1 Introduction

In this paper, we consider the following discrete reaction-diffusion equation

$$u_t(x,t) = u(x+1,t) - 2u(x,t) + u(x-1,t) + f(u(x,t)), \tag{1.1}$$

which is a discrete version of the following semilinear parabolic equation

$$u_t = u_{xx} + f(u). \tag{1.2}$$

When the function $f(u)$ is such that $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$ and $f(u) > 0$ for any $0 < u < 1$, (1.2) is called the Fisher's equation [4] or Kolmogorov, Petrovsky and Piskunov (KPP) equation [6], and it describes the propagation of an advantageous gene within an one-dimensional habitat. When $f(u) = u^m(1-u)$, where m is an integer greater than two, it is called the mth-order Fisher's equation. In particular, it is called the Zeldovich equation if $m = 2$. For a cubic nonlinearly $f(u) = u(1-u)(u-a)$, it is called the Allen-Cahn equation ($a = 1/2$) in phase transition and also the Nagumo equation ($a \in (0,1)$) in propagation of nerve excitation. A great deal of work has been carried out to extend this equation to take into account other biological, chemical or physical factors.

A solution $u(x,t)$ of (1.1) is called a traveling wave with speed c if there exists a function $U : \mathbb{R} \to [0,1]$ such that $u(x,t) = U(x+ct)$, which connects two equilibria $u = 0, 1$. Such solution (c, U) satisfies the following traveling wave problem and it is unique up to translation

$$
\begin{cases}
cU'(\cdot) = U(\cdot + 1) + U(\cdot - 1) - 2U(\cdot) + f(U(\cdot)) \quad on \;\; \mathbb{R}, \\
U(-\infty) = 0, \quad U(\infty) = 1, \quad 0 \le U \le 1 \;\; on \;\; \mathbb{R}.
\end{cases} \tag{1.3}
$$

When f is Lipschitz continuous on [0,1] with $f(0) = f(1) = 0 < f(u)$ for all $u \in (0,1)$, it has been shown in [2] that there exists $c_{min} > 0$ such that (1.3) admits a solution if and only if $c \ge c_{min}$. The existence, uniqueness and asymptotic stability of traveling waves, we refer the readers to [2, 3] and the references therein.

From the dynamical point of view, the traveling wave solution is not enough to understand the whole dynamics of a reaction-diffusion equation. Therefore, there have been many studies done recently for other types of entire solutions. For example, Chen and Guo in [2] constructed entire solutions which behave as two opposite wave fronts coming from both sides of x-axis and then annihilating in a finite time. Here the entire solution is meant by a solution which is defined for all $(x,t) \in \mathbb{R}^2$. Entire solutions play an important role in the whole dynamics. The study for entire solutions is crucial in the following sense: firstly, it helps us for the mathematical understanding of transient dynamics. As mentioned above, some transient dynamics can be characterized by the behavior of the past $t \approx -\infty$, even though we cannot describe the whole transient behavior. Secondly, structure of the maximal invariant set (or the global attractor) is one of the ultimate goal.

In [5], Guo and Morita studied (1.1) and (1.2) where $f(0) = f(1) = 0$, $f'(1) < 0$, and $f'(0) \neq 0$. They proved there exist entire solutions which behave as two opposite wave fronts coming from both sides of x-axis. The technique they used was to characterize the asymptotic behavior of the solutions as $t \to \pm\infty$ in terms of appropriate subsolutions and supersolutions and use the comparison argument. This argument can apply not only to a general bistable reaction-diffusion equation but also to the Fisher-KPP equation. They also extended it to a discrete diffusive Fisher-KPP equation.

In this paper, we focus on (1.1), where $f(u) = u^2(1-u)$. We note that $f'(0) = 0$ in this case. Following the method of [5], we prove the existence of entire solutions for $c = c_{min}$ in the following theorem.

Theorem 1.1 *Consider* (1.1)*, where* $f(u) = u^2(1-u)$*. Let* U *be a solution of* (1.3) *with* $c = c_{min}$*. Then, for any given constants* θ_1*,* θ_2*, there exists an entire solution* $u(x,t)$ *of* (1.1) *such that*

$$\lim_{t\to-\infty}\{\sup_{x\geq 0}|u(x,t) - U(x+ct+\theta_1)| + \sup_{x\leq 0}|u(x,t) - U(-x+ct+\theta_2)|\} = 0. \quad (1.4)$$