Chapter 1

Introduction

Consider a Quasi-Birth-and-Death (QBD) process with phase-type of interarrival and service times distributions. In this thesis, we investigate a vector product-form of the steady-state solution of the single server queueing system $C_k/C_m/1$, where C_k and C_m belong to the Coxian class of probability density functions with k and m stages respectively. Customers are served under the First-come First-served discipline (FCFS).

Starting from the papers of Evans [5] and Wallace [11] much attention was paid to present stationary probability vector of QBD process in a matrix-geometric form. The general approach was given by Neuts in [9] where the matrix-geometric method relies on determining the minimal nonnegative matrix solution R of a matrix-quadratic equation; the invariant vector is expressed in terms of powers of R. Instead of solving R, we will take a different approach in this thesis.

In [2] Bertimas solved the queueing system $C_k/C_m/s$ by using a generating function technique. He proved that the probability of n customers being in the system is a linear combination of geometric terms when it is saturated. In [3], he showed that the waiting time distribution under FCFS for the $MGE_k/MGE_m/s$ system, where MGE_n is the class of mixed generalized Erlang probability density function of order n, can be expressed as mixture of exponential distributions.

Le Boudec [4] studied a $PH/PH/1$ queue. He showed that the steady-state solution is a linear combination of product-forms which can be expressed in terms of roots of the associated characteristic polynomial. He showed that all the eigenvectors used in the expression of the steady-state probability of $PH/PH/1$ are Kronecker products and gave a simple formula for computing the stationary probability of the number of customers in the system.

Wang considered a $PH/PH/1/N$ open queueing system containing finite number of customers N in [10]. She showed that the number of roots of the associated characteristic polynomial depends on the traffic intensity but independently of N.

In the studies of Bertimas [2], Le Boudec [4], and Wang [10], when analyzing the equilibrium probability for unbounded state, they all presented that the Laplace transforms of interarrival and service times distributions satisfy an equation of a simple form, and used the roots of the equation to express the steady-state solution under the assumption all the roots are distinct.

Our interest is to study the property for the steady-state probability when multiple roots occur in the equation involving the Laplace transforms. As a result, in the $E_k/E_m/1$ queueing system, it is proved that the m roots of the characteristic polynomial are distinct if the arrival and service rates are reals, and when multiple roots occur, one needs to solve a set of equations of matrix polynomials.

This thesis is organized in the following. In chapter 2, we introduce the model of $C_k/C_m/1$ queues. Chapter 3 attempts to illustrate the solution space for the saturated probability and give a method to construct a solution basis that involves the left Jordan chains for $\mathbf{Q}(\omega)$ corresponding to the singularities of $\mathbf{Q}(\omega)$. In chapter 4, we discuss the relation between the nonzero singularities of a fundamental matrix polynomial $\mathbf{Q}(\omega)$ and the roots of the associated characteristic polynomial.

Chapter 5 exhibits a procedure for solving stationary probabilities and provides a method to describe those vectors as the linear combination of Kronecker products.