Chapter 5

A Method of Constructing Solution Spaces

It was observed in Theorem 3.3.3 that the vectors used in the expression of the saturated probability are described by canonical sets of the left Jordan chains for $\mathbf{Q}(\omega)$. In addition to this, from the preceding chapter, we see that the singularities of $\mathbf{Q}(\omega)$ has a close connection to the roots of (4.1). In this chapter, we want to find those vectors used in the expression of saturated probabilities. If we are able to find some vectors such that $\sum_{\alpha=1}^{s} dim\mathcal{L}(\omega_\alpha)$ is equal to m, then by Corollary 3.3.6 these vectors are sufficient to construct the solution space for the saturated probabilities.

5.1 Cases of Simple roots

If the m roots, $x_1, x_2, \ldots, x_m$, of (4.1) with positive real part are distinct and $f_{Ta}^*(x_i) \neq f_{Ta}^*(x_j)$ for each $i \neq j$, then according to Theorem 4.1.2, we set $\omega_\alpha =$

$f^*_{Ta}(x_\alpha)$ for $\alpha = 1, \ldots, m$. Given x_α, define $\mathbf{u}_1^{(\alpha)}$ and $\mathbf{v}_1^{(\alpha)}$ as follows (See Wang [10]),

$$\mathbf{u}_1^{(\alpha)} = a_1 \boldsymbol{\beta}_1 (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-1}, \tag{5.1}$$

$$\mathbf{v}_1^{(\alpha)} = a_2 \boldsymbol{\beta}_2 (\mathbf{S}_2 + x_\alpha \mathbf{I}_2)^{-1} \tag{5.2}$$

where a_1, a_2 are constants such that $\mathbf{u}_1^{(\alpha)} \mathbf{1} = \mathbf{v}_1^{(\alpha)} \mathbf{1} = 1$. Simply, set

$$a_1 = \tfrac{x_\alpha}{\omega_\alpha - 1}, \quad a_2 = \tfrac{x_\alpha \omega_\alpha}{\omega_\alpha - 1}.$$

We will show that $\mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)}$ forms the left null space of $\mathbf{Q}(\omega_\alpha)$ for $\alpha = 1, \ldots, m$.

Lemma 5.1.1 *If* $\mathbf{u}_1 \neq \mathbf{0}$, $\mathbf{v}_1 \neq \mathbf{0}$ *, and* $\mathbf{u}_2 \otimes \mathbf{v}_1 + \mathbf{u}_1 \otimes \mathbf{v}_2 = \mathbf{0}$ *, then* $\mathbf{u}_2 = c\,\mathbf{u}_1$ *and* $\mathbf{v}_2 = -c\,\mathbf{v}_1$ *for some constant c.*

Proof. Suppose $\mathbf{u}_i = (u_{i1}, u_{i2}, \ldots, u_{ik})$ and $\mathbf{v}_i = (v_{i1}, v_{i2}, \ldots, v_{im})$ for $i = 1, 2$, and

$$\mathbf{u}_2 \otimes \mathbf{v}_1 = -\mathbf{u}_1 \otimes \mathbf{v}_2,$$

i.e.,

$$(u_{21}\mathbf{v}_1, u_{22}\mathbf{v}_1, \ldots, u_{2k}\mathbf{v}_1) = -(u_{11}\mathbf{v}_2, u_{12}\mathbf{v}_2, \ldots, u_{1k}\mathbf{v}_2).$$

Without loss of generality, assume $\mathbf{u}_1 = (u_{11}, \ldots, u_{1j}, 0, \ldots, 0)$ where $u_{11}, \ldots, u_{1j}$ are not zeros, we have

$$\begin{aligned} \mathbf{v}_2 &= \frac{-u_{21}}{u_{11}} \mathbf{v}_1, \\ \mathbf{v}_2 &= \frac{-u_{22}}{u_{12}} \mathbf{v}_1, \\ &\vdots \\ \mathbf{v}_2 &= \frac{-u_{2j}}{u_{1j}} \mathbf{v}_1, \end{aligned}$$

$$u_{2,j+1} = u_{2,j+2} = \ldots = u_{2,k} = 0,$$

and set

$$c \triangleq \frac{u_{21}}{u_{11}} = \frac{u_{22}}{u_{12}} = \cdots = \frac{u_{2j}}{u_{1j}}.$$

Hence, we obtain

$$\mathbf{u}_2 = c\,\mathbf{u}_1 \quad \text{and} \quad \mathbf{v}_2 = -c\,\mathbf{v}_1. \tag{5.3}$$

$\square$

Theorem 5.1.2 *Let $\omega \neq 0$ and $\boldsymbol{\varphi} = \mathbf{u} \otimes \mathbf{v} \neq \mathbf{0}$. Then $\boldsymbol{\varphi}\mathbf{Q}(\omega) = \mathbf{0}$ if and only if $\mathbf{u}$ is the left eigenvector of $\mathbf{a}(\omega)$ corresponding to eigenvalue $x\omega$ and $\mathbf{v}$ is the left eigenvector of $\mathbf{b}(\omega)$ corresponding to eigenvalue $-x\omega$, where x is the root of (4.1) and $\omega = f^*_{Ta}(x)$.*

Proof. Suppose $(\mathbf{u} \otimes \mathbf{v})\mathbf{Q}(\omega) = 0$ or $\mathbf{u}\,\mathbf{a}(\omega) \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{v}\,\mathbf{b}(\omega) = \mathbf{0}$. Then there exists c such that

$$\mathbf{u}\,\mathbf{a}(\omega) = c\mathbf{u}$$

and

$$\mathbf{v}\,\mathbf{b}(\omega) = -c\mathbf{v},$$

i.e.,

$$\mathbf{u}(\mathbf{a}(\omega) - c\mathbf{I}_1) = \mathbf{0},$$

and

$$\mathbf{v}(\mathbf{b}(\omega) + c\mathbf{I}_2) = \mathbf{0}.$$

From Theorem 4.1.2, we see $det(\mathbf{a}(\omega) - c\mathbf{I}_1) = det(\mathbf{b}(\omega) + c\mathbf{I}_2) = 0$ if and only if $c = x\omega$ where $\omega = f^*_{Ta}(x)$ and x satisfies (4.1). Hence, the theorem holds true. $\square$

Theorem 5.1.3 *$\mathbf{u}_1^{(\alpha)}$ and $\mathbf{v}_1^{(\alpha)}$ are the left eigenvectors of $\mathbf{a}(\omega_\alpha)$ and $\mathbf{b}(\omega_\alpha)$ corresponding to the eigenvalues $x_\alpha\omega_\alpha$ and $-x_\alpha\omega_\alpha$ respectively.*

Proof. It is easy to verify

$$\begin{aligned}
& \mathbf{u}_1^{(\alpha)}(\omega_\alpha\mathbf{S}_1 + \boldsymbol{\gamma}_1\boldsymbol{\beta}_1 - x_\alpha\omega_\alpha\mathbf{I}_1) \\
= \;& a_1\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\{\omega_\alpha(\mathbf{S}_1 - x_\alpha\mathbf{I}_1) + \boldsymbol{\gamma}_1\boldsymbol{\beta}_1\} \\
= \;& a_1\{\omega_\alpha\boldsymbol{\beta}_1 + \boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1\boldsymbol{\beta}_1\} \\
= \;& a_1(\omega_\alpha\boldsymbol{\beta}_1 - \omega_\alpha\boldsymbol{\beta}_1) \\
= \;& \mathbf{0}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{v}_1^{(\alpha)}(\omega_\alpha \mathbf{S}_2 + \omega_\alpha^2 \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 + x_\alpha \omega_\alpha \mathbf{I}_2) \\
= \; & a_2 \boldsymbol{\beta}_2 (\mathbf{S}_2 + x_\alpha \mathbf{I}_2)^{-1} \{\omega_\alpha (\mathbf{S}_2 + x_\alpha \mathbf{I}_2) + \omega_\alpha^2 \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2\} \\
= \; & a_2 \{\omega_\alpha \boldsymbol{\beta}_2 + \omega_\alpha^2 \boldsymbol{\beta}_2 (\mathbf{S}_2 + x_\alpha \mathbf{I}_2)^{-1} \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2\} \\
= \; & a_2 (\omega_\alpha \boldsymbol{\beta}_2 - \omega_\alpha^2 \frac{1}{\omega_\alpha} \boldsymbol{\beta}_2) \\
= \; & \mathbf{0}.
\end{aligned}$$

□

From Theorems 5.1.2 and 5.1.3, we conclude that $\mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)}$ is in the left null space of $\mathbf{Q}(\omega_\alpha)$ for $\alpha = 1, \ldots, m$. From Theorem 3.3.3, we know that $(\boldsymbol{\varphi}_1^{(\alpha)}, \omega_\alpha \boldsymbol{\varphi}_1^{(\alpha)}, \omega_\alpha^2 \boldsymbol{\varphi}_1^{(\alpha)}, \ldots)$ is contained in $\mathcal{L}^*(\omega_\alpha)$ where $\boldsymbol{\varphi}_1^{(\alpha)} = \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)}$ for $\alpha = 1, \ldots, m$, and it follows that $\sum_{\alpha=1}^m dim \mathcal{L}^*(\omega_\alpha) = m$.

Suppose $\omega_1 = f_{Ta}^*(x_1) = f_{Ta}^*(x_2) = \cdots = f_{Ta}^*(x_d)$. Then

$$(\boldsymbol{\varphi}_1^{(\alpha)}, \omega_1 \boldsymbol{\varphi}_1^{(\alpha)}, \omega_1^2 \boldsymbol{\varphi}_1^{(\alpha)}, \ldots)$$

is contained in $\mathcal{L}^*(\omega_1)$ where $\mathbf{u}_1^{(\alpha)}$ and $\mathbf{v}_1^{(\alpha)}$ are defined in (5.1) and (5.2) for $\alpha = 1, \ldots, d$. It follows that $dim \mathcal{L}^*(\omega_1) \geq d$ by the fact $\mathbf{u}_1^{(1)} \otimes \mathbf{v}_1^{(1)}, \ldots, \mathbf{u}_1^{(d)} \otimes \mathbf{v}_1^{(d)}$ are linearly independent. It is because $\mathbf{u}_1^{(\alpha)}$ (resp. $\mathbf{v}_1^{(\alpha)}$) is the eigenvector of $\mathbf{a}(\omega_1)$ (resp. $\mathbf{b}(\omega_1)$) corresponding to $x_\alpha \omega_1$ (resp. $-x_\alpha \omega_1$) for $\alpha = 1, \ldots, d$. As a result, we have $dim \mathcal{L}^*(\omega_1) = d$ from Corollary 3.3.6.

In this section, we conclude that if the m roots of (4.1) are distinct, the saturated probabilities for $n \geq 2$ can be expressed as:

$$\boldsymbol{\pi}_n = \sum_{\alpha=1}^{m} c_\alpha \omega_\alpha^{n-1} \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)}. \tag{5.4}$$

Equality (5.4) is verified by comparing with the results in [4] and [10].

5.2 Cases of Multiple Roots

In this section, we discuss the situation when multiple roots occur in (4.1). Denote the multiplicities of x_α by ℓ_α for $\alpha = 1, \ldots, s$, $s \leq m$. Note that $\sum_{\alpha=1}^{s} \ell_\alpha = m$ when $\rho < 1$. We consider the following equations to find the vectors used in the expression of the saturated probabilities.

$$\begin{aligned}
&\boldsymbol{\varphi}_1^{(\alpha)}\mathbf{Q}(\omega_\alpha) = \mathbf{0}, \\
&\boldsymbol{\varphi}_1^{(\alpha)}\mathbf{Q}'(\omega_\alpha) + \boldsymbol{\varphi}_2^{(\alpha)}\mathbf{Q}(\omega_\alpha) = \mathbf{0}, \\
&\tfrac{1}{2!}\boldsymbol{\varphi}_1^{(\alpha)}\mathbf{Q}''(\omega_\alpha) + \boldsymbol{\varphi}_2^{(\alpha)}\mathbf{Q}'(\omega_\alpha) + \boldsymbol{\varphi}_3^{(\alpha)}\mathbf{Q}(\omega_\alpha) = \mathbf{0}, \\
&\qquad\vdots \\
&\tfrac{1}{(\ell_\alpha-1)!}\boldsymbol{\varphi}_1^{(\alpha)}\tfrac{d^{\ell_\alpha-1}}{d\omega^{\ell_\alpha-1}}\mathbf{Q}(\omega)|_{\omega=\omega_\alpha} + \cdots + \boldsymbol{\varphi}_{\ell_\alpha}^{(\alpha)}\mathbf{Q}(\omega_\alpha) = \mathbf{0}.
\end{aligned} \tag{5.5}$$

In the next section, we give several examples to explain how to find the vectors used in the expression of the saturated probability when multiple roots occur.

5.3 Examples of $\ell_\alpha \leq 4$

We will describe those vectors taken in the expression of the saturated probabilities as the linear combination of product-forms if the multiplicity of x_α is not exceeding 4 for $\alpha = 1, \ldots, s$, $s \leq m$.

If x_α is the root of (4.1) then we have

$$(\mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)})\mathbf{Q}(\omega_\alpha) = \mathbf{0},$$

where $\mathbf{u}_1^{(\alpha)}$, $\mathbf{v}_1^{(\alpha)}$ are defined in (5.1), (5.2) and

$$\mathbf{u}_1^{(\alpha)}\mathbf{a}(\omega_\alpha) = x_\alpha\omega_\alpha\mathbf{u}_1^{(\alpha)}, \tag{5.6}$$

$$\mathbf{v}_1^{(\alpha)}\mathbf{b}(\omega_\alpha) = -x_\alpha\omega_\alpha\mathbf{v}_1^{(\alpha)}. \tag{5.7}$$

Multiplying (5.1), (5.2) by $(\mathbf{S}_1 - x_\alpha \mathbf{I}_1)$, $(\mathbf{S}_2 + x_\alpha \mathbf{I}_2)$ respectively, it is easy to derive

$$\begin{aligned} \mathbf{v}_1^{(\alpha)} \boldsymbol{\gamma}_2 \boldsymbol{\beta}_2 &= \frac{-a_2}{\omega_\alpha} \boldsymbol{\beta}_2, &(5.8)\\ \mathbf{u}_1^{(\alpha)} \mathbf{a}'(\omega_\alpha) &= a_1 \boldsymbol{\beta}_1 + x_\alpha \mathbf{u}_1^{(\alpha)}, &(5.9)\\ \mathbf{v}_1^{(\alpha)} \mathbf{b}'(\omega_\alpha) &= -a_2 \boldsymbol{\beta}_2 - x_\alpha \mathbf{v}_1^{(\alpha)}. &(5.10) \end{aligned}$$

If x_α is a roots of (4.1) with multiplicity 2, then

$$\frac{d}{dx}\{f_{Ta}^*(x) f_{Ts}^*(-x) - 1\}|_{x=x_\alpha} = 0$$

or

$$\begin{aligned} &\boldsymbol{\beta}_1 (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-2} \boldsymbol{\gamma}_1 \boldsymbol{\beta}_2 (\mathbf{S}_2 + x_\alpha \mathbf{I}_2)^{-1} \boldsymbol{\gamma}_2 \\ &-\boldsymbol{\beta}_1 (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-1} \boldsymbol{\gamma}_1 \boldsymbol{\beta}_2 (\mathbf{S}_2 + x_\alpha \mathbf{I}_2)^{-2} \boldsymbol{\gamma}_2 = 0. \end{aligned} \quad (5.11)$$

Equation (5.11) can be divided into two cases.

Case 1: $\boldsymbol{\beta}_1 (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-2} \boldsymbol{\gamma}_1 = 0.$

Case 2: $\boldsymbol{\beta}_1 (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-2} \boldsymbol{\gamma}_1 \neq 0.$

In case 1, obviously, if $\boldsymbol{\beta}_1 (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-2} \boldsymbol{\gamma}_1 = 0$, then $\boldsymbol{\beta}_2 (\mathbf{S}_2 + x_\alpha \mathbf{I}_2)^{-2} \boldsymbol{\gamma}_2 = 0$ follows from (5.11). If $\boldsymbol{\beta}_1 (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-2} \boldsymbol{\gamma}_1 \neq 0$, it is easy to verify $\boldsymbol{\beta}_2 (\mathbf{S}_2 + x_\alpha \mathbf{I}_2)^{-2} \boldsymbol{\gamma}_2 \neq 0$ by (5.11).

Theorem 5.3.1 *If x_α is a root of (4.1) with multiplicity 2, and $\boldsymbol{\beta}_1 (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-2} \boldsymbol{\gamma}_1 = 0$, then*

$$(\mathbf{u}_0^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_0^{(\alpha)}) \mathbf{Q}(\omega_\alpha) = \mathbf{0} \quad (5.12)$$

where

$$\mathbf{u}_0^{(\alpha)} = \frac{1}{\omega_\alpha} \mathbf{u}_1^{(\alpha)} (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-1}, \quad (5.13)$$

$$\mathbf{v}_0^{(\alpha)} = \frac{1}{\omega_\alpha} \mathbf{v}_1^{(\alpha)} (\mathbf{S}_2 + x_\alpha \mathbf{I}_2)^{-1}, \quad (5.14)$$

and $\mathbf{u}_0^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_0^{(\alpha)}$, $\mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)}$ *are linearly independent.*

Proof. Since

$$\begin{aligned}
\mathbf{u}_0^{(\alpha)} \left(\mathbf{a}(\omega_\alpha) - x_\alpha \omega_\alpha \mathbf{I}_1\right) &= \frac{1}{\omega_\alpha} \mathbf{u}_1^{(\alpha)} (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-1} (\omega_\alpha \mathbf{S}_1 + \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 - x_\alpha \omega_\alpha \mathbf{I}_1) \\
&= \frac{1}{\omega_\alpha} a_1 \boldsymbol{\beta}_1 (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-2} (\omega_\alpha \mathbf{S}_1 + \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 - x_\alpha \omega_\alpha \mathbf{I}_1) \\
&= \frac{1}{\omega_\alpha} a_1 \boldsymbol{\beta}_1 (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-2} \boldsymbol{\gamma}_1 \boldsymbol{\beta}_1 + a_1 \boldsymbol{\beta}_1 (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-1} \\
&= \mathbf{0} + a_1 \boldsymbol{\beta}_1 (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-1} \\
&= \mathbf{u}_1^{(\alpha)},
\end{aligned}$$

we obtain

$$\begin{aligned}
\left(\mathbf{u}_0^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)}\right) \left(\mathbf{a}(\omega_\alpha) \oplus \mathbf{b}(\omega_\alpha)\right) &= \mathbf{u}_0^{(\alpha)} \mathbf{a}(\omega_\alpha) \otimes \mathbf{v}_1^{(\alpha)} + \mathbf{u}_0^{(\alpha)} \otimes \left(-x_\alpha \omega_\alpha \mathbf{v}_1^{(\alpha)}\right) \\
&= \mathbf{u}_0^{(\alpha)} \left(\mathbf{a}(\omega_\alpha) - x_\alpha \omega_\alpha \mathbf{I}_1\right) \otimes \mathbf{v}_1^{(\alpha)} \\
&= \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)}.
\end{aligned}$$

Similarly,

$$\mathbf{v}_0^{(\alpha)} \left(\mathbf{b}(\omega_\alpha) + x_\alpha \omega_\alpha \mathbf{I}_2\right) = \mathbf{v}_1^{(\alpha)},$$

and

$$\left(\mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_0^{(\alpha)}\right) \left(\mathbf{a}(\omega_\alpha) \oplus \mathbf{b}(\omega_\alpha)\right) = \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)}.$$

Thus, (5.12) holds. To show

$$\mathbf{u}_0^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_0^{(\alpha)}, \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)}$$

are linearly independent, it is sufficient to claim that $\mathbf{u}_0^{(\alpha)}$ is independent of $\mathbf{u}_1^{(\alpha)}$, and $\mathbf{v}_0^{(\alpha)}$ is independent of $\mathbf{v}_1^{(\alpha)}$. If

$$c_1 \mathbf{u}_0^{(\alpha)} + c_2 \mathbf{u}_1^{(\alpha)} = \mathbf{0},$$

then

$$(c_1 \mathbf{u}_0^{(\alpha)} + c_2 \mathbf{u}_1^{(\alpha)}) \boldsymbol{\gamma}_1 = \mathbf{0}.$$

Since

$$\mathbf{u}_0^{(\alpha)} \boldsymbol{\gamma}_1 = \frac{1}{\omega_\alpha} a_1 \boldsymbol{\beta}_1 (\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-2} \boldsymbol{\gamma}_1 = 0,$$

and

$$\mathbf{u}_1^{(\alpha)}\boldsymbol{\gamma}_1 = -a_1\omega_\alpha \neq 0,$$

it follows $c_2 = c_1 = 0$. Therefore, $\mathbf{u}_0^{(\alpha)}$ is independent of $\mathbf{u}_1^{(\alpha)}$. Similarly, $\mathbf{v}_0^{(\alpha)}$ is independent of $\mathbf{v}_1^{(\alpha)}$. $\square$

Theorem 5.3.2 *If x_α is a root of (4.1) with multiplicity 2 and $\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1 \neq 0$, then there exists $\mathbf{u}_2^{(\alpha)}$ and $\mathbf{v}_2^{(\alpha)}$ such that*

$$\boldsymbol{\varphi}_1^{(\alpha)}\mathbf{Q}'(\omega_\alpha) + \boldsymbol{\varphi}_2^{(\alpha)}\mathbf{Q}(\omega_\alpha) = \mathbf{0},$$

where

$$\begin{aligned}\boldsymbol{\varphi}_1^{(\alpha)} &= \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)},\\ \boldsymbol{\varphi}_2^{(\alpha)} &= \mathbf{u}_2^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_2^{(\alpha)}.\end{aligned}$$

Proof.

$$\begin{aligned}\mathbf{0} &= (\mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)})\mathbf{Q}'(\omega_\alpha) + (\mathbf{u}_2^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_2^{(\alpha)})\mathbf{Q}(\omega_\alpha)\\ &= (\mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)})(\mathbf{a}'(\omega_\alpha) \oplus \mathbf{b}'(\omega_\alpha)) + (\mathbf{u}_2^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_2^{(\alpha)})(\mathbf{a}(\omega_\alpha)\oplus\\ &\quad \mathbf{b}(\omega_\alpha))\\ &= \mathbf{u}_1^{(\alpha)}\mathbf{a}'(\omega_\alpha) \otimes \mathbf{v}_1^{(\alpha)} + \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)}\mathbf{b}'(\omega_\alpha) + \mathbf{u}_2^{(\alpha)}\mathbf{a}(\omega_\alpha) \otimes \mathbf{v}_1^{(\alpha)} + \mathbf{u}_2^{(\alpha)}\otimes\\ &\quad \mathbf{v}_1^{(\alpha)}\mathbf{b}(\omega_\alpha) - \mathbf{u}_1^{(\alpha)}\mathbf{a}(\omega_\alpha) \otimes \mathbf{v}_2^{(\alpha)} - \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_2^{(\alpha)}\mathbf{b}(\omega_\alpha).\end{aligned}$$

Inserting (5.6)$\sim$ (5.10) into the equations, it becomes

$$\begin{aligned}&(a_1\boldsymbol{\beta}_1 + x_\alpha\mathbf{u}_1^{(\alpha)}) \otimes \mathbf{v}_1^{(\alpha)} + \mathbf{u}_1^{(\alpha)} \otimes (-a_2\boldsymbol{\beta}_2 - x_\alpha\mathbf{v}_1^{(\alpha)}) + \mathbf{u}_2^{(\alpha)}\mathbf{a}(\omega_\alpha) \otimes \mathbf{v}_1^{(\alpha)}\\ &+\mathbf{u}_2^{(\alpha)} \otimes (-x_\alpha\omega_\alpha\mathbf{v}_1^{(\alpha)}) - (x_\alpha\omega_\alpha\mathbf{u}_1^{(\alpha)}) \otimes \mathbf{v}_2^{(\alpha)} - \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_2^{(\alpha)}\mathbf{b}(\omega_\alpha) = \mathbf{0}\end{aligned}$$

or

$$\begin{aligned}&\{\mathbf{u}_2^{(\alpha)}(\mathbf{a}(\omega_\alpha) - x_\alpha\omega_\alpha\mathbf{I}_1) + a_1\boldsymbol{\beta}_1\} \otimes \mathbf{v}_1^{(\alpha)} + \mathbf{u}_1^{(\alpha)} \otimes \{\mathbf{v}_2^{(\alpha)}(-\mathbf{b}(\omega_\alpha) - x_\alpha\omega_\alpha\mathbf{I}_2)\\ &-a_2\boldsymbol{\beta}_2\} = \mathbf{0}.\end{aligned}$$

Then there exists $b_1 \in \mathbb{R}$ such that

$$\mathbf{u}_2^{(\alpha)}(\mathbf{a}(\omega_\alpha) - x_\alpha\omega_\alpha\mathbf{I}_1) + a_1\boldsymbol{\beta}_1 = b_1\mathbf{u}_1^{(\alpha)}, \tag{5.15}$$

$$\mathbf{v}_2^{(\alpha)}(-\mathbf{b}(\omega_\alpha) - x_\alpha\omega_\alpha\mathbf{I}_2) - a_2\boldsymbol{\beta}_2 = -b_1\mathbf{v}_1^{(\alpha)}, \tag{5.16}$$

i.e.,

$$\mathbf{u}_2^{(\alpha)}(\omega_\alpha \mathbf{S}_1 + \boldsymbol{\gamma}_1\boldsymbol{\beta}_1 - x_\alpha\omega_\alpha \mathbf{I}_1) = b_1\mathbf{u}_1^{(\alpha)} - a_1\boldsymbol{\beta}_1, \tag{5.17}$$

$$\mathbf{v}_2^{(\alpha)}\{-(\omega_\alpha \mathbf{S}_2 + \omega_\alpha^2\boldsymbol{\gamma}_2\boldsymbol{\beta}_2) - x_\alpha\omega_\alpha \mathbf{I}_2\} = -b_1\mathbf{v}_1^{(\alpha)} + a_2\boldsymbol{\beta}_2. \tag{5.18}$$

Considering (5.17), we let

$$\begin{cases} \omega_\alpha \mathbf{u}_2^{(\alpha)}(\mathbf{S}_1 - x_\alpha \mathbf{I}_1) = b_1\mathbf{u}_1^{(\alpha)} \\ \mathbf{u}_2^{(\alpha)}\boldsymbol{\gamma}_1 = -a_1 \end{cases},$$

and get

$$\mathbf{u}_2^{(\alpha)} = \frac{b_1}{\omega_\alpha}\mathbf{u}_1^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1} \quad \text{and} \quad b_1 = \frac{-a_1\omega_\alpha}{\mathbf{u}_1^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}. \tag{5.19}$$

On the other hand, let

$$\begin{cases} \omega_\alpha \mathbf{v}_2^{(\alpha)}(\mathbf{S}_2 + x_\alpha \mathbf{I}_2) = b_1\mathbf{v}_1^{(\alpha)} \\ \omega_\alpha^2\mathbf{v}_2^{(\alpha)}\boldsymbol{\gamma}_2 = -a_2 \end{cases} \tag{5.20}$$

in (5.18), and we get

$$\mathbf{v}_2^{(\alpha)} = \frac{b_1}{\omega_\alpha}\mathbf{v}_1^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1} \quad \text{and} \quad b_1 = \frac{-a_2}{\omega_\alpha\mathbf{v}_1^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}. \tag{5.21}$$

Note that $\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1 \neq 0$, thus, $\mathbf{u}_1^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1 \neq 0$ and $\mathbf{v}_1^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2 \neq 0$. The proof will be completed if

$$\frac{-a_1\omega_\alpha}{\mathbf{u}_1^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1} = \frac{-a_2}{\omega_\alpha\mathbf{v}_1^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}.$$

In fact,

$$\begin{aligned} \frac{-a_1\omega_\alpha}{\mathbf{u}_1^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1} &= \frac{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1} && (5.22) \\ &= \frac{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2} && (5.23) \\ &= \frac{-a_2}{\omega_\alpha\mathbf{v}_1^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}. && (5.24) \end{aligned}$$

Equalities (5.22) and (5.24) follow by the fact $\omega_\alpha a_1 = a_2$, $\omega_\alpha = \boldsymbol{\beta}_1(x_\alpha\mathbf{I}_1 - \mathbf{S}_1)^{-1}\boldsymbol{\gamma}_1$, (5.1), and (5.2); (5.23) follows directly by (5.11). □

Remark 5.3.3

(1) In the proof of Theorem 5.3.2, we write $\mathbf{u}_2^{(\alpha)}$ *and* $\mathbf{v}_2^{(\alpha)}$ *as follows,*

$$\mathbf{u}_2^{(\alpha)} = \frac{b_1}{\omega_\alpha}\mathbf{u}_1^{(\alpha)}(\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-1}, \tag{5.25}$$

$$\mathbf{v}_2^{(\alpha)} = \frac{b_1}{\omega_\alpha}\mathbf{v}_1^{(\alpha)}(\mathbf{S}_2 + x_\alpha \mathbf{I}_2)^{-1}, \tag{5.26}$$

where

$$b_1 = \frac{-a_1\omega_\alpha}{\mathbf{u}_1^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}\left(or\ \frac{-a_2}{\omega_\alpha\mathbf{v}_1^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}\right). \tag{5.27}$$

(2) From (5.15), (5.16) and (5.20) we can easily derive

$$\mathbf{u}_2^{(\alpha)}\mathbf{a}(\omega_\alpha) = b_1\mathbf{u}_1^{(\alpha)} - a_1\boldsymbol{\beta}_1 + x_\alpha\omega_\alpha\mathbf{u}_2^{(\alpha)}, \tag{5.28}$$

$$\mathbf{v}_2^{(\alpha)}\mathbf{b}(\omega_\alpha) = b_1\mathbf{v}_1^{(\alpha)} - a_2\boldsymbol{\beta}_2 - x_\alpha\omega_\alpha\mathbf{v}_2^{(\alpha)}, \tag{5.29}$$

and

$$\mathbf{v}_2^{(\alpha)}\boldsymbol{\gamma}_2\boldsymbol{\beta}_2 = \frac{-a_2}{\omega_\alpha^2}\boldsymbol{\beta}_2. \tag{5.30}$$

(3) From (5.25), (5.26), and (5.30), it is easy to verify

$$\mathbf{u}_2^{(\alpha)}\mathbf{a}'(\omega_\alpha) = \frac{b_1}{\omega_\alpha}\mathbf{u}_1^{(\alpha)} + x_\alpha\mathbf{u}_2^{(\alpha)}, \tag{5.31}$$

$$\mathbf{v}_2^{(\alpha)}\mathbf{b}'(\omega_\alpha) = \frac{b_1}{\omega_\alpha}\mathbf{v}_1^{(\alpha)} - x_\alpha\mathbf{v}_2^{(\alpha)} - \frac{2a_2}{\omega_\alpha}\boldsymbol{\beta}_2. \tag{5.32}$$

Theorem 5.3.4 *If* x_α *is a root of (4.1) with multiplicity* 3 *and* $\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1 \neq 0$*, then there exists* $\mathbf{u}_3^{(\alpha)}$ *and* $\mathbf{v}_3^{(\alpha)}$ *such that*

$$\frac{1}{2!}\boldsymbol{\varphi}_1^{(\alpha)}\mathbf{Q}''(\omega_\alpha) + \boldsymbol{\varphi}_2^{(\alpha)}\mathbf{Q}'(\omega_\alpha) + \boldsymbol{\varphi}_3^{(\alpha)}\mathbf{Q}(\omega_\alpha) = \mathbf{0}, \tag{5.33}$$

where

$$\boldsymbol{\varphi}_1^{(\alpha)} = \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)},$$
$$\boldsymbol{\varphi}_2^{(\alpha)} = \mathbf{u}_2^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_2^{(\alpha)},$$
$$\boldsymbol{\varphi}_3^{(\alpha)} = \mathbf{u}_3^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_2^{(\alpha)} \otimes \mathbf{v}_2^{(\alpha)} + \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_3^{(\alpha)}.$$

Proof. Inserting (5.6)∼(5.8) and (5.28)∼(5.30) into (5.33), it becomes

$$(\mathbf{u}_3^{(\alpha)}\mathbf{a}(\omega_\alpha) - x_\alpha\omega_\alpha\mathbf{u}_3^{(\alpha)} - b_1\mathbf{u}_2^{(\alpha)}) \otimes \mathbf{v}_1^{(\alpha)} + \mathbf{u}_1^{(\alpha)} \otimes (x_\alpha\omega_\alpha\mathbf{v}_3^{(\alpha)} + \mathbf{v}_3^{(\alpha)}\mathbf{b}(\omega_\alpha) + \frac{a_2}{\omega_\alpha}\boldsymbol{\beta}_2 - b_1\mathbf{v}_2^{(\alpha)}) = \mathbf{0}.$$

Then there exists scalar b_2 such that

$$\mathbf{u}_3^{(\alpha)}\mathbf{a}(\omega_\alpha) - x_\alpha\omega_\alpha\mathbf{u}_3^{(\alpha)} - b_1\mathbf{u}_2^{(\alpha)} = b_2\mathbf{u}_1^{(\alpha)}, \tag{5.34}$$

$$x_\alpha\omega_\alpha\mathbf{v}_3^{(\alpha)} + \mathbf{v}_3^{(\alpha)}\mathbf{b}(\omega_\alpha) + \frac{a_2}{\omega_\alpha}\boldsymbol{\beta}_2 - b_1\mathbf{v}_2^{(\alpha)} = -b_2\mathbf{v}_1^{(\alpha)}, \tag{5.35}$$

i.e.,

$$\mathbf{u}_3^{(\alpha)}(\omega_\alpha\mathbf{S}_1 + \boldsymbol{\gamma}_1\boldsymbol{\beta}_1 - x_\alpha\omega_\alpha\mathbf{I}_1) = b_1\mathbf{u}_2^{(\alpha)} + b_2\mathbf{u}_1^{(\alpha)}, \tag{5.36}$$

$$\mathbf{v}_3^{(\alpha)}(\omega_\alpha\mathbf{S}_2 + \omega_\alpha^2\boldsymbol{\gamma}_2\boldsymbol{\beta}_2 + x_\alpha\omega_\alpha\mathbf{I}_2) = -b_2\mathbf{v}_1^{(\alpha)} + b_1\mathbf{v}_1^{(\alpha)} - \frac{a_2}{\omega_\alpha}\boldsymbol{\beta}_2. \tag{5.37}$$

Consider (5.36), and let

$$\begin{cases} \omega_\alpha\mathbf{u}_3^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1) = b_1\mathbf{u}_2^{(\alpha)} + b_2\mathbf{u}_1^{(\alpha)} \\ \mathbf{u}_3^{(\alpha)}\boldsymbol{\gamma}_1 = 0 \end{cases}.$$

We obtain

$$\begin{aligned} \mathbf{u}_3^{(\alpha)} &= \tfrac{1}{\omega_\alpha}(b_2\mathbf{u}_1^{(\alpha)} + b_1\mathbf{u}_2^{(\alpha)})(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}, \\ b_2 &= \tfrac{-b_1\mathbf{u}_2^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}{\mathbf{u}_1^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}. \end{aligned} \tag{5.38}$$

On the other hand, we let

$$\begin{cases} \omega_\alpha\mathbf{v}_3^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2) = b_1\mathbf{v}_2^{(\alpha)} - b_2\mathbf{v}_1^{(\alpha)} \\ \omega_\alpha^2\mathbf{v}_3^{(\alpha)}\boldsymbol{\gamma}_2 = \frac{-a_2}{\omega_\alpha} \end{cases} \tag{5.39}$$

in (5.37), and obtain

$$\begin{aligned} \mathbf{v}_3^{(\alpha)} &= \tfrac{1}{\omega_\alpha}(-b_2\mathbf{v}_1^{(\alpha)} + b_1\mathbf{v}_2^{(\alpha)})(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}, \\ b_2 &= \tfrac{\frac{a_2}{\omega_\alpha^2} + b_1\mathbf{v}_2^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}{\mathbf{v}_1^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}. \end{aligned} \tag{5.40}$$

The proof will be completed if

$$\frac{-b_1\mathbf{u}_2^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}{\mathbf{u}_1^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1} = \frac{\frac{a_2}{\omega_\alpha^2} + b_1\mathbf{v}_2^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}{\mathbf{v}_1^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}. \tag{5.41}$$

Since x_α is a root of (4.1) with multiplicity 3, it implies

$$\begin{aligned} 0 &= \tfrac{d^2}{dx^2}\{f^*_{Ta}(x)f^*_{Ts}(-x)-1\}|_{x=x_\alpha} \\ &= 2\boldsymbol{\beta}_1(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1\boldsymbol{\beta}_2(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-3}\boldsymbol{\gamma}_2 \\ &\quad -2\boldsymbol{\beta}_1(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1\boldsymbol{\beta}_2(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2 \\ &\quad +2\boldsymbol{\beta}_1(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1\boldsymbol{\beta}_2(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2. \end{aligned} \tag{5.42}$$

Dividing (5.42) by $2\boldsymbol{\beta}_1(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1\boldsymbol{\beta}_2(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2$, we obtain

$$\begin{aligned} &\frac{\boldsymbol{\beta}_2(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2}\cdot\frac{\boldsymbol{\beta}_1(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1} \\ &= 1-\frac{\boldsymbol{\beta}_1(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1}\cdot\frac{\boldsymbol{\beta}_2(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-3}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2}. \end{aligned} \tag{5.43}$$

It is equivalent to

$$b_1\frac{\boldsymbol{\beta}_1(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1} = 1-b_1\frac{\boldsymbol{\beta}_2(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-3}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2}. \tag{5.44}$$

Therefore, we have

$$\begin{aligned} \frac{-b_1\mathbf{u}_2^{(\alpha)}(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}{\mathbf{u}_1^{(\alpha)}(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1} &= \frac{-b_1^2}{-\omega_\alpha}\cdot\frac{\boldsymbol{\beta}_1(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1}{\boldsymbol{\beta}_1(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1} && (5.45) \\ &= \frac{-b_1}{\omega_\alpha}\left(1-b_1\frac{\boldsymbol{\beta}_2(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-3}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2}\right) && (5.46) \\ &= \frac{-b_1}{\omega_\alpha}+\frac{b_1^2}{\omega_\alpha}\frac{\boldsymbol{\beta}_2(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-3}\boldsymbol{\gamma}_2}{\boldsymbol{\beta}_2(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-2}\boldsymbol{\gamma}_2} \\ &= \frac{\frac{a_2}{\omega_\alpha^2}+b_1\mathbf{v}_2^{(\alpha)}(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}{\mathbf{v}_1^{(\alpha)}(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}. && (5.47) \end{aligned}$$

Equalities (5.45) and (5.47) follow by (5.25), (5.26) and (5.27) ; (5.46) follows directly from (5.44). □

Remark 5.3.5

(1) In the proof of Theorem 5.3.4, we write $\mathbf{u}_3^{(\alpha)}$ and $\mathbf{v}_3^{(\alpha)}$ as follows,

$$\mathbf{u}_3^{(\alpha)} = \frac{1}{\omega_\alpha}(b_2\mathbf{u}_1^{(\alpha)}+b_1\mathbf{u}_2^{(\alpha)})(\mathbf{S}_1-x_\alpha\mathbf{I}_1)^{-1}, \tag{5.48}$$

$$\mathbf{v}_3^{(\alpha)} = \frac{1}{\omega_\alpha}(-b_2\mathbf{v}_1^{(\alpha)}+b_1\mathbf{v}_2^{(\alpha)})(\mathbf{S}_2+x_\alpha\mathbf{I}_2)^{-1}, \tag{5.49}$$

where

$$b_2 = \frac{-b_1\mathbf{u}_2^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}{\mathbf{u}_1^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}\left(\mathit{or}\ \frac{\frac{a_2}{\omega_\alpha^2} + b_1\mathbf{v}_2^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}{\mathbf{v}_1^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}\right).$$

(2) From (5.36), (5.37) and (5.39) we can easily derive

$$\mathbf{u}_3^{(\alpha)}\mathbf{a}(\omega_\alpha) = b_2\mathbf{u}_1^{(\alpha)} + b_1\mathbf{u}_2^{(\alpha)} + x_\alpha\omega_\alpha\mathbf{u}_3^{(\alpha)}, \quad (5.50)$$

$$\mathbf{v}_3^{(\alpha)}\mathbf{b}(\omega_\alpha) = -b_2\mathbf{v}_1^{(\alpha)} + b_1\mathbf{v}_2^{(\alpha)} - x_\alpha\omega_\alpha\mathbf{v}_3^{(\alpha)} - \frac{a_2}{\omega_\alpha}\boldsymbol{\beta}_2, \quad (5.51)$$

and

$$\mathbf{v}_3^{(\alpha)}\boldsymbol{\gamma}_2\boldsymbol{\beta}_2 = \frac{-a_2}{\omega_\alpha^3}\boldsymbol{\beta}_2. \quad (5.52)$$

(3) From (5.48), (5.49), and (5.52), it is easy to verify

$$\mathbf{u}_3^{(\alpha)}\mathbf{a}'(\omega_\alpha) = \frac{b_2}{\omega_\alpha}\mathbf{u}_1^{(\alpha)} + \frac{b_1}{\omega_\alpha}\mathbf{u}_2^{(\alpha)} + x_\alpha\mathbf{u}_3^{(\alpha)}, \quad (5.53)$$

$$\mathbf{v}_3^{(\alpha)}\mathbf{b}'(\omega_\alpha) = \frac{-b_2}{\omega_\alpha}\mathbf{v}_1^{(\alpha)} + \frac{b_1}{\omega_\alpha}\mathbf{v}_2^{(\alpha)} - x_\alpha\mathbf{v}_3^{(\alpha)} - \frac{2a_2}{\omega_\alpha^2}\boldsymbol{\beta}_2. \quad (5.54)$$

Theorem 5.3.6 *If x_α is a root of (4.1) with multiplicity 3, $\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1 = 0$, and $\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1 = 0$, then there exists $\mathbf{u}_{-1}^{(\alpha)}$ and $\mathbf{v}_{-1}^{(\alpha)}$ such that*

$$(\mathbf{u}_{-1}^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_0^{(\alpha)} \otimes \mathbf{v}_0^{(\alpha)} + \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_{-1}^{(\alpha)})\mathbf{Q}(\omega_\alpha) = 0 \quad (5.55)$$

where

$$\mathbf{u}_{-1}^{(\alpha)} = \frac{1}{\omega_\alpha}\mathbf{u}_0^{(\alpha)}(\mathbf{S}_1 - x_\alpha\mathbf{I}_1)^{-1}, \quad (5.56)$$

$$\mathbf{v}_{-1}^{(\alpha)} = \frac{1}{\omega_\alpha}\mathbf{v}_0^{(\alpha)}(\mathbf{S}_2 + x_\alpha\mathbf{I}_2)^{-1}, \quad (5.57)$$

and

$$\{\mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)}, \mathbf{u}_0^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_0^{(\alpha)}, \mathbf{u}_{-1}^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_0^{(\alpha)} \otimes \mathbf{v}_0^{(\alpha)} + \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_{-1}^{(\alpha)}\}$$

is a linear independent set.

Theorem 5.3.7 *If x_α is a root of (4.1) with multiplicity* 3*, $\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1 = 0$, and $\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1 \neq 0$, then*

$$\begin{aligned}\boldsymbol{\varphi}_{11}^{(\alpha)}\mathbf{Q}(\omega_\alpha) &= \mathbf{0},\\ \boldsymbol{\varphi}_{21}^{(\alpha)}\mathbf{Q}(\omega_\alpha) &= \mathbf{0},\\ \boldsymbol{\varphi}_{21}^{(\alpha)}\mathbf{Q}'(\omega_\alpha) + \boldsymbol{\varphi}_{22}^{(\alpha)}\mathbf{Q}(\omega_\alpha) &= \mathbf{0},\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{\varphi}_{11}^{(\alpha)} &= \mathbf{u}_0^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_0^{(\alpha)},\\ \boldsymbol{\varphi}_{21}^{(\alpha)} &= \left(\frac{-\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-3}\boldsymbol{\gamma}_1}{\omega_\alpha^2}\right)\mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)}\\ \boldsymbol{\varphi}_{22}^{(\alpha)} &= \mathbf{u}_{-1}^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_0^{(\alpha)} \otimes \mathbf{v}_0^{(\alpha)} + \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_{-1}^{(\alpha)}.\end{aligned}$$

The proofs of Theorem 5.3.6 and Theorem 5.3.7 are given in Appendix C and Appendix D respectively.

Theorem 5.3.8 *If x_α is a root of (4.1) with multiplicity* 4 *and $\boldsymbol{\beta}_1(\mathbf{S}_1 - x_\alpha \mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1 \neq 0$, then there exists $\mathbf{u}_4^{(\alpha)}$ and $\mathbf{v}_4^{(\alpha)}$ such that*

$$\frac{1}{3!}\boldsymbol{\varphi}_1^{(\alpha)}\frac{d^3}{d\omega^3}\mathbf{Q}(\omega)|_{\omega=\omega_\alpha} + \frac{1}{2!}\boldsymbol{\varphi}_2^{(\alpha)}\mathbf{Q}''(\omega_\alpha) + \boldsymbol{\varphi}_3^{(\alpha)}\mathbf{Q}'(\omega_\alpha) + \boldsymbol{\varphi}_4^{(\alpha)}\mathbf{Q}(\omega_\alpha) = \mathbf{0}, \qquad (5.58)$$

where

$$\begin{aligned}\boldsymbol{\varphi}_1^{(\alpha)} &= \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)},\\ \boldsymbol{\varphi}_2^{(\alpha)} &= \mathbf{u}_2^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_2^{(\alpha)},\\ \boldsymbol{\varphi}_3^{(\alpha)} &= \mathbf{u}_3^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_2^{(\alpha)} \otimes \mathbf{v}_2^{(\alpha)} + \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_3^{(\alpha)},\\ \boldsymbol{\varphi}_4^{(\alpha)} &= \mathbf{u}_4^{(\alpha)} \otimes \mathbf{v}_1^{(\alpha)} - \mathbf{u}_3^{(\alpha)} \otimes \mathbf{v}_2^{(\alpha)} + \mathbf{u}_2^{(\alpha)} \otimes \mathbf{v}_3^{(\alpha)} - \mathbf{u}_1^{(\alpha)} \otimes \mathbf{v}_4^{(\alpha)}.\end{aligned}$$

The proof of Theorem 5.3.8 is given in the Appendix E.

Because we use a similar approach in the process of proving Theorems 5.3.2, 5.3.4, and 5.3.8, we summary it in the following four steps. For $\alpha = 1, \ldots, m$, it proceeds with,

step 1. given α, write

$$\boldsymbol{\varphi}_i^{(\alpha)} = \sum_{t=0}^{i-1}(-1)^t \mathbf{u}_{i-t}^{(\alpha)} \otimes \mathbf{v}_{t+1}^{(\alpha)};$$

step 2. use the equation

$$\tfrac{1}{(i-1)!}\boldsymbol{\varphi}_1^{(\alpha)} \tfrac{d^{i-1}}{d\omega^{i-1}}\mathbf{Q}(\omega)|_{\omega=\omega_\alpha} + \tfrac{1}{(i-2)!}\boldsymbol{\varphi}_2^{(\alpha)} \tfrac{d^{i-2}}{d\omega^{i-2}}\mathbf{Q}(\omega)|_{\omega=\omega_\alpha} + \cdots$$
$$+\boldsymbol{\varphi}_{\ell_\alpha}^{(\alpha)}\mathbf{Q}(\omega_\alpha) = \mathbf{0}$$

and Lemma 5.1.1 to obtain two equations (5.3);

step 3. separate each of the equations in (5.3) into two parts to obtain $\mathbf{u}_i^{(\alpha)}$ and $\mathbf{v}_i^{(\alpha)}$;

(e.g. (5.19) and (5.20) in $\ell_\alpha = 2$*; (E.3) and (5.39) in* $\ell_\alpha = 3$*.)*

step 4. use the equation

$$\frac{d^k}{dx^k}\{\boldsymbol{\beta}_1(\mathbf{S}_1 - x\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1\boldsymbol{\beta}_2(\mathbf{S}_2 + x\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2 - 1\} = 0 \quad \text{for } k = 0, \ldots, i-1$$

to verify step 3 ;

step 5. replace i by $i+1$ and repeat **step 1** $\sim$ **step 4** until $i = \ell_\alpha$.

From Theorems 5.3.2, 5.3.4, and 5.3.8, when the multiplicity of x_α does not exceed 4 for $\alpha = 1, \ldots, s$, $s \leq m$, we can construct m vectors in the left Jordan chains for $\mathbf{Q}(\omega)$. Therefore, the saturated probability for $n \geq 1$ can be described as the representation mentioned in Theorem 3.3.3. It implies the equality holds in Corollary 3.3.6, i.e., $\sum_{\alpha=1}^{s} \mathcal{L}^*(\omega) = m$.