Chapter 4

Singularities of $\mathbf{Q}(\omega)$ in the Open Unit Disk

We will present in this chapter that there is a close connection between the singularity of $\mathbf{Q}(\omega)$ and an equation involving the Laplace transforms. Define

$$f^*_{Ta}(x) f^*_{Ts}(-x) = 1. \tag{4.1}$$

We shall show that all the roots of (4.1) are simple in the $E_k/E_m/1$ system.

4.1 $\mathbf{Q}(\omega)$ and Laplace Transform Equation

In this section, our goal is to find the condition such that $\mathbf{Q}(\omega) = \mathbf{a}(\omega) \oplus \mathbf{b}(\omega)$ has eigenvalue 0. Since the eigenvalue of Kronecker sum of $\mathbf{a}(\omega) \oplus \mathbf{b}(\omega)$ is the sum of the eigenvalues of $\mathbf{a}(\omega)$ and $\mathbf{b}(\omega)$ respectively, (see Theorem **4.4.5** in Chapter **4.4** of [8]) $det\,\mathbf{Q}(\omega) = 0$ if and only if there exists θ such that $det\,(\mathbf{a}(\omega) - \theta \mathbf{I}_1) = 0$ and $det\,(\mathbf{b}(\omega) + \theta \mathbf{I}_2) = 0$.

Lemma 4.1.1 *Let $\omega \neq 0$. we shall show*

$$(1)\quad det(\tfrac{1}{\omega}\mathbf{a}(\omega) - x\mathbf{I}_1) = (1 - \tfrac{1}{\omega} f^*_{Ta}(x)) \textstyle\prod_{i=1}^{k}(-\lambda_i - x),$$

$$(2)\quad det(\tfrac{1}{\omega}\mathbf{b}(\omega) + x\mathbf{I}_2) = (1 - \omega f^*_{Ts}(-x)) \textstyle\prod_{i=1}^{m}(x - \mu_i).$$

Proof.

(1) Let U_n be the upper left $n \times n$ block of $\frac{1}{\omega}\boldsymbol{\gamma}_1\boldsymbol{\beta}_1 + \mathbf{S}_1 - x\mathbf{I}_1$, and it gives

$$U_k = \begin{bmatrix} \frac{1}{\omega}p_1\lambda_1 - \lambda_1 - x & (1-p_1)\lambda_1 & 0 & \cdots & 0 \\ \frac{1}{\omega}p_2\lambda_2 & -\lambda_2 - x & (1-p_2)\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{1}{\omega}p_{k-1}\lambda_{k-1} & 0 & 0 & -\lambda_{k-1} - x & (1-p_{k-1})\lambda_{k-1} \\ \frac{1}{\omega}\lambda_k & 0 & 0 & 0 & -\lambda_k - x \end{bmatrix}.$$

It is clear that for $k = 1$

$$\begin{aligned} det(U_1) &= \frac{1}{\omega}p_1\lambda_1 - \lambda_1 - x \\ &= (-\lambda_1 - x)\left(1 - \frac{p_1\lambda_1}{\omega(\lambda_1 + x)}\right). \end{aligned}$$

For $k = n$, expending along the n-th row, we obtain

$$det(U_n) = (-\lambda_n - x)det(U_{n-1}) - (-1)^n \frac{p_n\lambda_n}{\omega} \prod_{i=1}^{n-1}(1 - p_i)\lambda_i, \quad \text{for } n \geq 2. \tag{4.2}$$

Having (4.2) divided by $\prod_{i=1}^{n}(-\lambda_i - x)$, we get

$$\begin{aligned} s_n &= s_{n-1} - (-1)^n \frac{p_n\lambda_n}{\omega(-\lambda_n - x)} \prod_{i=1}^{n-1} \frac{(1-p_i)\lambda_i}{(-\lambda_i - x)} \\ &= s_{n-1} - \frac{p_n\lambda_n}{\omega(\lambda_n + x)} \prod_{i=1}^{n-1} \frac{(1-p_i)\lambda_i}{(\lambda_i + x)} \end{aligned}$$

where

$$s_n \triangleq \frac{det(U_n)}{\prod_{i=1}^{n}(-\lambda_i - x)}.$$

Thus, we have

$$\begin{aligned} s_k &= s_1 - \sum_{i=2}^{k} \frac{p_i\lambda_i}{\omega(\lambda_i + x)} \prod_{j=1}^{i-1} \frac{(1-p_j)\lambda_j}{(\lambda_j + x)} \\ &= \left(1 - \frac{p_1\lambda_1}{\omega(\lambda_1 + x)}\right) - \sum_{i=2}^{k} \frac{p_i\lambda_i}{\omega(\lambda_i + x)} \prod_{j=1}^{i-1} \frac{(1-p_j)\lambda_j}{(\lambda_j + x)} \\ &= 1 - \frac{1}{\omega} \sum_{i=1}^{k} \frac{p_i\lambda_i}{(\lambda_i + x)} \prod_{j=1}^{i-1} \frac{(1-p_j)\lambda_j}{(\lambda_j + x)}, \end{aligned}$$

and the theorem follows directly by $det(U_k) = s_k \prod_{i=1}^{k}(-\lambda_i - x)$.

The proof of (2) is similar. □

Theorem 4.1.2 *Let $\omega \neq 0$, $det(\mathbf{a}(\omega) \oplus \mathbf{b}(\omega)) = 0$ if and only if $\omega = f^*_{Ta}(x)$ where x satisfies (4.1).*

Proof. Since

$$det(\mathbf{a}(\omega) \oplus \mathbf{b}(\omega)) = \omega^{mk} det(\frac{1}{\omega}\mathbf{a}(\omega) \oplus \frac{1}{\omega}\mathbf{b}(\omega)) = 0$$

if and only if there exists x such that

$$\begin{cases} det(\frac{1}{\omega}\mathbf{a}(\omega) - x\mathbf{I}_1) = 0 \\ det(\frac{1}{\omega}\mathbf{b}(\omega) + x\mathbf{I}_2) = 0 \end{cases}. \tag{4.3}$$

For $x \neq -\lambda_i, \mu_i$ for all i, then (4.3) holds if and only if $\omega = f^*_{Ta}(x)$ where x satisfies $f^*_{Ta}(x) f^*_{Ts}(-x) = 1$ by Lemma 4.1.1. To show that this theorem is valid for all values of x, for $x = -\lambda_i$ or μ_i, one needs to consider $\mathbf{S}_1 - \varepsilon\mathbf{I}_1$ and $\mathbf{S}_2 + \varepsilon\mathbf{I}_2$ in place of $\mathbf{S}_1$ and $\mathbf{S}_2$ with a small constant ε. Then

$$\begin{cases} det(\frac{1}{\omega}\mathbf{a}(\omega) - (x+\varepsilon)\mathbf{I}_1) = 0 \\ det(\frac{1}{\omega}\mathbf{b}(\omega) + (x+\varepsilon)\mathbf{I}_2) = 0 \end{cases}$$

if and only if

$$f^*_{Ta}(x+\varepsilon) f^*_{Ts}(-x-\varepsilon) = 0.$$

Let $\varepsilon \to 0$ and by continuity argument, the theorem is proved. □

Thus, for every *nonzero* singularity of $\mathbf{Q}(\omega)$, there exists a value x such that $\omega = f^*_{Ta}(x)$ where x satisfies (4.1). Therefore, we solve equation (4.1) in place of computing $det\mathbf{Q}(\omega) = 0$. When the roots x with positive real part then

$$\begin{aligned} |\omega| = |f^*_{Ta}(x)| &= |\int_0^\infty e^{-xs} f_{Ta}(s)ds| \\ &\leq \int_0^\infty |e^{-xs}| f_{Ta}(s)ds \\ &= \int_0^\infty e^{Re\{-x\}s} f_{Ta}(s)ds \\ &< \int_0^\infty f_{Ta}(s)ds = 1. \end{aligned}$$

When x has negative real part, we have

$$|\omega| = |f^*_{Ta}(x)| > 1,$$

since $|f^*_{Ts}(-x)| < 1$ and $f^*_{Ta}(x) f^*_{Ts}(-x) = 1$. Here we are only interested in the root x with positive real part, so that $\boldsymbol{\pi}_n$ can be normalised to become of the probability distribution. It has been proved in [10] that if $\rho < 1$ then (4.1) has exactly m roots with positive real part.

4.2 A Special Case of Simple Roots

In this section, we discuss the property of multiplicity for the roots of (4.1) and show that, in $E_k/E_m/1$ system, the multiple roots of (4.1) only occur at $x = 0$.

We assume that the arrival rate λ and service rate μ are positive real numbers. The Laplace transforms of the probability density functions of the interarrival and service times are

$$f^*_{Ta}(x) = \left(\frac{\lambda}{x+\lambda}\right)^k$$

and

$$f^*_{Ts}(x) = \left(\frac{\mu}{x+\mu}\right)^m.$$

Theorem 4.2.1 *The nonzero roots of*

$$\left(\frac{\lambda}{x+\lambda}\right)^k \left(\frac{\mu}{\mu-x}\right)^m = 1$$

are simple.

Before proving this result, we first recall the Jensen's inequality theorem.

Theorem 4.2.2 *(***Jensen's inequality***) Let ϕ be convex over real line (a,b). Let $\{x_j\}_{j=1}^N$ be points of (a,b) and $\{p_j\}_{j=1}^N$ satisfy $p_j \geq 0$ and $\sum p_j > 0$. Then we have*

$$\phi\left(\frac{\sum p_j x_j}{\sum p_j}\right) \leq \frac{\sum p_j \phi(x_j)}{\sum p_j}.$$

Proof of Theorem 4.2.1:

It is sufficient to consider the function

$$g(x) \triangleq \lambda^k \mu^m - (x+\lambda)^k(\mu-x)^m.$$

Since

$$g'(x) = (\lambda+x)^{k-1}(\mu-x)^{m-1}[(m+k)x - (k\mu - m\lambda)],$$

it follows that the possible multiple roots only occur at $x = -\lambda, \mu$, and $\frac{k\mu-m\lambda}{m+k}$. Obviously, $g(-\lambda) \neq 0$ and $g(\mu) \neq 0$. We only need to consider $\frac{k\mu-m\lambda}{m+k}$ and denote it by x_0.

Let

$$t \triangleq \frac{m}{k}.$$

Then t is a positive real number and

$$g(x_0) = (\lambda\mu^t)^k - \left[\left(\frac{\lambda+\mu}{1+t}\right)^{1+t} t^t\right]^k.$$

Since λ, μ, and t are real, $g(x_0) = 0$ leads to

$$\lambda\mu^t = \left(\frac{\lambda+\mu}{1+t}\right)^{1+t} t^t.$$

That is, $g(x_0) = 0$ if and only if λ, μ, and t satisfy the equation

$$\lambda \left(\frac{\mu}{t}\right)^t = \left(\frac{1+\mu}{1+t}\right)^{1+t}. \tag{4.4}$$

In the following argument, we use the *Jensen's inequality* to find all positive real numbers λ, μ, and t such that (4.4) holds.

Given $t > 0$, let

$$\phi(x) = e^{(t+1)x}.$$

Because of

$$\phi'(x) = (t+1)e^{(t+1)x} > 0$$

and

$$\phi''(x) = (t+1)^2 e^{(t+1)x} > 0,$$

it follows that $\phi(x)$ is convex over $(-\infty, \infty)$. Let $x_1 = \ln \lambda^{\frac{1}{t+1}}$, $x_2 = \ln\left(\frac{\mu}{t}\right)^{\frac{1}{t+1}}$, $p_1 = \frac{1}{t+1}$, and $p_2 = \frac{t}{t+1}$. Jensen's inequality theorem implies

$$\phi\left(\frac{p_1 x_1 + p_2 x_2}{p_1 + p_2}\right) \le \frac{p_1\phi(x_1) + p_2\phi(x_2)}{p_1 + p_2}. \tag{4.5}$$

$$\begin{aligned}
\text{Left of (4.5)} &= \phi\left(\frac{1}{t+1}\ln\lambda^{\frac{1}{t+1}} + \frac{t}{t+1}\ln\left(\frac{\mu}{t}\right)^{\frac{1}{t+1}}\right) \\
&= e^{\ln\lambda^{\frac{1}{t+1}} + t\,\ln\left(\frac{\mu}{t}\right)^{\frac{1}{t+1}}} \\
&= e^{\ln\lambda^{\frac{1}{t+1}}} \cdot e^{\ln\left(\frac{\mu}{t}\right)^{\frac{t}{t+1}}} \\
&= \left[\lambda\left(\frac{\mu}{t}\right)^t\right]^{\frac{1}{t+1}}.
\end{aligned}$$

$$\begin{aligned}
\text{Right of (4.5)} &= \frac{1}{t+1}e^{(1+t)\ln\lambda^{\frac{1}{t+1}}} + \frac{t}{t+1}e^{(t+1)\ln\left(\frac{\mu}{t}\right)^{\frac{1}{t+1}}} \\
&= \frac{1}{t+1}e^{\ln\lambda} + \frac{t}{t+1}e^{\ln\left(\frac{\mu}{t}\right)} \\
&= \frac{1}{t+1}\left[\lambda + t\left(\frac{\mu}{t}\right)\right] \\
&= \frac{1}{t+1}(\lambda + \mu).
\end{aligned}$$

We obtain

$$\lambda \left(\frac{\mu}{t}\right)^t \leq \left(\frac{\lambda + \mu}{t+1}\right)^{t+1}.$$

Since $\phi(x)$ is convex, the equality holds in (4.5) if and only if $x_1 = x_2$. Hence, it yields

$$\lambda \left(\frac{\mu}{t}\right)^t = \left(\frac{\lambda + \mu}{t+1}\right)^{t+1}$$

if and only if $\frac{\mu}{t} = \lambda$, i.e., $x_0 = 0$. □