Chapter 2

Analysis of $C_k/C_m/1$

2.1 Interarrival and Service Times

We assume that both interarrival and service times are Coxian with k and m stages respectively. It means that an arrival has to go through at most up to k phases, and the length of phase j is exponential with a given rate λ_j for $j = 1, \ldots, k$. After phase j, $j = 1, \ldots, k$ the interarrival time comes to an end with probability p_j and enters to the next phase with probability $1 - p_j$. A similar notation for μ_j, q_j,$j = 1, \ldots, m$ in the service distribution is assumed. Obviously, set $p_k = 1$ and $q_m = 1$ for the last stage.

We denote the interarrival time distribution by $F_{Ta}(\cdot)$ (See Neuts [9]), its representation of dimension k by $(\boldsymbol{\beta}_1, \mathbf{S}_1)$, and its mean by λ', where the $1 \times k$ vector

$$\boldsymbol{\beta}_1 = (1, 0, \cdots, 0)$$

is the initial probability and

$$\mathbf{S}_1 = \begin{bmatrix} -\lambda_1 & (1-p_1)\lambda_1 & & & 0 \\ & -\lambda_2 & (1-p_2)\lambda_2 & & \\ & & \ddots & \ddots & \\ & & & -\lambda_{k-1} & (1-p_{k-1})\lambda_{k-1} \\ 0 & & & & -\lambda_k \end{bmatrix}$$

is a squared matrix of order k. The distribution function is given by

$$F_{Ta}(t) = 1 - \boldsymbol{\beta}_1 exp(\mathbf{S}_1 t)\mathbf{1},$$

where $\mathbf{1}$ is a column vector of all entries equal to 1.

Similarly, the service time distribution $F_{Ts}(\cdot)$ has mean μ' and representation $(\boldsymbol{\beta}_2, \mathbf{S}_2)$ of dimension m, where

$$\boldsymbol{\beta}_2 = (1, 0, \ldots, 0)$$

is a $1 \times m$ vector and

$$\mathbf{S}_2 = \begin{bmatrix} -\mu_1 & (1-q_1)\mu_1 & & & 0 \\ & -\mu_2 & (1-q_2)\mu_2 & & \\ & & \ddots & \ddots & \\ & & & -\mu_{m-1} & (1-q_{m-1})\mu_{m-1} \\ 0 & & & & -\mu_m \end{bmatrix}$$

is the squared matrix of order m. The distribution is given by

$$F_{Ts}(t) = 1 - \boldsymbol{\beta}_2 exp(\mathbf{S}_2 t)\mathbf{1}.$$

The Laplace transform of the interarrival time has the form

$$\begin{aligned} f^*_{Ta}(x) &= \int_0^\infty e^{-xt} dF_{Ta}(t) \\ &= \int_0^\infty e^{-xt}\{-\boldsymbol{\beta}_1 exp(\mathbf{S}_1 t)\mathbf{S}_1\mathbf{1}\}dt \\ &= \int_0^\infty -\boldsymbol{\beta}_1 exp\{(\mathbf{S}_1 - x\mathbf{I}_1)t\}\mathbf{S}_1\mathbf{1}\}dt \\ &= -\boldsymbol{\beta}_1(\mathbf{S}_1 - x\mathbf{I}_1)^{-1} exp(\mathbf{S}_1 - x\mathbf{I}_1)|_0^\infty \mathbf{S}_1\mathbf{1} \\ &= \boldsymbol{\beta}_1(x\mathbf{I}_1 - \mathbf{S}_1)^{-1}\boldsymbol{\gamma}_1, \end{aligned}$$

where $\boldsymbol{\gamma}_1 = -\mathbf{S}_1\mathbf{1}$. Similarly, the Laplace transform of the service time has the form

$$f^*_{Ts}(x) = \boldsymbol{\beta}_2(x\mathbf{I}_2 - \mathbf{S}_2)^{-1}\boldsymbol{\gamma}_2,$$

where $\boldsymbol{\gamma}_2 = -\mathbf{S}_2\mathbf{1}$. Let $\mathbf{I}_i$ for $i = 1, 2$, denote an identity matrix with a proper dimension in equation. The utilization factor is defined as

$$\rho = \frac{\mu'}{\lambda'}.$$

Since

$$\lambda' = \int_0^\infty t dF_{Ta}(t) = f^{*'}_{Ta}(0)$$

and

$$\mu' = \int_0^\infty t dF_{Ts}(t) = f^{*'}_{Ts}(0),$$

we have

$$\rho = \frac{f^{*'}_{Ts}(0)}{f^{*'}_{Ta}(0)}.$$

2.2 Assumption and Problem Description

A state of system is denoted by (n, i, j), where n is the number of customers in the service or in the waiting room, $n \geq 0$, and i (resp. j) is the phase of the customer presents in the interarrival fictitious center (resp. the service center), $1 \leq i \leq k$, $1 \leq j \leq m$. The states with $n \geq 1$ and $n = 0$ are called *unboundary* and *boundary* states, respectively. The stationary probability is denoted by $\boldsymbol{\pi}$ as

$$\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots)$$

where $\boldsymbol{\pi}_n$ is a row-vector of stationary probability when there are n customers in the system. The $\boldsymbol{\pi}_n$ with $n \geq 1$ and $n = 0$ are called *saturated* and *unsaturated* probabilities, respectively. For this system, a steady state will exist if the utilization factor $\rho < 1$.

2.3 Matrix of Transition Rates

The $C_k/C_m/1$ queue may be studied as a QBD process on the state space

$$E = \{(0,i,0), 1 \leq i \leq k\} \cup \{(n,i,j), n \geq 1, 1 \leq i \leq k, 1 \leq j \leq m\}.$$

We arrange the states (n,i,j) in lexicographic order, that is, $(0,1), \ldots, (0,k)$, $(1,1,1), \ldots, (1,1,m), (1,2,1), \ldots, (1,2,m), \ldots$. Then the transition rate matrix (or infinitesimal generator) $\mathbf{Q}$ is of the block-tridiagonal form and can be written as

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B}_0 & \overline{\mathbf{A}}_0 & & & \\ \mathbf{C}_0 & \mathbf{A}_1 & \mathbf{A}_0 & & \\ & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \\ & & \mathbf{A}_2 & \mathbf{A}_1 & \ddots \\ & & & \mathbf{A}_2 & \ddots \\ & & & & \ddots \end{bmatrix}. \tag{2.1}$$

The submatrices could be written as Kronecker products and Kronecker sums which were defined in Bellman [1] and denoted by $\otimes$ and $\oplus$ respectively (See Appendix A). They are expressed by

$$\begin{aligned} \mathbf{B}_0 &= \mathbf{S}_1, & \mathbf{A}_1 &= \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \mathbf{I}_2 \\ \mathbf{C}_0 &= \mathbf{I}_1 \otimes \boldsymbol{\gamma}_2, & \mathbf{A}_1 &= \mathbf{S}_1 \oplus \mathbf{S}_2 \\ \overline{\mathbf{A}}_0 &= \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2, & \mathbf{A}_2 &= \mathbf{I}_1 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 \\ \boldsymbol{\gamma}_1 &= -\mathbf{S}_1 \mathbf{1}, & \boldsymbol{\gamma}_2 &= -\mathbf{S}_2 \mathbf{1}. \end{aligned} \tag{2.2}$$

where $\mathbf{A}_i$, $i = 0,1,2$ are squared matrices of dimension mk, $\mathbf{B}_0$ is a squared matrix of dimension k, $\mathbf{C}_0$ a matrix of dimension mk by k, and $\overline{\mathbf{A}}_0$ a k by mk matrix.

Define a fundamental matrix polynomial

$$\mathbf{Q}(\omega) \triangleq \mathbf{A}_0 + \omega \mathbf{A}_1 + \omega^2 \mathbf{A}_2,$$

and we have

$$\mathbf{Q}(\omega) = \mathbf{a}(\omega) \oplus \mathbf{b}(\omega)$$

where

$$\mathbf{a}(\omega) \quad \stackrel{\triangle}{=} \omega \mathbf{S}_1 + \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1,$$

$$\mathbf{b}(\omega) \quad \stackrel{\triangle}{=} \omega \mathbf{S}_2 + \omega^2 \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2.$$

If ω_o satisfies $det\mathbf{Q}(\omega_o) = 0$, then ω_o is called a singularity of $\mathbf{Q}(\omega)$.

2.4 State Balance Equations

For the state balance equations $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$ and the normalization condition $\boldsymbol{\pi}\mathbf{1} = 1$, we give the following equations:

$$\boldsymbol{\pi}_0 \mathbf{B}_0 + \boldsymbol{\pi}_1 \mathbf{C}_0 = \mathbf{0}, \tag{2.3}$$

$$\boldsymbol{\pi}_0 \overline{\mathbf{A}}_0 + \boldsymbol{\pi}_1 \mathbf{A}_1 + \boldsymbol{\pi}_2 \mathbf{A}_2 = \mathbf{0}, \tag{2.4}$$

$$\boldsymbol{\pi}_{n-1} \mathbf{A}_0 + \boldsymbol{\pi}_n \mathbf{A}_1 + \boldsymbol{\pi}_{n+1} \mathbf{A}_2 = \mathbf{0}, \quad n \geq 2, \tag{2.5}$$

$$\boldsymbol{\pi}\mathbf{1} = 1. \tag{2.6}$$

Substituting (2.2) into (2.3)$\sim$(2.5), we rewrite the balance equations as:

$$\boldsymbol{\pi}_0 \mathbf{S}_1 + \boldsymbol{\pi}_1 (\mathbf{I}_1 \otimes \boldsymbol{\gamma}_2) = \mathbf{0}, \tag{2.7}$$

$$\boldsymbol{\pi}_0 (\boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2) + \boldsymbol{\pi}_1 (\mathbf{S}_1 \oplus \mathbf{S}_2) + \boldsymbol{\pi}_2 (\mathbf{I}_1 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2) = \mathbf{0}, \tag{2.8}$$

$$\boldsymbol{\pi}_{n-1} (\gamma_1 \boldsymbol{\beta}_1 \otimes \mathbf{I}_2) + \boldsymbol{\pi}_n (\mathbf{S}_1 \oplus \mathbf{S}_2) + \boldsymbol{\pi}_{n+1} (\mathbf{I}_1 \otimes \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2) = \mathbf{0}, \; n \geq 2. \tag{2.9}$$