6 附錄

6.1 附錄一

(3.3),(3.4) 及 (3.5) 式之推導

$$
\begin{aligned}
D^2 &= \|Xa-C\|^2+\|[Xa-l(Xab+\mathbf{1}d)]-[C-lS]\|^2 \\
&\quad +\|[Xa+l(Xab+\mathbf{1}d)]-[C+lS]\|^2 \\
&= a'X'Xa-C'Xa-a'X'C+C'C+2a'X'C+l^2b^2a'X'Xa+2l^2bda'X'\mathbf{1} \\
&\quad -2l^2ba'X'S+2l^2bd\mathbf{1}'Xa+2l^2d^2\mathbf{1}'\mathbf{1}-2l^2d\mathbf{1}'S \\
&\quad -2C'Xa+2C'C-2l^2bS'Xa-2l^2S'd\mathbf{1}+2l^2S'S \\
&= 3a'X'Xa-6C'Xa+3C'C+2l^2b^2a'X'Xa+4l^2bda'X'1-4l^2ba'X'S \\
&\quad +2l^2nd^2-4l^2dn\bar{S}+2l^2S'S
\end{aligned}
$$

令 $\frac{\partial D^2}{\partial a}=0,\frac{\partial D^2}{\partial b}=0,\frac{\partial D^2}{\partial d}=0$ 得到

$$
\frac{\partial D^2}{\partial a} = 3X'Xa-3X'C+2l^2b^2X'Xa+2l^2bdX'\mathbf{1}-2l^2bX'S=0 \quad (6.1)
$$

$$
\frac{\partial D^2}{\partial b} = ba'X'Xa+da'X'\mathbf{1}-a'X'S=0 \quad (6.2)
$$

$$
\frac{\partial D^2}{\partial d} = ba'X'\mathbf{1}+nd-\mathbf{1}'S=0 \quad (6.3)
$$

由 (6.1) 知 $a=\frac{1}{3+2l^2b^2}(X'X)^{-1}[3X'C+2l^2bX'S-2l^2bdX'\mathbf{1}]$，將 a 之值代入 (6.2) 與 (6.3) 中，可得

$$
\begin{aligned}
\frac{\partial D^2}{\partial b} &= 9b\|\hat{C}\|^2+6l^2b^2\hat{C}'S-6l^2b^2dn\bar{C}+9n\bar{C}d \\
&\quad +12l^2n\bar{S}bd-6l^2nbd^2-9C'\hat{S}-6l^2\|\hat{S}\|^2b \quad (6.4)
\end{aligned}
$$

$$
\frac{\partial D^2}{\partial d} = b\bar{C}+d-\bar{S}=0 \quad (6.5)
$$

其中 $X(X'X)^{-1}X'C=\hat{C}X(X'X)^{-1}X'S=\hat{S},\ \bar{\hat{S}}=\bar{S}$。

再由 (6.5) 式中得知 $d=\bar{S}-b\bar{C}$，代入 (6.4) 中可以簡化成一個 b 的二次式

$$
K_1b^2+K_2b+K_3=0
$$

故可得 $\hat{b}=\frac{-K_1\pm\sqrt{K_2^2-4K_1K_2}}{2K_1}$

6.2 附錄二

(4.2),(4.3),(4.4) 公式之推導

簡單距離公式下：

$$e_{(i)}^2 = (c_i - x_i\hat{a}_{(i)})^2 + (s_i - x_i\hat{r}_{(i)})^2$$

因 $\hat{a}_{(i)} = \hat{a} - \frac{e_i^c}{1-h_{ii}}(X'X)^{-1}x_i'$，且 $\hat{r}_{(i)} = \hat{r} - \frac{e_i^s}{1-h_{ii}}(X'X)^{-1}x_i'$，故

$$\begin{aligned} e_{(i)}^2 &= (c_i - x_i\hat{a} + \frac{e_i^c}{1-h_{ii}}x_i(X'X)^{-1}x_i')^2 + \\ &\quad (s_i - x_i\hat{r} + \frac{e_i^s}{1-h_{ii}}x_i(X'X)^{-1}x_i')^2 \\ &= (\frac{e_i^c}{1-h_{ii}})^2 + (\frac{e_i^s}{1-h_{ii}})^2 \\ &= (\frac{e_i}{1-h_{ii}})^2 \end{aligned}$$

Yang 和 Ko 距離公式下：

$$\begin{aligned} e_i^2 &= (c_i - x_i\hat{a})^2 + ((c_i - ls_i) - (x_i\hat{a} - lx_i\hat{r}))^2 + \\ &\quad ((c_i + ls_i) - (x_i\hat{a} + lx_i\hat{r}))^2 \\ &= 3(c_i - x_i\hat{a})^2 + 2(l(s_i - x_i\hat{r}))^2 \\ &= 3(e_i^c)^2 + 2l^2(e_i^s)^2 \end{aligned}$$

$$\begin{aligned} e_{(i)}^2 &= (c_i - x_i\hat{a}_{(i)})^2 + ((c_i - ls_i) - (x_i\hat{a}_{(i)} - lx_i'\hat{r}_{(i)}))^2 + \\ &\quad ((c_i + ls_i) - (x_i'\hat{a}_{(i)} + lx_i'\hat{r}_{(i)}))^2 \\ &= 3(c_i - x_i\hat{a}_{(i)})^2 + 2(ls_i - l(x_i\hat{r}_{(i)}))^2 \\ &= 3(\frac{e_i^c}{1-h_{ii}})^2 + 2l^2(\frac{e_i^s}{1-h_{ii}})^2 \\ &= (\frac{e_i}{1-h_{ii}})^2 \end{aligned}$$

6.3 附錄三

引理 4.1 之證明

證明：欲證明 $\tilde{d}_{LR}$ 爲距離測度，即需證明以下三性質成立：

1. $\forall\ \mathbb{X}$，$\mathbb{Y} \in \tilde{F}_{LR}(\Re)$，$\tilde{d}_{LR}(\mathbb{X},\mathbb{Y}) \geq 0$，如果$\tilde{d}_{LR}(\mathbb{X},\mathbb{Y}) = 0$則$\mathbb{X} = \mathbb{Y}$。

2. $\forall\ \mathbb{X}$，$\mathbb{Y} \in \tilde{F}_{LR}(\Re)$，$\tilde{d}_{LR}(\mathbb{X},\mathbb{Y}) = \tilde{d}_{LR}(\mathbb{Y},\mathbb{X})$。

3. $\forall\ \mathbb{X}$，$\mathbb{Y}$，$\mathbb{Z} \in \tilde{F}_{LR}(\Re)$，$\tilde{d}_{LR}(\mathbb{X},\mathbb{Y}) \leq \tilde{d}_{LR}(\mathbb{X},\mathbb{Z}) + \tilde{d}_{LR}(\mathbb{Z},\mathbb{Y})$

因爲d是距離測度，可以很容易證明1和2兩性質成立，故僅需證明第3性質：

$$\begin{aligned}
\text{因}\tilde{d}^2_{LR}(\mathbb{X},\mathbb{Y}) &= \sum_{i=1}^{p} d^2(X_i,Y_i) \\
&\leq \sum_{i=1}^{p} d^2(X_i,Z_i) + \sum_{i=1}^{p} d^2(Z_i,Y_i) \\
&\leq \tilde{d}^2_{LR}(\mathbb{X},\mathbb{Z}) + \tilde{d}^2_{LR}(\mathbb{Z},\mathbb{Y}) + 2\sqrt{\sum_{i=1}^{p} d^2(X_i,Z_i)}\sqrt{\sum_{i=1}^{p} d^2(Z_i,Y_i)} \\
&= (\tilde{d}_{LR}(\mathbb{X},\mathbb{Z}) + \tilde{d}_{LR}(\mathbb{Z},\mathbb{Y}))^2 \\
\text{故}\tilde{d}_{LR}(\mathbb{X},\mathbb{Y}) &\leq \tilde{d}_{LR}(\mathbb{X},\mathbb{Z}) + \tilde{d}_{LR}(\mathbb{Z},\mathbb{Y})
\end{aligned}$$

其次，設d是完備距離測度。
令$\{\mathbb{X}^m\}_{m=1}^{\infty}$是一個在$\tilde{F}_{LR}(\Re)$上的柯西數列，即
$\forall\varepsilon > 0$，$\exists k \in \mathbb{N} \ni m,m' > k$恆有$\tilde{d}_{LR}(\mathbb{X}^m,\mathbb{X}^{m'}) < \varepsilon$。
則$\forall m,m' > k$，$d(X_j^m,X_j^{m'}) < \sqrt{\sum_{i=1}^{p} d^2(X_i^m,X_i^{m'})} = \tilde{d}_{LR}(\mathbb{X}^m,\mathbb{X}^{m'}) < \varepsilon$。
因此$\forall 1 \leq j \leq p,\{X_j^m\}_{m=1}^{\infty}$是一個在$F_{LR}(\Re)$上的柯西數列。
故$\exists X_j \in F_{LR}(\Re), \ni X_j^m \to X_j$。令$\mathbb{X} = (X_1,X_2,...,X_p)'$。
$\because X_j^m \to X_j \therefore \forall\varepsilon > 0, \exists n_j \in \mathbb{N} \ni m > n_j$恆有$d(X_j^m,X_j) < \frac{\varepsilon}{\sqrt{p}}, j = 1,2,...,p$。
令$n = max\{n_1,n_2,...,n_p\}$，則$\forall m > n,$恆有$\tilde{d}_{LR}(\mathbb{X}^m,\mathbb{X}) = \sqrt{\sum_{i=1}^{p} d^2(X_i^m,X_i)} < \varepsilon$,即$\mathbb{X}^m \to \mathbb{X}$。

6.4 附錄四

(4.6),(4.7) 式之推導

簡單距離公式下：

$$
\begin{aligned}
CD_i &= \frac{1}{ps^2}\tilde{d}_{LR}(\hat{\mathbb{Y}}, \hat{\mathbb{Y}}_{(i)}) \\
&= \frac{1}{ps^2}\sum_{i=1}^{n} d^2(\hat{Y}_i, \hat{Y}_{(i)}) \\
&= \frac{1}{ps^2}\{\sum_{i=1}^{n}(x_i\hat{a} - x_i\hat{a}_{(i)})^2 + \sum_{i=1}^{n}(x_i\hat{r} - x_i\hat{r}_{(i)})^2 \\
&= \frac{1}{ps^2}\{(\frac{e_i^c}{1-h_{ii}})^2 h_{ii} + (\frac{e_i^s}{1-h_{ii}})^2 h_{ii}\} \\
&= \frac{1}{ps^2}\frac{e_i^2 h_{ii}}{(1-h_{ii})^2}
\end{aligned}
$$

Yang 和 Ko 距離公式下：

$$
\begin{aligned}
CD_i &= \frac{1}{ps^2}\tilde{d}_{LR}(\hat{\mathbb{Y}}, \hat{\mathbb{Y}}_{(i)}) \\
&= \frac{1}{ps^2}\sum_{i=1}^{n} d^2(\hat{Y}_i, \hat{Y}_{(i)}) \\
&= \frac{1}{ps^2}\{\sum_{i=1}^{n}(x_i\hat{a} - x_i\hat{a}_{(i)})^2 + \sum_{i=1}^{n}((x_i\hat{a} - lx_i\hat{r}) - (x_i\widehat{a_{(i)}} - lx_i\hat{r}_{(i)}))^2 \\
&\quad + \sum_{i=1}^{n}((x_i\hat{a} + lx_i\hat{r}) - (x_i\widehat{a_{(i)}} + lx_i\hat{r}_{(i)}))^2\} \\
&= \frac{1}{ps^2}\{3(\frac{e_i^c}{1-h_{ii}})^2 h_{ii} + 2l^2(\frac{e_i^s}{1-h_{ii}})^2 h_{ii}\} \\
&= \frac{1}{ps^2}\frac{e_i^2 h_{ii}}{(1-h_{ii})^2}
\end{aligned}
$$