\sum_{i=1}^{n}a_is_1^{(i)}z+\cdots+\sum_{i=1}^{n}a_is_{n-1}^{(i)}z^{n-1}=s_1+2s_2z+\cdots+ns_nz^{n-1}.$$

By comparing coefficients, we have

$$ks_k=\sum_{i=1}^{n}a_is_{k-1}^{(i)},\ \text{for } 1\le k\le n,$$

where $s_0^{(i)}=1$. Clearly, $s_k=s_k^{(i)}+a_is_{k-1}^{(i)}$. So,

$$s_k^{(i)}=s_k-a_is_{k-1}^{(i)},\ \text{for } 1\le k\le n.$$

Therefore,

$$s_k^{(i)}=s_k-a_is_{k-1}^{(i)}=s_k-a_i\left(s_{k-1}-a_is_{k-2}^{(i)}\right).$$

Continuing in this way and using the fact that $s_0^{(i)}=1$, we conclude that

$$s_k^{(i)}=s_k-a_is_{k-1}+a_i^2s_{k-2}+\cdots+(-1)^ka_i^k,\ \text{for } 1\le i,\ k\le n.$$

Replacing $k$ by $k-1$ and multiplying $a_i$, we get

$$a_is_{k-1}^{(i)}=a_i\left(s_{k-1}-a_is_{k-2}+a_i^2s_{k-3}+\cdots+(-1)^{k-1}a_i^{k-1}\right),\ \text{for } k>1.$$

Adding this expression from $i=1$ to $n$ to get

$$\sum_{i=1}^{n}a_is_{k-1}^{(i)}=s_{k-1}\sum_{i=1}^{n}a_i-s_{k-2}\sum_{i=1}^{n}a_i^2+\cdots+(-1)^{k-1}\sum_{i=1}^{n}a_i^k$$

and obtain

$$ks_k=s_{k-1}t_1-s_{k-2}t_2+\cdots+(-1)^{k-1}t_k,\ \text{for } k>1.$$

❑

Now, we begin to state and prove our main result:

**Theorem 2.14** *If* $Q(z) = \dfrac{(z-a_1)^{p_1}(z-a_2)^{p_2}\cdots(z-a_n)^{p_n}}{(z-b_1)^{q_1}(z-b_2)^{q_2}\cdots(z-b_m)^{q_m}}$, *where* $a_1$, $a_2$, $\ldots$, $a_n$, $b_1$, $b_2$, $\ldots$, $b_m$ *are distinct complex numbers,* $p_1$, $p_2$, $\ldots$, $p_n$, $q_1$, $q_2$, $\ldots$, $q_m$ *are distinct prime integers. Then* $Q(z)$ *is prime.*

***Proof***. By Theorem 2.7, 2.9, 2.11 and their corollaries, we may only consider the case when $m+n \geq 4$. If $Q(z)$ is not prime, by Proposition 2.6, we may assume $Q(z) = f \circ g(z)$, where $f(z)$, $g(z)$ are both non-bilinear rational functions. Write

$$f(z) = \frac{(z-c_1)^{l_1}(z-c_2)^{l_2}\cdots(z-c_r)^{l_r}}{(z-d_1)^{k_1}(z-d_2)^{k_2}\cdots(z-d_s)^{k_s}},$$

where $c_1, c_2, \ldots, c_r$ are distinct zeros of $f(z)$ with multiplicity $l_1, l_2, \ldots, l_r$, respectively, and $d_1, d_2, \ldots, d_s$ are distinct poles of $f(z)$ with multiplicity $k_1, k_2, \ldots, k_s$, respectively, and $g(z) = \dfrac{\alpha(z)}{\beta(z)}$, where $\alpha(z), \beta(z)$ are relative prime. Furthermore, we assume the leader coefficient of $\beta(z)$ is 1. So

$$\begin{aligned}\frac{(z-a_1)^{p_1}(z-a_2)^{p_2}\cdots(z-a_n)^{p_n}}{(z-b_1)^{q_1}(z-b_2)^{q_2}\cdots(z-b_m)^{q_m}} &= Q(z) = f\circ g(z)\\ &= \frac{(g(z)-c_1)^{l_1}(g(z)-c_2)^{l_2}\cdots(g(z)-c_r)^{l_r}}{(g(z)-d_1)^{k_1}(g(z)-d_2)^{k_2}\cdots(g(z)-d_s)^{k_s}},\end{aligned}$$

i.e.,

$$\begin{aligned}&(z-a_1)^{p_1}\cdots(z-a_n)^{p_n}(\alpha(z)-d_1\beta(z))^{k_1}\cdots(\alpha(z)-d_s\beta(z))^{k_s}\beta^w(z)\\ &= (z-b_1)^{q_1}\cdots(z-b_m)^{q_m}(\alpha(z)-c_1\beta(z))^{l_1}\cdots(\alpha(z)-c_r\beta(z))^{l_r}.\end{aligned}$$

Let $w = \sum_{i=i}^{r} l_i - \sum_{j=1}^{s} k_j \geq 0$. If $w \leq 0$, then we consider

$$\begin{aligned}&(z-b_1)^{q_1}\cdots(z-b_m)^{q_m}(\alpha(z)-c_1\beta(z))^{l_1}\cdots(\alpha(z)-c_r\beta(z))^{l_r}\beta^w(z)\\ &= (z-a_1)^{p_1}\cdots(z-a_n)^{p_n}(\alpha(z)-d_1\beta(z))^{k_1}\cdots(\alpha(z)-d_s\beta(z))^{k_s}.\end{aligned}$$

Therefore, we may assume that $w \geq 0$. Note that the zeros of $\beta(z)$, $\alpha(z) - d_j\beta(z)$ and $\alpha(z) - c_i\beta(z)$ are all distinct for $1 \leq j \leq s$, $1 \leq i \leq r$. If $w = 0$, we have the identity

$$\begin{aligned}&(z-a_1)^{p_1}\cdots(z-a_n)^{p_n}(\alpha(z)-d_1\beta(z))^{k_1}\cdots(\alpha(z)-d_s\beta(z))^{k_s}\\ &= (z-b_1)^{q_1}\cdots(z-b_m)^{q_m}(\alpha(z)-c_1\beta(z))^{l_1}\cdots(\alpha(z)-c_r\beta(z))^{l_r}.\end{aligned}$$

If $w > 0$ and $\beta(z)$ is a constant, we also get the same identity since $\beta(z) \equiv 1$. If $w > 0$ and $\beta(z)$ is a nonconstant polynomial, then $\beta^w(z)$ is a finite product of $(z - b_j)^{q_j}$. We may assume that

$$\beta^w(z) = (z - b_1)^{q_1} \cdots (z - b_u)^{q_u},$$

where $0 < u \leq m$. If $u = m$, then, by dividing the common factors on both sides, we get

$$\begin{aligned}&(z - a_1)^{p_1} \cdots (z - a_n)^{p_n} (\alpha(z) - d_1\beta(z))^{k_1} \cdots (\alpha(z) - d_s\beta(z))^{k_s} \\ &= (\alpha(z) - c_1\beta(z))^{l_1} \cdots (\alpha(z) - c_r\beta(z))^{l_r}.\end{aligned}$$

which is impossible since $(\alpha(z) - d_1\beta(z))^{k_1} \cdots (\alpha(z) - d_s\beta(z))^{k_s}$ is a nonconstant polynomial. So $0 < u < m$. Again, by dividing the common factors on both sides, we get

$$\begin{aligned}&(z - a_1)^{p_1} \cdots (z - a_n)^{p_n} (\alpha(z) - d_1\beta(z))^{k_1} \cdots (\alpha(z) - d_s\beta(z))^{k_s} \\ &= (z - b_1)^{q_{u+1}} \cdots (z - b_m)^{q_m} (\alpha(z) - c_1\beta(z))^{l_1} \cdots (\alpha(z) - c_r\beta(z))^{l_r}.\end{aligned}$$

Hence, without lose of generality, we only consider the case when $w = 0$.

Now, if

$$\alpha(z) - c_i\beta(z) = (z - a_1)^{\zeta_1} \cdots (z - a_v)^{\zeta_v},$$

where $v > 1$, then

$$(\alpha(z) - c_i\beta(z))^{l_i} = (z - a_1)^{l_i\zeta_1} \cdots (z - a_v)^{l_i\zeta_v},$$

it implies that $l_i\zeta_1 = p_1, \cdots, l_i\zeta_v = p_v$. Since $p_i$ are all distinct prime integers, we get $l_i = 1$. If

$$\alpha(z) - c_i\beta(z) = (z - a_i)^{\zeta_i},$$

then

$$(\alpha(z) - c_i\beta(z))^{l_i} = (z - a_i)^{l_i\zeta_i},$$

it implies that $l_i\zeta_i = p_i$, so $l_i = 1$ or $p_i$. If there are two or more $l_i = p_i$, say, $l_{i_1}$ and $l_{i_2}$, then $\zeta_{i_1} = \zeta_{i_2} = 1$ and we have

$$\alpha(z) - c_{i_1}\beta(z) = z - a_{i_1}$$

and

$$\alpha(z) - c_{i_2}\beta(z) = z - a_{i_2},$$

which implies $\beta(z)$ is constant, so $g(z)$ is linear, a contradiction. Therefore, there are at most one $l_i = p_i$. Similarly, there are at most one $k_j = q_j$. In fact, there are at most one $l_i = p_i$ or $k_j = q_j$. Otherwise, we will still get $g(z)$ is linear. Hence, by dividing the common factors on both sides, we may assume $l_i = k_j = 1$ for all $1 \leq i \leq r$, $1 \leq j \leq s$. Then

$$\begin{aligned}
&(z-a_1)^{p_1}\cdots(z-a_n)^{p_n}(\alpha(z)-d_1\beta(z))\cdots(\alpha(z)-d_s\beta(z)) \\
&= (z-b_1)^{q_1}\cdots(z-b_m)^{q_m}(\alpha(z)-c_1\beta(z))\cdots(\alpha(z)-c_r\beta(z)).
\end{aligned}$$

Therefore, we conclude that

$$\begin{cases}
\alpha(z)-c_1\beta(z) = (z-a_1)^{p_1}(z-a_2)^{p_2}\cdots(z-a_{N_1})^{p_{N_1}} \\
\alpha(z)-c_2\beta(z) = (z-a_{N_1+1})^{p_{N_1+1}}(z-a_{N_1+2})^{p_{N_1+2}}\cdots(z-a_{N_2})^{p_{N_2}} \\
\quad\vdots \\
\alpha(z)-c_r\beta(z) = (z-a_{N_{r-1}+1})^{p_{N_{r-1}+1}}(z-a_{N_{r-1}+2})^{p_{N_{r-1}+2}}\cdots(z-a_{N_r})^{p_{N_r}} \\
\alpha(z)-d_1\beta(z) = (z-b_1)^{q_1}(z-b_2)^{q_2}\cdots(z-b_{M_1})^{q_{M_1}} \\
\alpha(z)-d_2\beta(z) = (z-b_{M_1+1})^{q_{M_1+1}}(z-b_{M_1+2})^{q_{M_1+2}}\cdots(z-b_{M_2})^{q_{M_2}} \\
\quad\vdots \\
\alpha(z)-d_s\beta(z) = (z-b_{M_{s-1}+1})^{q_{M_{s-1}+1}}(z-b_{M_{s-1}+2})^{q_{M_{s-1}+2}}\cdots(z-b_{M_s})^{q_{M_s}},
\end{cases} \quad (*)$$

where $N_r = n$, $M_s = m$. Clearly, $r + s \geq 3$. Otherwise, $f(z)$ is bilinear which is impossible by assumption. Therefore, $(*)$ has as least three different equations, Without lose of generality, we consider

$$\begin{aligned}
\alpha(z)-c_1\beta(z) &= (z-a_1)^{p_1}(z-a_2)^{p_2}\cdots(z-a_k)^{p_k} \\
\alpha(z)-c_2\beta(z) &= (z-a_{k+1})^{p_{k+1}}(z-a_{k+2})^{p_{k+2}}\cdots(z-a_{k+l})^{p_{k+l}} \\
\alpha(z)-c_3\beta(z) &= (z-a_{k+l+1})^{p_{k+l+1}}(z-a_{k+l+2})^{p_{k+l+2}}\cdots(z-a_{k+l+r})^{p_{k+l+r}},
\end{aligned}$$

where $k = N_1$, $k+l = N_2$, $k+l+r = N_3$ for simplicity. Furthermore, we assume $p_{k+l+r} = \max_{1\leq i\leq k+l+r}\{p_i\}$. Since

$$\begin{aligned}
(c_3-c_1)\beta(z) &= (z-a_1)^{p_1}\cdots(z-a_k)^{p_k} - (z-a_{k+l+1})^{p_{k+l+1}}\cdots(z-a_{k+l+r})^{p_{k+l+r}} \\
(c_3-c_2)\beta(z) &= (z-a_{k+1})^{p_{k+1}}\cdots(z-a_{k+l})^{p_{k+l}} - (z-a_{k+l+1})^{p_{k+l+1}}\cdots(z-a_{k+l+r})^{p_{k+l+r}},
\end{aligned}$$

we have

$$\frac{(z-a_1)^{p_1}\cdots(z-a_k)^{p_k}-(z-a_{k+l+1})^{p_{k+l+1}}\cdots(z-a_{k+l+r})^{p_{k+l+r}}}{c_3-c_1}$$
$$=\frac{(z-a_{k+1})^{p_{k+1}}\cdots(z-a_{k+l})^{p_{k+l}}-(z-a_{k+l+1})^{p_{k+l+1}}\cdots(z-a_{k+l+r})^{p_{k+l+r}}}{c_3-c_2},$$

i.e.,

$$\begin{aligned}&(c_2-c_1)(z-a_{k+l+1})^{p_{k+l+1}}\cdots(z-a_{k+l+r})^{p_{k+l+r}}\\&=(c_2-c_3)(z-a_1)^{p_1}\cdots(z-a_k)^{p_k}+(c_3-c_1)(z-a_{k+1})^{p_{k+1}}\cdots(z-a_{k+l})^{p_{k+l}}.\end{aligned}$$

Replacing $z$ by $z+a_{k+l+r}$, we get

$$\begin{aligned}&(c_2-c_1)(z+e_{k+l+1})^{p_{k+l+1}}\cdots(z+e_{k+l+r-1})^{p_{k+l+r-1}}z^{p_{k+l+r}}\\&=(c_2-c_3)(z+e_1)^{p_1}\cdots(z+e_k)^{p_k}+(c_3-c_1)(z+e_{k+1})^{p_{k+1}}\cdots(z+e_{k+l})^{p_{k+l}}\\&=\sigma(1+\frac{z}{e_1})^{p_1}\cdots(1+\frac{z}{e_k})^{p_k}+\sigma^{'}(1+\frac{z}{e_{k+1}})^{p_{k+1}}\cdots(1+\frac{z}{e_{k+l}})^{p_{k+l}},\end{aligned}$$

where $e_i=a_{k+l+r}-a_i$, $i=1,2,\ldots,k+l+r$ and $\sigma=(c_2-c_3)(e_1^{p_1}\cdots e_k^{p_k})$, $\sigma^{'}=(c_3-c_1)(e_{k+1}^{p_{k+1}}\cdots e_{k+l}^{p_{k+l}})$. Since $p_{k+l+r}>k+l$, the coefficients of $1,z,\ldots,z^{k+l}$ in the left hand side of the last equation must be zero. Write the right hand side of the last equation as follows:

$$\begin{aligned}&\sigma\underbrace{(1+\frac{z}{e_1})\cdots(1+\frac{z}{e_1})}_{p_1}\cdots\underbrace{(1+\frac{z}{e_k})\cdots(1+\frac{z}{e_k})}_{p_k}\\&+\sigma^{'}\underbrace{(1+\frac{z}{e_{k+1}})\cdots(1+\frac{z}{e_{k+1}})}_{p_{k+1}}\cdots\underbrace{(1+\frac{z}{e_{k+l}})\cdots(1+\frac{z}{e_{k+l}})}_{p_{k+l}}\\&=\sigma(1+s_1z+s_2z^2+\cdots+s_{k+l}z^{k+l}+\cdots+s_{\sum_{i=1}^{k}p_i}z^{\sum_{i=1}^{k}p_i})\\&+\sigma^{'}(1+s_1^{'}z+s_2^{'}z^2+\cdots+s_{k+l}^{'}z^{k+l}+\cdots+s^{'}_{\sum_{i=k+1}^{k+l}p_i}z^{\sum_{i=k+1}^{k+l}p_i}),\end{aligned}$$

where $s_i$ and $s_i'$ are defined as in Lemma 2.13. Then we have $\sigma+\sigma^{'}=0$ and $\sigma s_j+\sigma^{'}s_j^{'}=0$ for all $1\le j\le k+l$. Let

$$t_j=\underbrace{(\frac{1}{e_1^j}+\cdots+\frac{1}{e_1^j})}_{p_1}+\cdots+\underbrace{(\frac{1}{e_k^j}+\cdots+\frac{1}{e_k^j})}_{p_k}=\frac{p_1}{e_1^j}+\cdots+\frac{p_k}{e_k^j}$$
$$t_j^{'}=\underbrace{(\frac{1}{e_{k+1}^j}+\cdots+\frac{1}{e_{k+1}^j})}_{p_{k+1}}+\cdots+\underbrace{(\frac{1}{e_{k+l}^j}+\cdots+\frac{1}{e_{k+l}^j})}_{p_{k+l}}=\frac{p_{k+1}}{e_{k+1}^j}+\cdots+\frac{p_{k+l}}{e_{k+l}^j}$$

for all $1 \leq j \leq k+l$, now we claim that

$$\sigma t_j + \sigma' t_j' = 0$$

for all $1 \leq j \leq k+l$.

For $j=1$, $t_1 = s_1$ and $t_1' = s_1'$ by definition, so

$$\sigma t_1 + \sigma' t_1' = \sigma s_1 + \sigma' s_1' = 0.$$

For $j=2$, by Lemma 2.13, we have

$$\begin{aligned} 2s_2 &= s_1 t_1 - t_2 = t_1^2 - t_2 \\ 2s_2' &= s_1' t_1' - t_2' = t_1'^2 - t_2', \end{aligned}$$

so

$$\begin{aligned} &\sigma t_2 + \sigma' t_2' = \sigma(t_1^2 - 2s_2) + \sigma'(t_1'^2 - 2s_2') \\ &= \sigma t_1^2 + \sigma' t_1'^2 = (\sigma t_1 + \sigma' t_1')(t_1 + t_1') - (\sigma + \sigma') t_1 t_1' = 0. \end{aligned}$$

Assume

$$\sigma t_j + \sigma' t_j' = 0, \quad 1 \leq j \leq u-1,$$

, where $u = 2, 4, \ldots, k+l$. If $j = u$, then

$$\begin{aligned} &\sigma(-1)^{u-1} t_u + \sigma'(-1)^{u-1} t_u' \\ &= \sigma[u s_u - s_{u-1} t_1 + \cdots + (-1)^{u-1} s_1 t_{u-1}] \\ &+\sigma'[u s_u' - s_{u-1}' t_1' + \cdots + (-1)^{u-1} s_1' t_{u-1}'], \end{aligned}$$

where we use Lemma 2.13. Note that

$$\begin{aligned} &\sigma s_{u-j} t_j + \sigma' s_{u-j}' t_j' \\ &= (\sigma s_{u-j} + \sigma' s_{u-j}')(t_j + t_j') - \sigma s_{u-j} t_j' - \sigma' s_{u-j}' t_j \\ &= -\sigma s_{u-j} t_j' - \sigma' s_{u-j}' t_j \end{aligned}$$

for all $1 \leq j \leq u-1$. Since $\sigma s_{u-j} + \sigma' s_{u-j}' = 0$ and $\sigma t_j + \sigma' t_j' = 0$ by induction hypothesis for all $1 \leq j \leq u-1$, we have $\sigma s_{u-j} = -\sigma' s_{u-j}'$ and $\sigma t_j = -\sigma' t_j'$, i.e.,

$$s_{u-j} t_j' = s_{u-j}' t_j$$

for all $1 \leq j \leq u-1$. This implies

$$\begin{aligned}&\sigma s_{u-j}t_j + \sigma' s'_{u-j}t'_j = -\sigma s_{u-j}t'_j - \sigma' s'_{u-j}t_j \\ &= -\sigma s'_{u-j}t_j - \sigma' s'_{u-j}t_j = -(\sigma+\sigma')s'_{u-j}t_j = 0.\end{aligned}$$

for all $1 \leq j \leq u-1$. Hence,

$$\begin{aligned}&\sigma(-1)^{u-1}t_u + \sigma'(-1)^{u-1}t'_u \\ &= \sigma u s_u + \sigma' u s'_u = 0.\end{aligned}$$

That is,

$$\sigma t_u + \sigma' t'_u = 0.$$

So, we have proved $\sigma t_j + \sigma' t'_j = 0$ for all $1 \leq j \leq k+l$ by induction. From which, we get a system of linear equations as follows:

$$\begin{cases} \sigma(\frac{p_1}{e_1} + \cdots + \frac{p_k}{e_k}) + \sigma'(\frac{p_{k+1}}{e_{k+1}} + \cdots + \frac{p_{k+l}}{e_{k+l}}) = 0 \\ \sigma(\frac{p_1}{e_1^2} + \cdots + \frac{p_k}{e_k^2}) + \sigma'(\frac{p_{k+1}}{e_{k+1}^2} + \cdots + \frac{p_{k+l}}{e_{k+l}^2}) = 0 \\ \qquad\qquad\qquad\vdots \\ \sigma(\frac{p_1}{e_1^{k+l}} + \cdots + \frac{p_k}{e_k^{k+l}}) + \sigma'(\frac{p_{k+1}}{e_{k+1}^{k+l}} + \cdots + \frac{p_{k+l}}{e_{k+l}^{k+l}}) = 0. \end{cases}$$

Since $e_j \neq 0$ for all $1 \leq j \leq k+l$, the determinant of the coefficient matrix

$$\begin{vmatrix} \frac{\sigma}{e_1} & \cdots & \frac{\sigma}{e_k} & \frac{\sigma'}{e_{k+1}} & \cdots & \frac{\sigma'}{e_{k+l}} \\ \frac{\sigma}{e_1^2} & \cdots & \frac{\sigma}{e_k^2} & \frac{\sigma'}{e_{k+1}^2} & \cdots & \frac{\sigma'}{e_{k+l}^2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma}{e_1^{k+l}} & \cdots & \frac{\sigma}{e_k^{k+l}} & \frac{\sigma'}{e_{k+1}^{k+l}} & \cdots & \frac{\sigma'}{e_{k+l}^{k+l}} \end{vmatrix}$$

$$= \frac{\sigma^k \sigma'^l}{\prod_{i=1}^{k+l} e_i} \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ \frac{1}{e_1} & \cdots & \frac{1}{e_k} & \frac{1}{e_{k+1}} & \cdots & \frac{1}{e_{k+l}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{e_1^{k+l-1}} & \cdots & \frac{1}{e_k^{k+l-1}} & \frac{1}{e_{k+1}^{k+l-1}} & \cdots & \frac{1}{e_{k+l}^{k+l-1}} \end{vmatrix}$$

and

$$\begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ \frac{1}{e_1} & \cdots & \frac{1}{e_k} & \frac{1}{e_{k+1}} & \cdots & \frac{1}{e_{k+l}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{e_1^{k+l-1}} & \cdots & \frac{1}{e_k^{k+l-1}} & \frac{1}{e_{k+1}^{k+l-1}} & \cdots & \frac{1}{e_{k+l}^{k+l-1}} \end{vmatrix}$$

is a Vandermonde determinant which equals to $\prod_{1\le i<j\le k+l}(\frac{1}{e_j}-\frac{1}{e_i})$, we conclude that

$$\begin{vmatrix} \frac{\sigma}{e_1} & \cdots & \frac{\sigma}{e_k} & \frac{\sigma'}{e_{k+1}} & \cdots & \frac{\sigma'}{e_{k+l}} \\ \frac{\sigma}{e_1^2} & \cdots & \frac{\sigma}{e_k^2} & \frac{\sigma'}{e_{k+1}^2} & \cdots & \frac{\sigma'}{e_{k+l}^2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma}{e_1^{k+l}} & \cdots & \frac{\sigma}{e_k^{k+l}} & \frac{\sigma'}{e_{k+1}^{k+l}} & \cdots & \frac{\sigma'}{e_{k+l}^{k+l}} \end{vmatrix} = \frac{\sigma^k \sigma'^l}{\prod_{i=1}^{k+l} e_i} \prod_{1\le i<j\le k+l}(\frac{1}{e_j}-\frac{1}{e_i})$$

which is nonzero. Hence, we obtain $p_j = 0$ for all $1 \le j \le k+l$, which is impossible. Thus, we conclude that $Q(z)$ is prime and we complete the proof. ❑