3 Model and problem formulation

We will deduce an equation of changing on sale intensity under the effect of diffusing by word of mouth and the promoting activities.

The changing of the sale intensity is due to various factors. To formulate the behavior of the changing of the sale intensity, we start with the basic structure

$$S_t = \sum F_i,$$

where S denotes the sale intensity, S_t is the derivative of S in time t and F_i is the $i-th$ factor that makes effect on the sale intensity.

(i) The spontaneous sale intensity

The first factor we consider is the spontaneous sale intensity. In Vidale-Wolfe's model, they suggest that there is a spontaneous decay constant $\kappa > 0$ in modeling the behavior of the spontaneous sale intensity in time and the function of spontaneous sale intensity would be

$$S\left(t\right) = S\left(0\right) exp\left(-\kappa t\right)$$

with the initial sale intensity $S\left(0\right)$.

In another point of view, the spontaneous sale is caused by "word-of-mouth". After making consumption on merchandise, the consumer will be able to give appraisal to the merchandise, and deliver the information of the merchandise to others by word of mouth. Moreover, before making consumption on merchandise, a consumer might seek for information from those who had already made consumption on the merchandise. These later quantity of sale is created by the quality and the quantity of previous sale without external forces, therefore the sale intensity changes by the quantity of the previous sale changes; hence it is a kind of diffusion. For this reason, we delineate the spontaneous sale intensity by the heat equation which is used to delineate the same behavior in heat transfers.

Define the function $S(x,t)$ as the sale intensity of the product where t represents the time and x represents the number of consumer in the target market, that is, for $x = x_1$, $t = t_{1,}$ then $S_1 = S(x_1, t_1)$ means that the quantity of consumptions as the number of consumer reaches x_1 in the target market at time t_1 is S_1 (in practical, x_1 may be viewed as the number of consumer in some certain geographic areas, except being able to model the diffusion behavior of sale intensity, the setting of the x parameter also have advantage on investigating sale intensity in certain geographic areas.). Similar to the heat transfer, the function $S(x,t)$ might satisfy the equation

$$S_t(x,t) = c^2 S_{xx}(x,t),$$

where c denotes the diffusion constant. (Just like the heat transfer problems, for different kinds of merchandises or manufacturers, there will be different kinds of diffusion constants.)

(ii)The effects of promotion

In most cases, merchandise will not be put to run its course, so the effect of promoting activity is added into the considerations necessarily. The equation becomes

$$S_t(x,t) = c^2 S_{xx}(x,t) + P(x,t).$$

Different from natural science, it is not a proper way to add the effect into the quantity in social science since the factor did not change the quantity directly. We prefer studying the effect on the ratio between sale intensity changing rate and sale intensity (the margin that the sale intensity changes) than those directly on the changing rate of sale intensity. The equation would be rewritten as

$$\frac{S_t(x,t)}{S(x,t)} = c^2 \frac{S_{xx}(x,t)}{S(x,t)} + P(x,t).$$

(iii) The boundary condition and initial condition

In marketing, there are some restrictions. To delineate these restrictions in mathematical method, we have the following initial condition

$$S(x,0) = S_0(x).$$

We defined this initial condition as the first quantity of sale that is not affected by the circulating of information and any promoting activities. Practically, this is the quantity of sale right after the store penetrations complete.

Undoubtedly, there will be no consumption without consumer, so the sale intensity at $x = 0$ is

$$S(0,t) = 0, \quad t \geq 0.$$

On the other hand, since the number of consumer of the target market is limited, we also have the boundary condition

$$S_x(L,t) = 0, \quad t \geq 0,$$

with the target market size $x = L$. If we say that the target market size of a merchandise is L, it means that there are L people that have demand on the merchandise; in other words, there will be no demand more than L people and hence no consuming even if the information is spread out more than L people. So far, the basic model is obtained.

Proposition 1 : *The change of sale intensity could be basically modeled as*

$$\frac{S_t(x,t)}{S(x,t)} = c^2 \frac{S_{xx}(x,t)}{S(x,t)} + P(x,t), \quad 0 \leq x \leq L, \quad t \geq 0.$$

Subjected to

$$S(x,0) = S_0(x), \quad 0 \leq x \leq L,$$

$$S(0,t) = 0 = S_x(L,t), \quad t \geq 0.$$

Denoted by

- $S(x,t)$ the sale intensity at time t with quantity of consumers x,
- $P(x,t)$ the effect of promotion at time t with quantity of purchasers x,
- c the diffusion constant of merchandise,
- L the size of the market.

Here we focus on the major promotion, advertising, and rewrite the equation as

$$\frac{S_t(x,t)}{S(x,t)} = c^2 \frac{S_{xx}(x,t)}{S(x,t)} + A(x,t),$$

where $A(x,t)$ is the effect that the advertising strategy brought. Unfortunately, the term $A(x,t)$ cannot be characterized generally because of the great difference between different kinds of advertising strategies. Therefore, the problem formulating and model solving are restricted to be discussed case by case.