Chapter 3

Solution Spaces

In this chapter, we introduce the solution space that satisfies the unboundary equations. We will construct a basis for the solution space and describe the general solution form of $\boldsymbol{\pi}_n$ by using a canonical set of left Jordan chains for the matrix polynomial $\mathbf{Q}(\omega)$ corresponding to ω_o (See [8]). We give a brief introduction to the left Jordan chains in section 3.1 and review some results in [6] in order to illustrate the solution space in section 3.2. The general solution space for the vector of stationary probability is discussed in section 3.3.

3.1 Left Jordan Chains

Definition 3.1.1 *(p.28 in [7]) The sequence of* $n-$*dimensional vectors* $\boldsymbol{\varphi}_1$, $\ldots, \boldsymbol{\varphi}_\ell$ ($\boldsymbol{\varphi}_1 \neq 0$) *is called a left Jordan chain of length* ℓ *for the matrix polynomial*

$L(\omega)$ corresponding to the complex number ω_o if the following equalities holds:

$$\begin{aligned}
&\boldsymbol{\varphi}_1 L(\omega_o) = \mathbf{0}, \\
&\boldsymbol{\varphi}_1 L'(\omega_o) + \boldsymbol{\varphi}_2 L_(\omega_o) = \mathbf{0}, \\
&\tfrac{1}{2!}\boldsymbol{\varphi}_1 L''(\omega_0) + \boldsymbol{\varphi}_2 L'(\omega_o) + \boldsymbol{\varphi}_3 L_(\omega_o) = \mathbf{0}, \\
&\qquad\qquad \vdots \\
&\tfrac{1}{(\ell-1)!}\boldsymbol{\varphi}_1 \tfrac{d^{\ell-1}}{d\omega^{\ell-1}} L(\omega)|_{\omega=\omega_o} + \tfrac{1}{(\ell-2)!}\boldsymbol{\varphi}_2 \tfrac{d^{\ell-2}}{d\omega^{\ell-2}} L(\omega)|_{\omega=\omega_o} + \cdots + \boldsymbol{\varphi}_\ell L(\omega_o) = \mathbf{0}.
\end{aligned} \tag{3.1}$$

where $\frac{d}{d\omega}L(\omega)$ denotes derivative with respect to each element of $L(\omega)$.

Example 3.1.2 *Let*

$$L(\omega) = \begin{bmatrix} \omega^2 & -\omega \\ 0 & \omega^2 \end{bmatrix}.$$

Since $detL(\omega) = \omega^4$, there exists one eigenvalue of $L(\omega)$, namely, $\omega_o = 0$. All left Jordan chains of length not exceeding 3 can be described as follows:

1. Left Jordan chains of length 1 are $\boldsymbol{\varphi}_1 = \begin{bmatrix} x_{11} & x_{12} \end{bmatrix}$, where $x_{11}, x_{12} \in \mathcal{C}$ are not both zero.

2. Left Jordan chains of length 2 are $\boldsymbol{\varphi}_1 = \begin{bmatrix} x_{11} & 0 \end{bmatrix}$, and $\boldsymbol{\varphi}_2$, where $x_{11} \neq 0$ and $\boldsymbol{\varphi}_2$ is arbitrary.

3. Left Jordan chains of length 3 are $\boldsymbol{\varphi}_1 = \begin{bmatrix} x_{11} & 0 \end{bmatrix}$, $\boldsymbol{\varphi}_2 = \begin{bmatrix} x_{21} & x_{11} \end{bmatrix}$, and $\boldsymbol{\varphi}_3$, where $x_{11} \neq 0$, and x_{21}, $\boldsymbol{\varphi}_3$ are arbitrary.

□

Note that $\boldsymbol{\varphi}_1$ is contained in the left null space of $L(\omega_o)$, and the vectors in a Jordan chain for matrix polynomial $L(\omega)$ are not necessarily linearly independent. The example shows the structure of Jordan chains for matrix polynomial can be quite complicated. To understand this structure better, it is useful to introduce canonical sets.

Proposition 3.1.3 *(Proposition 1.15 in [7]) Let*

$$\{\boldsymbol{\varphi}_{i1}, \cdots, \boldsymbol{\varphi}_{i\kappa_i}, i = 1, \ldots, p\} \tag{3.2}$$

be a set of left Jordan chains for the matrix polynomial $L(\omega)$ *corresponding to* ω_o*. Then the following conditions are equivalent.*

(i) The set (3.2) is canonical.

(ii) The vectors $\boldsymbol{\varphi}_{11}, \ldots, \boldsymbol{\varphi}_{p1}$ *are linearly independent and* $\sum_{i=1}^{p} \kappa_i = r$*, where* r *is the multiplicity of* ω_0 *which is a zero of* $\det L(\omega)$*.*

(iii) Let

$$\mathbf{U} = \begin{bmatrix} L(\omega_o) & L'(\omega_o) & \cdots & \frac{1}{(\ell-1)!}\frac{d^{\ell-1}}{d\omega^{\ell-1}}L(\omega)|_{\omega=\omega_o} \\ 0 & L(\omega_o) & \cdots & \frac{1}{(\ell-2)!}\frac{d^{\ell-2}}{d\omega^{\ell-2}}L(\omega)|_{\omega=\omega_o} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L(\omega_o) \end{bmatrix}$$

and denote the left null space of $\mathbf{U}$ *by* $\mathcal{N}$*. The sequences of vectors*

$$\boldsymbol{\gamma}_{ij} = (\mathbf{0}, \ldots, \mathbf{0}, \boldsymbol{\varphi}_{i1}, \ldots, \boldsymbol{\varphi}_{ij}), \ \ j = 1, \ldots, \kappa_i, \ \ i = 1, \ldots, p, \tag{3.3}$$

form a basis in $\mathcal{N}$*, where the number of zero vectors preceding* $\boldsymbol{\varphi}_{i1}$ *in* $\boldsymbol{\gamma}_{ij}$ *is* $\ell - j$*, (*ℓ*= max* $\{\kappa_1 \ldots, \kappa_p\}$*).*

Note that the canonical set is not unique, but the lengths $\kappa_1, \ldots, \kappa_p$ of left Jordan chains in a canonical set is uniquely determined. In fact, $\kappa_1, \ldots, \kappa_p$ are known as nonzero *partial multiplicities* (see Appendix B) of $L(\omega)$ at ω_o. Refers to [7] for details.

3.2 The Structure of the Solution Space

In [6], Gail expressed the solution space for saturated probabilities of $G/M/1$ system in detail. We review some of important results from [6].

Lemma 3.2.1 *Define the matrix* $\mathbf{P} = \mathbf{I} + \frac{\mathbf{Q}}{q}$ *with*

$$\mathbf{P} = \begin{bmatrix} \boldsymbol{\mathcal{B}}_0 & \overline{\boldsymbol{\mathcal{A}}}_0 & & & \\ \boldsymbol{\mathcal{C}}_0 & \boldsymbol{\mathcal{A}}_1 & \boldsymbol{\mathcal{A}}_0 & & \\ & \boldsymbol{\mathcal{A}}_2 & \boldsymbol{\mathcal{A}}_1 & \ddots & \\ & & \boldsymbol{\mathcal{A}}_2 & \ddots & \\ & & & \ddots & \end{bmatrix}. \tag{3.4}$$

where q *is any nonzero real number satisfying*

$$q \geq \max_j \{-(\mathbf{B}_0)_{jj}, -(\mathbf{A}_1)_{jj}\}.$$

Then the matrix $\mathbf{P}$ *is stochastic and nonnegative. Furthermore, in a positive recurrent case, the stationary distribution* $\boldsymbol{\pi} = [\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots]$*, where* $\boldsymbol{\pi}_j$ *is* $1 \times mk$ *row vector, satisfies* $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$ *equivalent to* $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$*.*

Proof. Since the off-diagonal elements of $\mathbf{Q}$ are nonnegative and its diagonal elements are all negative, the elements of $\mathbf{P}$ are nonnegative by the definition of $\mathbf{P}$. The matrix $\mathbf{P}$ is stochastic (i.e., $\mathbf{P1} = \mathbf{1}$) follows directly from the fact $\mathbf{Q1} = \mathbf{0}$. Obviously, the equations $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$ are equivalent to $\boldsymbol{\pi}(\mathbf{P} - \mathbf{I}) = \mathbf{0}$, i.e., $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$. □

Lemma 3.2.2 *Let*

$$\widetilde{\mathbf{P}} = \begin{bmatrix} \boldsymbol{\mathcal{A}}_1 + \boldsymbol{\mathcal{C}}_0(I - \boldsymbol{\mathcal{B}}_0)^{-1}\overline{\boldsymbol{\mathcal{A}}}_0 & \boldsymbol{\mathcal{A}}_0 & & & \\ \boldsymbol{\mathcal{A}}_2 & \boldsymbol{\mathcal{A}}_1 & \boldsymbol{\mathcal{A}}_0 & & \\ & \boldsymbol{\mathcal{A}}_2 & \boldsymbol{\mathcal{A}}_1 & \ddots & \\ & & \boldsymbol{\mathcal{A}}_2 & \ddots & \\ & & & \ddots & \end{bmatrix}.$$

Then $\widetilde{\mathbf{P}}$ *is nonnegative stochastic and the solutions* $(\boldsymbol{\pi}_0, \widetilde{\boldsymbol{\pi}})$ *of*

$$\begin{cases} \widetilde{\boldsymbol{\pi}}\widetilde{\mathbf{P}} = \widetilde{\boldsymbol{\pi}} \\ \boldsymbol{\pi}_0 = \boldsymbol{\pi}_1\boldsymbol{\mathcal{C}}_0(\mathbf{I} - \boldsymbol{\mathcal{B}}_0)^{-1} \end{cases} \tag{3.5}$$

are the same as the solutions $\boldsymbol{\pi}$ *of* $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$ *where* $\widetilde{\boldsymbol{\pi}} = (\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots)$ *and* $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \widetilde{\boldsymbol{\pi}})$*.*

Proof. To prove that the matrix is nonnegative, it is sufficient to show that $\boldsymbol{\mathcal{A}}_1 + \boldsymbol{\mathcal{C}}_0(\mathbf{I} - \boldsymbol{\mathcal{B}}_0)^{-1}\overline{\boldsymbol{\mathcal{A}}}_0$ is nonnegative. Since $(\mathbf{I} - \boldsymbol{\mathcal{B}}_0)^{-1} = \sum_{j=0}^{\infty} \boldsymbol{\mathcal{B}}_0^j \geq 0$ and $\boldsymbol{\mathcal{A}}_1$, $\boldsymbol{\mathcal{C}}_0$, $\overline{\boldsymbol{\mathcal{A}}}_0$ are nonnegative, it follows that $\boldsymbol{\mathcal{A}}_1 + \boldsymbol{\mathcal{C}}_0(\mathbf{I} - \boldsymbol{\mathcal{B}}_0)^{-1}\overline{\boldsymbol{\mathcal{A}}}_0 \geq 0$. Note that $\mathbf{P1} = \mathbf{1}$, and

$$\begin{aligned}
(\boldsymbol{\mathcal{A}}_1 + \boldsymbol{\mathcal{C}}_0(\mathbf{I} - \boldsymbol{\mathcal{B}}_0)^{-1}\overline{\boldsymbol{\mathcal{A}}}_0 + \boldsymbol{\mathcal{A}}_0)\mathbf{1} &= (\boldsymbol{\mathcal{A}}_0 + \boldsymbol{\mathcal{A}}_1)\mathbf{1} + \boldsymbol{\mathcal{C}}_0(\mathbf{I} - \boldsymbol{\mathcal{B}}_0)^{-1}\overline{\boldsymbol{\mathcal{A}}}_0\mathbf{1} \\
&= (\mathbf{1} - \boldsymbol{\mathcal{C}}_0\mathbf{1}) + \boldsymbol{\mathcal{C}}_0(\mathbf{I} - \boldsymbol{\mathcal{B}}_0)^{-1}(\mathbf{1} - \boldsymbol{\mathcal{B}}_0\mathbf{1}) \\
&= \mathbf{1} - \boldsymbol{\mathcal{C}}_0\mathbf{1} + \boldsymbol{\mathcal{C}}_0(\mathbf{I} - \boldsymbol{\mathcal{B}}_0)^{-1}(\mathbf{I} - \boldsymbol{\mathcal{B}}_0)\mathbf{1} \\
&= \mathbf{1}.
\end{aligned}$$

Hence, $\widetilde{\mathbf{P}}$ is stochastic. It is easy to verify that (3.5) is equivalent to $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$. Therefore, the conclusion follows directly from Lemma 3.2.1. □

We transform the original system into $\widetilde{\boldsymbol{\pi}}\widetilde{\mathbf{P}} = \widetilde{\boldsymbol{\pi}}$ where $\widetilde{\mathbf{P}}$ has the same form as the transition probability matrix defined in [6]. The results in [6] are summaried as follows. Write

$$\widetilde{\mathbf{P}} = \begin{bmatrix} \mathbf{B} & \mathbf{D} \\ \mathbf{R} & \mathbf{L} \end{bmatrix}$$

where $\mathbf{B}$ is $mk \times mk$. Replacing $\mathbf{B}$, $\mathbf{R}$ in $\widetilde{\mathbf{P}}$ by $\mathbf{I}$, $\mathbf{O}$, we have

$$\widetilde{\mathbf{P}}^* = \begin{bmatrix} \mathbf{I} & \mathbf{D} \\ \mathbf{O} & \mathbf{L} \end{bmatrix}.$$

Define

$$\mathcal{L}^* \triangleq \{\mathbf{y} \in l^\infty : \mathbf{y}\widetilde{\mathbf{P}}^* = \mathbf{y}\}.$$

We also define the operator τ as " shift left mk columns " on row vectors $\mathbf{y} \in l^\infty$ by

$$\tau\,(\,\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots\,) = (\mathbf{y}_2, \mathbf{y}_3, \ldots)$$

for each $\mathbf{y}_i$ is $1 \times mk$. Then $\mathcal{L}^*$ is τ-invariant (i.e., $\tau(\mathcal{L}^*) \subset \mathcal{L}^*$). For an eigenvalue ω_o of $\tau : \mathcal{L}^* \to \mathcal{L}^*$ the corresponding generalized eigenspace is

$$\mathcal{L}^*(\omega_o) = \{\mathbf{y} \in \mathcal{L}^* : (\tau - \omega_o I)^d \mathbf{y} = \mathbf{0} \quad \text{for some } d\},$$

where I is the identity operator, and from the results in linear algebra, we know $\mathcal{L}^*$ is the direct sum of these generalized eigenspaces, i.e.,

$$\mathcal{L}^* = \bigoplus_{|\omega_\alpha| \leq 1} \mathcal{L}^*(\omega_\alpha). \tag{3.6}$$

Since $\|\tau \mathbf{y}\|_\infty \leq \|\mathbf{y}\|_\infty$, any eigenvalues ω_α must satisfy $|\omega_\alpha| \leq 1$. Recall that $\mathcal{L}^*(\omega_\alpha)$ has a basis consisting of one or more cycles of generalized eigenvectors. Suppose $\{\boldsymbol{\chi}_{1,1}, \cdots, \boldsymbol{\chi}_{1,\kappa_1}, \cdots, \boldsymbol{\chi}_{p,1}, \cdots, \boldsymbol{\chi}_{p,\kappa_p}\}$ is a basis for $\mathcal{L}^*(\omega_o)$, namely,

$$\begin{aligned} (\tau - \omega_o I)\boldsymbol{\chi}_{i,1} &= \mathbf{0} \\ (\tau - \omega_o I)\boldsymbol{\chi}_{i,j} &= \boldsymbol{\chi}_{i,j-1}, \quad j = 2\ldots, \kappa_i. \end{aligned} \tag{3.7}$$

For $i = 1, \ldots, p$, we have

$$\boldsymbol{\chi}_{i,j} = \mathbf{z}_{i,j}\, e(\omega_o) + \frac{1}{1!}\mathbf{z}_{i,j-1} e'(\omega_o) + \cdots + \frac{1}{(j-1)!}\mathbf{z}_{i,1} \frac{d^{j-1}}{d\omega^{j-1}} e(\omega)|_{\omega=\omega_o} \quad j = 1, \ldots, \kappa_i, \tag{3.8}$$

where $\mathbf{z}_{i,j}$ is the vector whose elements are taken from the first mk entries of $\boldsymbol{\chi}_{i,j}$, and $e(\omega)$ is a $mk \times \infty$ matrix defined by

$$e(\omega) = \left[\mathbf{I}, \omega\mathbf{I}, \omega^2\mathbf{I}, \cdots\right].$$

From (3.6) and (3.8), we visualize that the l^1 solution space $\mathcal{L} = \{(\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots)\}$ satisfying normalization condition (2.6) and the unboundary equations (2.5) is

$$\mathcal{L} = \bigoplus_{|\omega_\alpha| < 1} \mathcal{L}^*(\omega_\alpha). \tag{3.9}$$

Define

$$\boldsymbol{\mathcal{A}}(\omega) = \omega\mathbf{I} - (\boldsymbol{\mathcal{A}}_0 + \omega\boldsymbol{\mathcal{A}}_1 + \omega^2\boldsymbol{\mathcal{A}}_2).$$

Then it is clear that $\boldsymbol{\mathcal{A}}(\omega)$ equals $-\frac{1}{q}\mathbf{Q}(\omega)$ and it follows $\boldsymbol{\mathcal{A}}(\omega)$ and $\mathbf{Q}(\omega)$ have the same singularities.

Proposition 3.2.3 *(Proposition 33 in [6]) If $|\omega_o| < 1$ is a zero of $det\boldsymbol{\mathcal{A}}(\omega) = 0$ of multiplicity r, then $\mathcal{L}^*(\omega_o)$ has dimension r.*

Theorem 3.2.4 *(Theorem 8 in [6]) Let $\widetilde{\mathbf{P}}$ be a transition matrix of an irreducible Markov chain. The system is ergodic if and only if $\det \boldsymbol{\mathcal{A}}(\omega) = 0$ has exactly mk zeros in the open unit disk.*

We conclude that if $\omega_1, \ldots, \omega_s, \omega_{s+1}$ are the distinct singularities of $\mathbf{Q}(\omega)$ in the open unit disk then the l^1 solution space $\mathcal{L} = \{(\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots)\}$ satisfying the unboundary equations (2.5) is the direct sum of vector space $\mathcal{L}^*(\omega_\alpha)$, for $\alpha = 1, \ldots, s, s+1$, and $\mathcal{L}^*(\omega_\alpha)$ has dimension r_α which is the algebraic multiplicity of ω_α. If $\{\boldsymbol{\chi}_{1,1}, \cdots, \boldsymbol{\chi}_{1,\kappa_1}, \cdots, \boldsymbol{\chi}_{p,1}, \cdots, \boldsymbol{\chi}_{p,\kappa_p}\}$ is a basis of $\mathcal{L}^*(\omega_\alpha)$ for some $|\omega_\alpha| < 1$, and $\kappa_1 + \ldots + \kappa_p = r_\alpha$, then these vectors satisfy equations (3.8).

3.3 General Solution Forms

Let $\{\boldsymbol{\chi}_{1,1}, \cdots, \boldsymbol{\chi}_{1,\kappa_1}, \cdots, \boldsymbol{\chi}_{p,1}, \cdots, \boldsymbol{\chi}_{p,\kappa_p}\}$ be a basis of $\mathcal{L}^*(\omega_o)$ for some $|\omega_o| < 1$. We are now able to explain that

$$\{\mathbf{z}_{i,1}, \cdots, \mathbf{z}_{i,\kappa_i}, \qquad i = 1, \ldots, p\} \tag{3.10}$$

is a canonical set of left Jordan chains for $\mathbf{Q}(\omega)$ corresponding to ω_o. Since $\boldsymbol{\chi}_{i,j} \in \mathcal{L}^*(\omega_o)$ satisfies the unboundary equation (2.5), thus $\boldsymbol{\chi}_{i,j}\mathbf{Q}^* = \mathbf{0}$, where

$$\mathbf{Q}^* = \begin{bmatrix} \mathbf{A}_0 & & & \\ \mathbf{A}_1 & \mathbf{A}_0 & & \\ \mathbf{A}_2 & \mathbf{A}_1 & \ddots & \\ & \mathbf{A}_2 & \ddots & \\ & & \ddots & \end{bmatrix}.$$

Multiplying (3.8) on the right by $\mathbf{Q}^*$, those equations become

$$\begin{aligned} &\mathbf{z}_{i,1}\mathbf{Q}(\omega_o) = \mathbf{0}, \\ &\mathbf{z}_{i,1}\frac{\mathbf{Q}'(\omega_o)}{1!} + \mathbf{z}_{i,2}\mathbf{Q}(\omega_o) = \mathbf{0}, \\ &\vdots \\ &\frac{1}{(\kappa_i-1)!}\mathbf{z}_{i,1}\frac{d^{\kappa_i-1}}{d\omega^{\kappa_i-1}}\mathbf{Q}(\omega)|_{\omega_o} + \frac{1}{(\kappa_i-2)!}\mathbf{z}_{i,1}\frac{d^{\kappa_i-2}}{d\omega^{\kappa_i-2}}\mathbf{Q}(\omega)|_{\omega_o} + \cdots + \mathbf{z}_{i,\kappa_i}\mathbf{Q}(\omega_o) = \mathbf{0}, \end{aligned} \tag{3.11}$$

since

$$e(\omega)\mathbf{Q}^* = \mathbf{A}_0 + \omega\mathbf{A}_1 + \omega^2\mathbf{A}_2 = \mathbf{Q}(\omega).$$

That is, $\{\mathbf{z}_{i,1}, \ldots, \mathbf{z}_{i,\kappa_i}\}$, is a left Jordan chain of length κ_i for the matrix polynomial $\mathbf{Q}(\omega)$ corresponding to the complex number ω_o, for $i = 1, \ldots, p$. Since $\mathbf{z}_{11}, \mathbf{z}_{21}, \ldots, \mathbf{z}_{p1}$ are linearly independent, and $\sum_{i=1}^{p} \kappa_i = r_o$, (3.10) is canonical by Proposition 3.1.3.

Conversely, if there is a set of vectors satisfies (3.11), it is not trivial that we can construct a basis of $\mathcal{L}^*(\omega_o)$ by these vectors since the vectors in a left Jordan chain for a matrix polynomial are not necessarily independent and not uniquely determined. However, we will see later that the basis of $\mathcal{L}^*(\omega_o)$ can be constructed by an arbitrary canonical set of left Jordan chains for the matrix polynomial $\mathbf{Q}(\omega)$ corresponding to ω_o.

Proposition 3.3.1 *Let*

$$\{\boldsymbol{\varphi}_{i1}, \cdots, \boldsymbol{\varphi}_{i,\,\kappa_i}, i = 1, \ldots, p\ \} \tag{3.12}$$

be an arbitrary canonical set of left Jordan chains for $\mathbf{Q}(\omega)$ *corresponding to* ω_o. *Define*

$$\{\boldsymbol{\delta}_{1,1}, \cdots, \boldsymbol{\delta}_{1,\,\kappa_1}, \cdots, \boldsymbol{\delta}_{p,1}, \cdots, \boldsymbol{\delta}_{p,\,\kappa_p}\} \tag{3.13}$$

as

$$\boldsymbol{\delta}_{i,\,j} = \boldsymbol{\varphi}_{i,\,j}\, e(\omega_o) + \frac{1}{1!}\boldsymbol{\varphi}_{i,j-1} e'(\omega_o) + \cdots + \frac{1}{\kappa_i - 1}\boldsymbol{\varphi}_{i,1}\frac{d^{\kappa_i - 1}}{d\omega^{\kappa_i - 1}} e(\omega)|_{\omega=\omega_o}. \tag{3.14}$$

Then $\{\boldsymbol{\delta}_{1,1}, \cdots, \boldsymbol{\delta}_{1,\,\kappa_1}, \cdots, \boldsymbol{\delta}_{p,1}, \cdots, \boldsymbol{\delta}_{p,\,\kappa_p}\}$ *is also a basis of* $\mathcal{L}^*(\omega_o)$.

Proof. Suppose

$$\{\boldsymbol{\chi}_{1,1}, \cdots, \boldsymbol{\chi}_{1,\,\kappa_1}, \cdots, \boldsymbol{\chi}_{p,1}, \cdots, \boldsymbol{\chi}_{p,\,\kappa_p}\} \tag{3.15}$$

is a basis for $\mathcal{L}^*(\omega)$. We claim that (3.13) and (3.15) span the same vector space.

Define the sequences of vectors

$$\boldsymbol{\gamma}_{ij} = (\mathbf{0}, \ldots, \mathbf{0}, \boldsymbol{\varphi}_{i1}, \ldots, \boldsymbol{\varphi}_{ij}), \tag{3.16}$$

and

$$\boldsymbol{\beta}_{ij} = (\mathbf{0}, \ldots, \mathbf{0}, \mathbf{z}_{i1}, \ldots, \mathbf{z}_{ij}), \tag{3.17}$$

for $j = 1, \ldots, \kappa_i$, $i = 1, \ldots, p$ in the same way as (3.3). Then (3.16) and (3.17) are both bases for $\mathcal{N}$ by Proposition 3.1.3. Let

$$\Psi_1 = \begin{bmatrix} \boldsymbol{\gamma}_{11} \\ \vdots \\ \boldsymbol{\gamma}_{p,\kappa_p} \end{bmatrix}$$

and

$$\Psi_2 = \begin{bmatrix} \boldsymbol{\beta}_{11} \\ \vdots \\ \boldsymbol{\beta}_{p,\kappa_p} \end{bmatrix},$$

which imply

$$\begin{bmatrix} \boldsymbol{\delta}_{11} \\ \vdots \\ \boldsymbol{\delta}_{p,\kappa_p} \end{bmatrix} = \Psi_1 \, \widetilde{e}(\omega_o) \quad \text{and} \quad \begin{bmatrix} \boldsymbol{\chi}_{11} \\ \vdots \\ \boldsymbol{\chi}_{p,\kappa_p} \end{bmatrix} = \Psi_2 \, \widetilde{e}(\omega_o)$$

where

$$\widetilde{e}(\omega_o) = \begin{bmatrix} \frac{1}{\ell-1} \frac{d^{\ell-1}}{d\omega^{\ell-1}} e(\omega)|_{\omega=\omega_o} \\ \vdots \\ \frac{1}{2!} e''(\omega_o) \\ \frac{1}{1!} e'(\omega_o) \\ e(\omega_o) \end{bmatrix}$$

and $\ell = \max \{\kappa_1 \ldots, \kappa_p\}$. Note that Ψ_1 and Ψ_2 have the same row space, $\mathcal{N}$. Therefore, the row space of $\Psi_1 \, \widetilde{e}(\omega_o)$ is the same as $\Psi_2 \, \widetilde{e}(\omega_o)$. It follows that (3.13) is also a basis of $\mathcal{L}^*(\omega_o)$. □

Example 3.3.2 *Suppose* ω_1 is a singularity of $\mathbf{Q}(\omega)$ of multiplicity 3, and nullity $\mathbf{Q}(\omega_1) = 2$, $\{\boldsymbol{\varphi}_{11}, \boldsymbol{\varphi}_{12}, \boldsymbol{\varphi}_{21}\}$ satisfying

$$\begin{cases} \boldsymbol{\varphi}_{11}\mathbf{Q}(\omega_1) = \mathbf{0}, \\ \boldsymbol{\varphi}_{21}\mathbf{Q}(\omega_1) = \mathbf{0}, \\ \boldsymbol{\varphi}_{11}\mathbf{Q}'(\omega_1) + \boldsymbol{\varphi}_{12}\mathbf{Q}(\omega_1) = \mathbf{0}, \end{cases}$$

where $\{\boldsymbol{\varphi}_{11}, \boldsymbol{\varphi}_{21}\}$ spans the left null space of $\mathbf{Q}(\omega_1)$. Then $\{\boldsymbol{\varphi}_{11}, \boldsymbol{\varphi}_{12}, \boldsymbol{\varphi}_{21}\}$ is a canonical set of the left Jordan chains for $\mathbf{Q}(\omega)$ corresponding to ω_1, and $\{\boldsymbol{\delta}_{11}, \boldsymbol{\delta}_{12}, \boldsymbol{\delta}_{21}\}$ is a basis for $\mathcal{L}^*(\omega_1)$, where

$$\begin{aligned}
\boldsymbol{\delta}_{11} &= \{\boldsymbol{\varphi}_{11}, \omega\boldsymbol{\varphi}_{11}, \omega^2\boldsymbol{\varphi}_{11}, \omega^3\boldsymbol{\varphi}_{11}, \cdots\}, \\
\boldsymbol{\delta}_{12} &= \{\boldsymbol{\varphi}_{12}, \omega\boldsymbol{\varphi}_{12} + \boldsymbol{\varphi}_{11}, \omega^2\boldsymbol{\varphi}_{12} + 2\omega\boldsymbol{\varphi}_{11}, \omega^3\boldsymbol{\varphi}_{12} + 3\omega^2\boldsymbol{\varphi}_{11}, \cdots\}, \\
\boldsymbol{\delta}_{21} &= \{\boldsymbol{\varphi}_{21}, \omega\boldsymbol{\varphi}_{21}, \omega^2\boldsymbol{\varphi}_{21}, \omega^3\boldsymbol{\varphi}_{21}, \cdots\}.
\end{aligned}$$

□

Theorem 3.3.3 *Let $\omega_1, \ldots, \omega_s, \omega_{s+1}$ be the $s+1$ distinct singularities of $\mathbf{Q}(\omega)$ with multiplicity $r_1, \ldots, r_s, r_{s+1}$ in the open unit disk. Then the probability $\boldsymbol{\pi}_n$ can be expressed as*

$$\boldsymbol{\pi}_n = \sum_{\alpha=1}^{s+1} \boldsymbol{\pi}_n(\omega_\alpha) \qquad \text{for } n \geq 1 \tag{3.18}$$

where

$$\boldsymbol{\pi}_n(\omega_\alpha) = \sum_{i=1}^{p(\alpha)} \sum_{t=1}^{\kappa_i(\alpha)} c_{i,t}\{\omega_\alpha^{n-1}\boldsymbol{\varphi}_{i,t}^{(\alpha)} + \tbinom{n-1}{1}\omega_\alpha^{n-2}\boldsymbol{\varphi}_{i,t-1}^{(\alpha)} + \cdots + \tbinom{n-1}{t-1}\omega_\alpha^{n-t}\boldsymbol{\varphi}_{i,1}^{(\alpha)}\},$$

and for $\alpha = 1, \ldots, s+1$, $\{\boldsymbol{\varphi}_{1,1}^{(\alpha)}, \ldots, \boldsymbol{\varphi}_{p(\alpha),\kappa_\alpha(\alpha)}^{(\alpha)}\}$ is an arbitrary canonical set of left Jordan chains for $\mathbf{Q}(\omega)$ corresponding to ω_α.

Proof. The result is obvious by (3.9) and Proposition 3.3.1. □

Example 3.3.4 *In* example 3.3.2, we see that

$$\begin{aligned}
\boldsymbol{\pi}_1(\omega_1) &= c_1\boldsymbol{\varphi}_{11} + c_2\boldsymbol{\varphi}_{12} + c_3\boldsymbol{\varphi}_{21}, \\
\boldsymbol{\pi}_2(\omega_1) &= c_1\omega_1\boldsymbol{\varphi}_{11} + c_2(\omega_1\boldsymbol{\varphi}_{12} + \boldsymbol{\varphi}_{11}) + c_3\omega_1\boldsymbol{\varphi}_{21}, \\
&\vdots \\
\boldsymbol{\pi}_n(\omega_1) &= c_1\omega_1^{n-1}\boldsymbol{\varphi}_{11} + c_2(\omega_1^{n-1}\boldsymbol{\varphi}_{12} + (n-1)\omega_1^{n-2}\boldsymbol{\varphi}_{11}) + c_3\omega_1^{n-1}\boldsymbol{\varphi}_{21}.
\end{aligned}$$

□

In the following Proposition, we will see that the multiplicity of 0 is at least $mk - m$.

Proposition 3.3.5 *nullity*$\mathbf{Q}(0) = mk - m$.

Proof. Since

$$\mathbf{Q}(0) = \mathbf{A}_0 = \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 \otimes \mathbf{I}_2 = \begin{bmatrix} \boldsymbol{\Gamma}_1 & 0 & \cdots & 0 \\ \boldsymbol{\Gamma}_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\Gamma}_k & 0 & \cdots & 0 \end{bmatrix}$$

where

$$\boldsymbol{\Gamma}_i = \begin{bmatrix} p_i\lambda_i & & & \\ & p_i\lambda_i & & \\ & & \ddots & \\ & & & p_i\lambda_i \end{bmatrix}$$

is a diagonal matrix of dimension m for $i = 1, \ldots, k$, and rank$\mathbf{Q}(0) = m$ that implies nullity$\mathbf{Q}(0) = mk - m$. $\square$

Thus, 0 is a singularity of $\mathbf{Q}(\omega)$. Without loss of generality, we assume that $\omega_{s+1} = 0$ and $\omega_1, \ldots, \omega_s$ are the distinct nonzero singularities of $\mathbf{Q}(\omega)$. From Proposition 3.3.5, it is clear that $r_{s+1} \geq mk - m$. Recall there are exactly mk singularities of $\mathbf{Q}(\omega)$ in the open unit disk which implies

$$\sum_{\alpha=1}^{s+1} dim\mathcal{L}^*(\omega_\alpha) = \sum_{\alpha=1}^{s+1} r_\alpha = mk$$

by Theorem 3.2.4 and Proposition 3.2.3. Therefore, we obtain the following result.

Corollary 3.3.6 *The sum of dimension of the generalized eigenspaces is equivalent to the total number of nonzero singularities of* $\mathbf{Q}(\omega)$ *which is at most* m*, i.e.,*

$$\sum_{\alpha=1}^{s} dim\mathcal{L}^*(\omega_\alpha) = \sum_{\alpha=1}^{s} r_\alpha \leq m.$$

Since $r_\alpha \geq 1$, we know $s \leq m$. It is clear that if we are able to find vectors in $\mathcal{L}^*(\omega_\alpha)$ such that $\sum_{\alpha=1}^{s} dim\mathcal{L}^*(\omega_\alpha) = m$, then it is sufficient to construct a solution space $\mathcal{L}$.

It is easy to see that if nullity$\mathbf{Q}(\omega_{s+1}) = r_{s+1}$, then

$$\boldsymbol{\pi}_n(\omega_{s+1}) = \mathbf{0}$$

for $n \geq 2$.

Corollary 3.3.7 *If nullity* $\mathbf{Q}(\omega_\alpha) = r_\alpha$ *for* $\alpha = 1, \ldots, s+1$, *then the saturated probabilities* $\boldsymbol{\pi}_n$ *for* $n \geq 2$ *can be expressed as the linear combination of the vectors in the left null space of* $\mathbf{Q}(\omega)$, *i.e.,*

$$\boldsymbol{\pi}_n = \sum_{\alpha=1}^{s} \sum_{t=1}^{r_\alpha} c_{\alpha,t} \omega_\alpha^{n-1} \boldsymbol{\varphi}_t^{(\alpha)}, \qquad \text{for } n \geq 2 \tag{3.19}$$

where $\boldsymbol{\varphi}_t^{(\alpha)}$ *is contained in the left null space of* $\mathbf{Q}(\omega_\alpha)$ *for* $t = 1, \ldots, r_\alpha$, $\alpha = 1, \ldots, s$, *and* $c_{\alpha,t}$ *is the coefficient with respect to* $\boldsymbol{\varphi}_t^{(\alpha)}$.

It has been shown that solution space can be described in terms of singularities and vectors of $\mathbf{Q}(\omega)$. We will discuss the nonzero roots of $det\mathbf{Q}(\omega) = 0$ in the next chapter.