3 Factorization of the Weierstrass $\wp$-Function

In this section, we will discuss the factorization of the Weierstrass $\wp$-function. First, we recall some basic definition.

Definition 3.1 *The Weierstrass $\wp$-function is a double period meromorphic function defined by*

$$\wp(u) = \frac{1}{u^2} + \sum_{\omega \in L'} \left(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2} \right)$$

where $\omega_1, \omega_2 \in \mathbb{C}$ with $Im(\omega_1/\omega_2) > 0$ and $L' = \{\omega = m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}, m, n$ not both zero.$\}$ is the set of all nonzero period of $\wp(u)$.

Clearly, $\wp(u)$ is an even function. Since 0 is a double pole of $\wp(u)$, we can get an expansion of of $\wp(u)$ near 0 as follows:

$$\wp(u) = \frac{1}{u^2} + c_1 u^2 + c_2 u^4 + \cdots + c_n u^{2n} + \cdots,$$

where $c_n = (2n+1) \sum_{\omega \in L'} \frac{1}{\omega^{2n+2}}$. In general, we denote

$$g_2 = 20c_1 = 60 \sum_{\omega \in L'} \frac{1}{\omega^4}$$
$$g_3 = 28c_2 = 140 \sum_{\omega \in L'} \frac{1}{\omega^6},$$

which are called the invariant of $\wp(u)$. In fact, they can be explicitly expressed as follows:

Proposition 3.2 [13] *Let g_2, g_3 be the invariant of the Weierstrass $\wp$-function with primitive periods ω_1 and ω_2. Then*

$$g_2 = (\frac{2\pi}{\omega_1})^4 (\frac{1}{12} + 20 \sum_{r=1}^{\infty} \frac{r^3 h^{2r}}{1-h^{2r}})$$
$$g_3 = (\frac{2\pi}{\omega_1})^6 (\frac{1}{216} - \frac{7}{3} \sum_{r=1}^{\infty} \frac{r^5 h^{2r}}{1-h^{2r}}),$$

where $h = e^{(\omega_2/\omega_1)\pi i}$.

We have the following well-known properties of $\wp$(u):

Proposition 3.3 **[13]** *The Weierstrass $\wp$-function satisfies the differential equation*

$$(\wp'(u))^2 = 4\wp(u)^3 - g_2\wp(u) - g_3,$$

where g_2 and g_3 are the invariant of $\wp(u)$.

Proposition 3.4 **[13]** *The Weierstrass $\wp$-function satisfies the formula*

$$\wp(2u) = -2\wp(u) + \frac{1}{4}\left(\frac{\wp''(u)}{\wp'(u)}\right)^2.$$

Proposition 3.5 **[13]** *Any elliptic function with the same periods of $\wp$-function can be expressed as a rational function of $\wp(u)$ and $\wp'(u)$.*

It is easy to see that any rational function can be factored into prime functions. Ritt[4] proved that if a polynomial $P(z)$ has two factorizations

$$P(z) = P_1 \circ \cdots \circ P_n(z)$$

and

$$P(z) = Q_1 \circ \cdots \circ Q_m(z),$$

where P_i and Q_j are prime functions, then $m = n$. However, this is not true for rational functions. In fact, Walter Bergeiler[9] give such an example which is constructed in terms of the Weierstrass $\wp$-function.

Now, our purpose is to give a detail study of this example because some non-trival details are skipped by Walter Bergeiler:

Theorem 3.6 **[9]** *There exists a rational function which has two factorizations into prime functions, and each having a different number of factors.*

Proof. Consider the Weierstrass $\wp$-function

$$\wp(u) = \frac{1}{u^2} + \sum_{\omega \in L'} \left(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2} \right)$$

with period 1 and $\sqrt{8}\,i$, where $L' = \{m + n \cdot \sqrt{8}\,i \mid m, n \in \mathbb{Z}, m, n \text{ not both zero.}\}$ is the set of all nonzero period of $p(u)$. From Proposition 3.3 and taking the derivative on both side, we have

$$2(\wp'(u)) \cdot \wp''(u) = 12\wp(u)^2 \cdot \wp'(u) - g_2 \cdot \wp'(u),$$

i.e.,

$$\wp''(u) = 6\wp(u)^2 - \frac{1}{2} g_2.$$

Substituting the identity into Proposition 3.4, we get

$$\wp(2u) = \frac{16\wp(u)^4 + 8g_2\wp(u)^2 + 32g_3\wp(u) + g_2^2}{16(4\wp(u)^3 - g_2\wp(u) - g_3)}.$$

Let

$$R(z) = \frac{16z^4 + 8g_2z^2 + 32g_3z + g_2^2}{16(4z^3 - g_2z - g_3)},$$

then $\wp(2u) = R(\wp(u))$. Also, by Proposition 3.5, we have $\wp(\sqrt{8}\,iu) = S(\wp(u))$ and $\wp(8u) = T(\wp(u))$, where S, T are both rational functions. Since $\wp(u)$ is an even function, we deduce that

$$R(R(R(\wp(u)))) = \wp(8u) = T(\wp(u)) = \wp(-8u) = S(S(\wp(u))),$$

i.e.,

$$T = R \circ R \circ R = S \circ S.$$

Suppose R is prime. If S is prime, then we find a rational function T which has two different number factorizations into prime functions. If S is not prime, since every rational function can be factored into prime functions, we can factor $S \circ S$ into an even number prime functions, hence T still has the desired properties.

Now, it remains to prove that R is prime. Suppose R is not prime, since the degree of R is 4, we may assume there exist two rational functions P and Q

of degree 2 such that $R = P \circ Q$. Without lose of generality, we also assume $P(0) = P(\infty) = \infty$, otherwise, we can consider $R = P \circ L \circ L^{-1} \circ Q$, where L is a suitable linear transformation. It follows that

$$P(z) = az + b + \frac{c}{z}$$

for some a, b ,$c \in \mathbb{C}$. On the other hand, if we denote by e_1, e_2, e_3 the zeros of $4z^3 - g_2 z - g_3$, we may assume without loss of generality that

$$Q(z) = \frac{(z-e_1)(z-e_2)}{z-e_3}.$$

Then

$$\begin{aligned}
R(z) &= P \circ Q(z) \\
&= a \cdot \frac{(z-e_1)(z-e_2)}{z-e_3} + b + \frac{c(z-e_3)}{(z-e_1)(z-e_2)} \\
&= \frac{a(z-e_1)^2(z-e_2)^2 + b(z-e_1)(z-e_2)(z-e_3) + c(z-e_3)^2}{(z-e_1)(z-e_2)(z-e_3)} \\
&= \frac{az^4 + (b - 2a(e_1+e_2))z^3 + (a(e_1^2 + 4e_1e_2 + e_2^2) - b(e_1+e_2+e_3) + c)z^2}{(z-e_1)(z-e_2)(z-e_3)} \\
&+ \frac{(b(e_1e_2 + e_2e_3 + e_1e_3) - 2ae_1e_2(e_1+e_2) - 2ce_3)z + (ae_1^2e_2^2 + be_1e_2e_3 + ce_3^2)}{(z-e_1)(z-e_2)(z-e_3)} \\
&= \frac{16z^4 + 8g_2z^2 + 32g_3z + g_2^2}{16(4z^3 - g_2z - g_3)}.
\end{aligned}$$

Comparing the coefficients of z^2, z and the constant term in the denominator, and the coefficients of z^4, z^3, z^2, z and the constant term in the numerator, we have the following system of equations:

$$\begin{cases}
e_1 + e_2 + e_3 = 0 \\
e_1e_2 + e_2e_3 + e_1e_3 = -\frac{1}{4}g_2 \\
e_1e_2e_3 = \frac{1}{4}g_3 \\
a = \frac{1}{4} \\
b - 2a(e_1+e_2) = 0 \\
a(e_1^2 + 4e_1e_2 + e_2^2) - b(e_1+e_2+e_3) + c = \frac{1}{8}g_2 \\
b(e_1e_2 + e_2e_3 + e_1e_3) - 2ae_1e_2(e_1+e_2) - 2ce_3 = \frac{1}{2}g_3 \\
ae_1^2e_2^2 + be_1e_2e_3 + ce_3^2 = \frac{1}{64}g_2^2.
\end{cases}$$

This implies $e_3 = -e_1 - e_2$, $g_2 = 4(e_1^2 + e_2^2 + e_1 e_2)$, $a = 1/4$, $b = (e_1 + e_2)/2$, $c = (e_1 - e_2)^2/4$, $g_3 = -4e_1 e_2(e_1 + e_2)$ and $e_1 e_2(e_1 + e_2) = 0$. That is, $g_3 = 0$.

Let

$$f(h) = 64\pi^6\left(\frac{1}{216} - \frac{7}{3}\sum_{r=1}^{\infty}\frac{r^5 h^{2r}}{1 - h^{2r}}\right),$$

by Proposition 3.2, we get $g_3 = f(e^{-\sqrt{8}\pi})$. On the other hand, we note that the g_3-value is zero if the primitive periods of the Weierstrass $\wp$-function are 1 and i. So we get $f(e^{-\pi}) = 0$ if we choose $\omega_1 = 1$ and $\omega_2 = i$ in Proposition 3.2. Since $f(h)$ is decreasing, we see that $g_3 = f(e^{-\sqrt{8}\pi}) > f(e^{-\pi}) = 0$. This contradiction completes the proof of the theorem. ❑

Now, we will discuss about the primeness of the Weierstrass $\wp$-function. First, we need some basic facts of Nevanlinna theory.

Definition 3.7 *Given a meromorphic function $f(z)$, define $n(r, f)$ be the number of zeros of $f(z)$ in $|z| \leq r$ with zeros of multiplicity p being counted p times, and denote*

$$\begin{aligned} N(r, f) &= \int_0^r \frac{n(x, 1/f) - n(0, 1/f)}{x}\,dx + n(0, \frac{1}{f})\log r \\ m(r, f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta, \end{aligned}$$

where $\log^+ \left| f\left(re^{i\theta}\right)\right| = max\{\left| f\left(re^{i\theta}\right)\right|,\ 0\}$. *Then the Nevanlinna characteristic function is defined by*

$$T(r, f) = m(r, f) + N(r, f).$$

The following Lemmas are well-known results:

Lemma 3.8 [8] *Let $f(z)$ be an entire function. Then*

$$\overline{\lim_{r\to\infty}}\frac{T(r, f')}{T(r, f)} \leq 1$$

outside a set of r of finite linear measure.

Lemma 3.9 [8] *Let $f(z)$ be a transcendental meromorphic function, and $g(z)$ be a transcendental entire function. Then*

$$\lim_{r\to\infty} \frac{T(r, f\circ g)}{T(r,g)} = \infty.$$

From these Lemmas, we can prove the primeness and pseudo-primeness of the Weierstrass $\wp$-function when the second invariant $g_2 = 0$.

Theorem 3.10 **[15]** *If $g_2 = 0$ in the Weierstrass $\wp$-function $\wp(u)$, then $\wp'(u)$ is pseudo-prime, and the only possible nonelliptic right factors are cubic polynomials.*

Proof. Write $\wp'(u) = f\circ g(u)$, where $f(z)$ is transcendental meromorphic and $g(z)$ is entire. By Proposition 3.3, $\wp'(u)$ satisfies

$$(w')^3 = P(w),$$

where $P(w)$ is a polynomial. So

$$(g'(u)f'(g(u)))^3 = P(f(g(u))),$$

i.e.,

$$(g'(u))^3 = F(g(u)),$$

where $F(z) = P(f(z))/(f'(z))^3$.

If $F(z)$ is transcendental meromorphic, by Lemma 3.8 and Lemma 3.9, we have

$$\infty = \lim_{r\to\infty}\frac{T(r, F\circ g)}{T(r,g)} = \lim_{r\to\infty}\frac{T(r,(g'))^3}{T(r,g)} \le \lim_{r\to\infty}\frac{3T(r,g')}{T(r,g)} \le 3$$

which is a contradiction. Hence, $F(z)$ is rational. Write

$$(g'(u))^3 = C\frac{(g(u)-a_1)^{n_1}\cdots(g(u)-a_k)^{n_k}}{(g(u)-b_1)^{m_1}\cdots(g(u)-b_l)^{m_l}},$$

where C is a constant, $a_1,\ldots,a_k, b_1,\ldots,b_l$ are distinct complex numbers, $n_1,\ldots,n_k$, $m_1,\ldots,m_l$ are nonnegative integers. Then

$$(g(u)-b_1)^{m_1}\cdots(g(u)-b_l)^{m_l}(g'(u))^3 = C(g(u)-a_1)^{n_1}\cdots(g(u)-a_k)^{n_k}.$$

Since $g(u)$ is entire, from Picard's Theorem, we may assume $n_2 = \cdots = n_k = m_1 = \cdots = m_l = 0$, and $g(u)$ has no Picard-exceptional value, i.e.,

$$(g'(u))^3 = C(g(u) - a_1)^{n_1}.$$

Clearly, $n_1 \leq 2$. since $g(u)$ is entire, we may assume $g(u_0) = a_1$ and write

$$g(u) = \sum_{j=0}^{\infty} \frac{g^{(j)}(u_0)}{j!}(u - u_0)^j.$$

If $n_1 = 1$, then

$$\left(\sum_{j=1}^{\infty} \frac{g^{(j)}(u_0)}{(j-1)!}(u - u_0)^{j-1}\right)^3 = C\left(\sum_{j=1}^{\infty} \frac{g^{(j)}(u_0)}{j!}(u - u_0)^j\right).$$

Compare the coefficients of $(u - u_0)^j$, we get $g^{(j)}(u_0) = 0$ for all $j = 1, 2, \cdots$, i.e., $g(u)$ is constant, a contradiction. Hence, $n_1 = 2$. Then

$$\left(\sum_{j=1}^{\infty} \frac{g^{(j)}(u_0)}{(j-1)!}(u - u_0)^{j-1}\right)^3 = C\left(\sum_{j=1}^{\infty} \frac{g^{(j)}(u_0)}{j!}(u - u_0)^j\right)^2.$$

Compare the coefficients of $(u - u_0)^j$ again, we first get $g'(u_0) = g''(u_0) = 0$. For the coefficient of $(u - u_0)^6$, we get

$$\left(\frac{g'''(u_0)}{2!}\right)^3 = C\left(\frac{g'''(u_0)}{3!}\right)^2,$$

i.e.,

$$g'''(u_0) = 0 \text{ or } \frac{2C}{9}.$$

If $g'''(u_0) = 0$, we will get $g^{(j)}(u_0) = 0$ for all $j = 1, 2, \cdots$, i.e., $g(u)$ is a constant, a contradiction. Therefore, $g'''(u_0) = 2C/9$. Then

$$\begin{aligned}&\left(\frac{1}{9} + \frac{g^{(4)}(u_0)}{3!}(u - u_0) + \frac{g^{(5)}(u_0)}{4!}(u - u_0)^2 + \cdots\right)^3 \\ &= C\left(\frac{1}{27} + \frac{g^{(4)}(u_0)}{4!}(u - u_0) + \frac{g^{(5)}(u_0)}{5!}(u - u_0)^2 + \cdots\right)^2.\end{aligned}$$

Compare the coefficients of $(u - u_0)^j$ again, we get $g^{(j)}(u_0) = 0$ for all $j = 4, 5, \cdots$. Thus, $g(u) = a_1 + C(u - u_0)^3/27$, a cubic polynomial. That is, we conclude that $\wp(u)$ is pseudo-prime. ❑

From Theorem 3.10, we can see that if $g_2 = 0$, $\wp'(u)$ is pseudo-prime, but not prime. In fact, $\wp'(u)$ is prime if and only if $g_2 \neq 0$. The proof is more complicated, so we formulate the result as a theorem and one can find the proof in [15].

Theorem 3.11 [15] *Let $\wp(u)$ be the Weierstrass $\wp$-function, g_2, g_3 be the invariant of $\wp(u)$. Then $\wp'(u)$ is prime if and only if $g_2 \neq 0$.*