Appendix E

Proof of Theorem 5.3.8 :

Inserting (5.6)∼(5.10), (5.28)∼(5.32), (5.50)∼(5.54), (5.48), and (5.49) into (5.58), it becomes

$$\begin{aligned}\mathbf{0} &= (\mathbf{u}_4^{(\alpha)}\mathbf{a}(\omega_\alpha) - x_\alpha\omega_\alpha\mathbf{u}_4^{(\alpha)} - b_1\mathbf{u}_3^{(\alpha)} - b_2\mathbf{u}_2^{(\alpha)}) \otimes \mathbf{v}_1^{(\alpha)} \\ &\quad + \mathbf{u}_1^{(\alpha)} \otimes (-x_\alpha\omega_\alpha\mathbf{v}_4^{(\alpha)} - \mathbf{v}_4^{(\alpha)}\mathbf{b}(\omega_\alpha) - \frac{a_2}{\omega_\alpha^2}\boldsymbol{\beta}_2 + b_1\mathbf{v}_3^{(\alpha)} - b_2\mathbf{v}_2^{(\alpha)}).\end{aligned}$$

Then there exists scalar b_3 such that

$$\mathbf{u}_4^{(\alpha)}\mathbf{a}(\omega_\alpha) - x_\alpha\omega_\alpha\mathbf{u}_4^{(\alpha)} - b_1\mathbf{u}_3^{(\alpha)} - b_2\mathbf{u}_2^{(\alpha)} = b_3\mathbf{u}_1^{(\alpha)}$$

$$-x_\alpha\omega_\alpha\mathbf{v}_4^{(\alpha)} - \mathbf{v}_4^{(\alpha)}\mathbf{b}(\omega_\alpha) - \frac{a_2}{\omega_\alpha^2}\boldsymbol{\beta}_2 + b_1\mathbf{v}_3^{(\alpha)} - b_2\mathbf{v}_2^{(\alpha)} = -b_3\mathbf{v}_1$$

i.e.,

$$\mathbf{u}_4^{(\alpha)}(\omega_\alpha\mathbf{S}_1 + \boldsymbol{\gamma}_1\boldsymbol{\beta}_1 - x_\alpha\omega_\alpha\mathbf{I}_1) = b_1\mathbf{u}_3^{(\alpha)} + b_2\mathbf{u}_2^{(\alpha)} + b_3\mathbf{u}_1^{(\alpha)}, \tag{E.1}$$

$$\mathbf{v}_4^{(\alpha)}(\omega_\alpha\mathbf{S}_2 + \omega_\alpha^2\boldsymbol{\gamma}_2\boldsymbol{\beta}_2 + x_\alpha\omega_\alpha\mathbf{I}_2) = b_1\mathbf{v}_3^{(\alpha)} - b_2\mathbf{v}_2^{(\alpha)} + b_3\mathbf{v}_1^{(\alpha)} - \frac{a_2}{\omega_\alpha^2}\boldsymbol{\beta}_2. \tag{E.2}$$

Consider (E.1) and let

$$\begin{cases} \omega_\alpha\mathbf{u}_4^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1) = b_1\mathbf{u}_3^{(\alpha)} + b_2\mathbf{u}_2^{(\alpha)} + b_3\mathbf{u}_1^{(\alpha)} \\ \mathbf{u}_4^{(\alpha)}\boldsymbol{\gamma}_1 = 0 \end{cases}.$$

We get

$$\mathbf{u}_4^{(\alpha)} = \frac{1}{\omega_\alpha}(b_1\mathbf{u}_3^{(\alpha)} + b_2\mathbf{u}_2^{(\alpha)} + b_3\mathbf{u}_1^{(\alpha)})(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1},$$

and

$$b_3 = \frac{-(b_1\mathbf{u}_3^{(\alpha)} + b_2\mathbf{u}_2^{(\alpha)})(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}{\mathbf{u}_1^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}.$$

On the other hand, we let

$$\begin{cases} \omega_\alpha\mathbf{v}_4^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2) = b_1\mathbf{v}_3^{(\alpha)} - b_2\mathbf{v}_2^{(\alpha)} + b_3\mathbf{v}_1^{(\alpha)} \\ \omega_\alpha^2\mathbf{v}_4^{(\alpha)}\boldsymbol{\gamma}_2 = \frac{-a_2}{\omega_\alpha^2} \end{cases}$$

in (E.2), and obtain

$$\mathbf{v}_4^{(\alpha)} = \frac{1}{\omega_\alpha}(b_1\mathbf{v}_3^{(\alpha)} - b_2\mathbf{v}_2^{(\alpha)} + b_3\mathbf{v}_1^{(\alpha)})(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1},$$
$$b_3 = \frac{-\frac{a_2}{\omega_\alpha^3} + (b_2\mathbf{v}_2^{(\alpha)} - b_1\mathbf{v}_3^{(\alpha)})(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}{\mathbf{v}_1^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}.$$

The proof will be completed if

$$\frac{-(b_1\mathbf{u}_3^{(\alpha)} + b_2\mathbf{u}_2^{(\alpha)})(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}{\mathbf{u}_1^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1} = \frac{-\frac{a_2}{\omega_\alpha^3} + (b_2\mathbf{v}_2^{(\alpha)} - b_1\mathbf{v}_3^{(\alpha)})(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}{\mathbf{v}_1^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}.$$

Since x_α is a root of (4.1) with multiplicity 4, then

$$\frac{d^3}{dx^3}\{f_{Ta}^*(x)f_{Ts}^*(-x) - 1\}|_{x=x_\alpha} = 0$$

implies

$$\begin{aligned}
& \boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-4}\boldsymbol{\gamma}_1\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2 \\
- & \boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2 \\
+ & \boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-3}\boldsymbol{\gamma}_3 \\
- & \boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-4}\boldsymbol{\gamma}_2 \\
= & \ 0.
\end{aligned} \tag{E.3}$$

Dividing (E.3) by $\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2$, we obtain

$$\begin{aligned}
& b_1\left(\frac{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-4}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1} - \frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-4}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2}\right) - \frac{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1} \\
& + \frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-3}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2} = 0.
\end{aligned} \tag{E.4}$$

Since

$$\mathbf{u}_1^{(\alpha)}(S_1 - x_\alpha\mathbf{I}_1)^{-n}\boldsymbol{\gamma}_1 = a_1\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-(n+1)}\boldsymbol{\gamma}_1,$$
$$\mathbf{v}_1^{(\alpha)}(S_2 + x_\alpha\mathbf{I}_2)^{-n}\boldsymbol{\gamma}_2 = a_2\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-(n+1)}\boldsymbol{\gamma}_2,$$
$$\mathbf{u}_2^{(\alpha)}(S_1 - x_\alpha\mathbf{I}_1)^{-n}\boldsymbol{\gamma}_1 = \tfrac{a_1b_1}{\omega_\alpha}\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-(n+2)}\boldsymbol{\gamma}_1,$$
$$\mathbf{v}_2^{(\alpha)}(S_2 + x_\alpha\mathbf{I}_2)^{-n}\boldsymbol{\gamma}_2 = \tfrac{a_2b_1}{\omega_\alpha}\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-(n+2)}\boldsymbol{\gamma}_2,$$

and

$$\begin{aligned}
& \mathbf{u}_3^{(\alpha)}(S_1 - x_\alpha\mathbf{I}_1)^{-n}\boldsymbol{\gamma}_1 \\
= & \tfrac{b_2a_1}{\omega_\alpha}\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-(n+2)}\boldsymbol{\gamma}_1 + \tfrac{b_1^2a_1}{\omega_\alpha}\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-(n+3)}\boldsymbol{\gamma}_1, \\
& \mathbf{v}_3^{(\alpha)}(S_2 + x_\alpha\mathbf{I}_2)^{-n}\boldsymbol{\gamma}_2 \\
= & \tfrac{-b_2a_2}{\omega_\alpha}\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-(n+2)}\boldsymbol{\gamma}_2 + \tfrac{b_1^2a_2}{\omega_\alpha^2}\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-(n+3)}\boldsymbol{\gamma}_2,
\end{aligned}$$

we have

$$
\begin{aligned}
&\frac{-(b_1\mathbf{u}_3^{(\alpha)} + b_2\mathbf{u}_2^{(\alpha)})(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}{\mathbf{u}_1^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1} \\
= \; &\frac{b_1}{\omega_\alpha}\left(-2b_2\frac{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1} - \frac{b_1^2}{\omega_\alpha}\frac{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-4}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{-\frac{a_2}{\omega_\alpha^3} + (b_2\mathbf{v}_2^{(\alpha)} - b_1\mathbf{v}_3^{(\alpha)})(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}{\mathbf{v}_1^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2} \\
= \; &\frac{b_1}{\omega_\alpha}\left(2b_2\frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-3}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2} - \frac{b_1^2}{\omega_\alpha}\frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-4}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2} + \frac{1}{\omega_\alpha}\right).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&-2b_2\frac{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1} - \frac{b_1^2}{\omega_\alpha}\frac{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-4}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1} \\
= \; &\frac{-2b_2}{b_1}\left(1 - b_1\frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-3}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2}\right) - \frac{b_1}{\omega_\alpha}\left(b_1\frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-4}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2}\right. \\
&\left.-\frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-3}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2} + \frac{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1}\right) \qquad \text{(E.5)} \\
= \; &2b_2\frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-3}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2} - \frac{b_1^2}{\omega_\alpha}\frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-4}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2} - \frac{2b_2}{b_1} \\
&+\frac{b_1}{\omega_\alpha}\frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-3}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2} - \frac{b_1}{\omega_\alpha}\frac{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1} \\
= \; &2b_2\frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-3}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2} - \frac{b_1^2}{\omega_\alpha}\frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-4}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2} \qquad \text{(E.6)} \\
&-\frac{2b_1}{\omega_\alpha}\frac{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1} + \frac{1}{\omega_\alpha}\left(1 - b_1\frac{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1}\right) \\
&-\frac{b_1}{\omega_\alpha}\frac{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1} \\
= \; &2b_2\frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-3}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2} - \frac{b_1^2}{\omega_\alpha}\frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-4}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2} + \frac{1}{\omega_\alpha}.
\end{aligned}
$$

Equality (E.5) follows by (5.44) and (E.4) ; (E.6) follows directly from (5.43).

□