Chapter 5

The Deficient Values of a Class of Meromorphic Functions

5.1 Introduction

A well-known result by Picard [27] says that any non-constant entire function f can omit at most one finite complex value, which we call a Picard exceptional value of f. Nevanlinna generalized the idea of omitting values, and define now called the Nevanlinna deficiency $\delta(a, f)$ to measure the degree of a meromorphic function f "misses" the value a. Recall that an extended complex value a is a deficient value of f if $\delta(a, f) > 0$. Under this terminology, if a is a Picard exceptional value of f, then $\delta(a, f) = 1$.

Yang [35, 37] proved that any non-constant rational function f has exactly one deficient value a. Also, we can easily calculate the Nevanlinna deficiency $\delta(a, f)$ for the corresponding deficient value a. For completeness, we will state Yang's results in section 5.2.

To construct a meromorphic function with two deficient values, our approach is as follows. First, we consider a meromorphic function g with two Piacrd exceptional values a and b. Then, take a polynomial $P(z)$, and consider the meromorphic function $f(z) = P(g(z))$. We will show that f has at most two deficient values, and the only possible deficient values are $P(a)$ and $P(b)$. If g is of finite order, both $P(a)$ and $P(b)$ are deficient values of f, and the corresponding deficiencies can be computed explicitly. While a polynomial $P(z)$ is fixed, we define $\nu(\alpha)$ to be the multiplicity of the zero of $P(z) - P(\alpha)$ at $z = \alpha$ if α is a finite complex number, and $\nu(\infty)$ to be the degree of $P(z)$.

Now, given a non-constant meromorphic function g with two Picard exceptional values 0 and ∞, then it is well-known that $g(z) = e^{h(z)}$, where $h(z)$ is an entire function. Moreover, if g is of finite order, then h must be a non-constant polynomial [28]. In this case, we have the following theorems.

Theorem A *Let h be a non-constant polynomial and $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_k z^k$ be a non-constant polynomial, where $k \geq 0$ and a_k, a_n are non-zero constants. Let $f(z) = P(e^{h(z)})$. We have*

(i) *If $k \geq 1$, then 0 and ∞ are the only two deficient values of f. Moreover, $\delta(0, f) = \frac{\nu(0)}{n} = \frac{k}{n}$ and $\delta(\infty, f) = \frac{\nu(\infty)}{n} = 1$.*

(ii) *If $k = 0$, then a_0 and ∞ are the only two deficient values of f. Moreover, $\delta(a_0, f) = \frac{\nu(0)}{n}$ and $\delta(\infty, f) = \frac{\nu(\infty)}{n} = 1$.*

Theorem B Let g be a non-constant meromorphic function of finite order, such that g has two Picard exceptional values a and b. Let $P(z)$ be a non-constant polynomial of degree n. We have

(i) If $P(a) = P(b)$, then $P(a)$ is the only deficient value of $P(g)$. Moreover, $\delta(P(a), P(g)) = \frac{\nu(a)+\nu(b)}{n}$.

(ii) If $P(a) \neq P(b)$ and a, b are finite, then $P(a)$ and $P(b)$ are the only two deficient values of $P(g)$. Moreover, $\delta(P(a), P(g)) = \frac{\nu(a)}{n}$ and $\delta(P(b), P(g)) = \frac{\nu(b)}{n}$.

(iii) If a is finite and $b = \infty$, then $P(a)$ and ∞ are the only deficient values of $P(g)$. Moreover, $\delta(P(a), P(g)) = \frac{\nu(a)}{n}$ and $\delta(\infty, P(g)) = \frac{\nu(\infty)}{n} = 1$.

When g is of infinite order, we can get similar but somewhat weaker results as Theorem A and B, which will be stated in section 5.3 and 5.4.

5.2 The Deficient Values of Rational Functions

Clearly, by definition 2.5.1, $0 \le \delta(a, f) \le 1$. If $\delta(a, f)$ is much more close to 1, this means $N(r, \frac{1}{f-a})$ much smaller than $T(r, f)$. In other words, the lack of f at a is much more acuter. In general, it is quite difficult to find the deficient values of an arbitrary meromorphic function. However, for rational function, C. C. Yang [37] proved the following.

Theorem 5.2.1 *Let f be a non-constant rational function defined by*

$$f(z) = \frac{a_p z^p + a_{p-1} z^{p-1} + \cdots + a_0}{b_q z^q + b_{q-1} z^{q-1} + \cdots + b_0},$$

where $a_p z^p + a_{p-1} z^{p-1} + \cdots + a_0$ and $b_q z^q + b_{q-1} z^{q-1} + \cdots + b_0$ are relatively prime. Then

(i) $N(r, f) = q \log r$ *and* $N(r, \frac{1}{f}) = p \log r$.

(ii) $$m(r, f) = \begin{cases} (p-q)\log r + O(1) & \text{if } p > q \\ O(1) & \text{if } p \le q \end{cases}$$

(iii) $$N(r, \tfrac{1}{f-a}) = \begin{cases} \max\{p, q\} \log r & \text{if } p \ne q \\ p \log r & \text{if } p = q \text{ and } a_p \ne a b_q \\ k \log r & \text{if } p = q \text{ and } a_p = a b_q \text{ for some } 0 \le k \le p-1, \end{cases}$$
where a is a non-zero complex number.

(iv) $T(r, f) = \max\{p, q\} \log r + O(1)$.

It follows from Theorem 5.2.1, we can completely classify all deficient values and their corresponding deficiency for rational functions as follows.

Corollary 5.2.2 *If f is a non-constant rational function, then f has only one deficient value $f(\infty)$. More precisely, we have the following cases:*

(i) *If $p > q$, then ∞ is the only deficient value of f and $\delta(\infty, f) = 1 - \frac{q}{p}$.*

(ii) *If $p < q$, then 0 is the only deficient value of f and $\delta(0, f) = 1 - \frac{p}{q}$.*

(iii) *If $p = q$, then $\frac{a_p}{b_q}$ is the only deficient value of f and $\delta(\frac{a_p}{b_q}, f) = 1 - \frac{k}{p}$, where k is the largest non-negative integer j such that $a_j \neq ab_j$.*

5.3 The Proof of Theorem A

Let g be a non-constant meromorphic function with two Picard exceptional values 0 and ∞, so $g(z) = e^{h(z)}$, where $h(z)$ is an entire function. In this section, we study the deficient values and deficiencies of $P(g)$, where $P(z)$ is a non-constant polynomial. First, we establish some lemmas.

Lemma 5.3.1 *Let h be a non-constant entire function and $f(z) = a + be^{h(z)}$, where a and b are non-zero complex numbers. Then*

$$m(r, \frac{1}{f}) = S(r, e^h).$$

Proof. By the Nevanlinna's second fundamental theorem,

$$\begin{aligned} T(r, \tfrac{1}{f}) &= T(r, f) + O(1) \\ &\leq \overline{N}(r, \tfrac{1}{f}) + \overline{N}(r, \tfrac{1}{f-a}) + \overline{N}(r, f) + S(r, f) \\ &\leq \overline{N}(r, \tfrac{1}{f}) + \overline{N}(r, \tfrac{1}{be^h}) + S(r, f) \\ &\leq N(r, \tfrac{1}{f}) + S(r, f). \end{aligned}$$

Hence,

$$m(r, \frac{1}{f}) = T(r, \frac{1}{f}) - N(r, \frac{1}{f}) = S(r, f) = S(r, e^h).$$

❑

Lemma 5.3.2 *Let h be a non-constant entire function and $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a polynomial, where a_0 and a_n are non-zero complex numbers. If $f(z) = P(e^{h(z)})$, then*

$$m(r, \frac{1}{f}) = S(r, e^h).$$

Proof. Write $P(z) = c\prod_{j=1}^{n}(z - \alpha_j)$. Clearly, $\alpha_j \neq 0$ for all $1 \leq j \leq n$. By Lemma 5.3.1, we have

$$\begin{aligned} m(r, \tfrac{1}{f}) &= m(r, \tfrac{1}{c\prod_{j=1}^{n}(e^h - \alpha_j)}) \\ &\leq \textstyle\sum_{j=1}^{n} m(r, \tfrac{1}{e^h - \alpha_j}) + O(1) \\ &= S(r, e^h). \end{aligned}$$

❑

In order to find $m(r, \frac{1}{P(e^h)})$, we need the following fact [37] about the characteristic function of polynomial in a meromorphic function.

Theorem 5.3.3 *Let g be a non-constant meromorphic function and $P(z) = a_n z^n + \cdots + a_0$, where $a_0, \ldots, a_n$ are small functions of g. Then*

$$T(r, P(g)) = nT(r, g) + S(r, g).$$

In particular, if g is of finite order, so is $P(g)$.

Now, we can express $m(r, \frac{1}{P(e^h)})$ in terms of $m(r, \frac{1}{e^h})$, which is fundamental to the proofs of Theorem A and B.

Theorem 5.3.4 *Let h be a non-constant entire function and $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_k z^k$ be a polynomial, where $k \geq 0$ and a_k, a_n are non-zero constants. If $f(z) = P(e^{h(z)})$, then*

$$m(r, \frac{1}{f}) = k\, m(r, \frac{1}{e^h}) + S(r, e^h).$$

Proof. Write $P(z) = z^k Q(z)$ and $Q(z) = c \prod_{j=1}^{n-k} (z - \alpha_j)$, where $\alpha_j \neq 0$ for all $1 \leq j \leq n-k$. Then, by Lemma 5.3.2 and Theorem 5.3.3,

$$\begin{aligned}
T(r, P(e^h)) &= T(r, \tfrac{1}{P(e^h)}) + O(1) \\
&= N(r, \tfrac{1}{P(e^h)}) + m(r, \tfrac{1}{P(e^h)}) + O(1) \\
&= N(r, \tfrac{1}{Q(e^h)}) + m(r, \tfrac{1}{P(e^h)}) + O(1) \\
&\leq \textstyle\sum_{j=1}^{n-k} N(r, \tfrac{1}{e^h - \alpha_j}) + m(r, \tfrac{1}{P(e^h)}) + O(1) \\
&\leq \textstyle\sum_{j=1}^{n-k} N(r, \tfrac{1}{e^h - \alpha_j}) + k\, m(r, \tfrac{1}{e^h}) + m(r, \tfrac{1}{Q(e^h)}) + O(1) \\
&\leq \textstyle\sum_{j=1}^{n-k} N(r, \tfrac{1}{e^h - \alpha_j}) + k\, m(r, \tfrac{1}{e^h}) + S(r, e^h) \\
&\leq \textstyle\sum_{j=1}^{n-k} T(r, \tfrac{1}{e^h - \alpha_j}) + k\, T(r, \tfrac{1}{e^h}) + S(r, e^h) \\
&= nT(r, e^h) + S(r, e^h) \\
&= T(r, P(e^h)) + S(r, e^h).
\end{aligned}$$

Therefore, we have equality everywhere. In particular,

$$m(r, \frac{1}{f}) = k\, m(r, \frac{1}{e^h}) + S(r, e^h).$$

❑

Now, we are ready to prove Theorem A.

Theorem A Let h be a non-constant polynomial and $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_k z^k$ be a non-constant polynomial, where $k \geq 0$ and a_k, a_n are non-zero constants. Let $f(z) = P(e^{h(z)})$. We have

(i) If $k \geq 1$, then 0 and ∞ are the only two deficient values of f. Moreover, $\delta(0, f) = \frac{\nu(0)}{n} = \frac{k}{n}$ and $\delta(\infty, f) = \frac{\nu(\infty)}{n} = 1$.

(ii) If $k = 0$, then a_0 and ∞ are the only two deficient values of f. Moreover, $\delta(a_0, f) = \frac{\nu(0)}{n}$ and $\delta(\infty, f) = \frac{\nu(\infty)}{n} = 1$.

Proof. Note that h is a polynomial, e^h is of finite order. We have $S(r, e^h) = o(T(r, e^h))$ as $r \to \infty$. Clearly, in any case, ∞ is a deficient value of f and $\delta(\infty, f) = \frac{\nu(\infty)}{n} = 1$.

For $k \geq 1$, we have $\nu(0) = k$ and, by Theorem 5.3.4,

$$\begin{aligned}
\delta(0, f) &= \liminf_{r\to\infty} \frac{m(r, \frac{1}{f})}{T(r, f)} \\
&= \liminf_{r\to\infty} \frac{km(r, \frac{1}{e^h}) + S(r, e^h)}{nT(r, e^h) + S(r, e^h)} \\
&= \tfrac{k}{n}.
\end{aligned}$$

On the other hand, for any $a \neq 0$, by Lemma 5.3.2, we have

$$\begin{aligned}
\delta(a, f) &= \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} \\
&= \liminf_{r\to\infty} \frac{S(r, e^h)}{nT(r, e^h) + S(r, e^h)} \\
&= 0.
\end{aligned}$$

Hence, 0 and ∞ are the only two deficient values of f and $\delta(0, f) = \frac{k}{n}$, $\delta(\infty, f) = 1$. This proves (i).

For $k = 0$, we can write $P(z) - a_0 = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_l z^l$, where $a_l \neq 0$ and $\nu(0) = l$. As above, we have

$$\begin{aligned}
\delta(a_0, f) &= \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a_0})}{T(r, f)} \\
&= \liminf_{r\to\infty} \frac{lm(r, \frac{1}{e^h}) + S(r, e^h)}{nT(r, e^h) + S(r, e^h)} \\
&= \tfrac{l}{n}.
\end{aligned}$$

Moreover, for any $a \neq a_0$, by Lemma 5.3.2, we have

$$\begin{aligned}
\delta(a, f) &= \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} \\
&= \liminf_{r\to\infty} \frac{S(r, e^h)}{nT(r, e^h) + S(r, e^h)} \\
&= 0.
\end{aligned}$$

Therefore, a_0 and ∞ are the only two deficient values of f and $\delta(a_0, f) = \frac{l}{n}$, $\delta(\infty, f) = 1$, which proves (ii). ❑

For general transcendental entire function h, due to the fact that $S(r, f) = o(T(r, f))$ as $r \to \infty$ and $r \notin E$, where E is a set of finite measure, we cannot get Theorem A. However, as in the proof of Theorem A, we have the following.

Theorem A′ Let h be a transcendental entire function of infinite order and $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_k z^k$ be a non-constant polynomial, where $k \geq 0$ and a_k, a_n are non-zero constants. Let $f(z) = P(e^{h(z)})$. We have

(i) If $k \geq 1$, then $\delta(0, f) \leq \frac{\nu(0)}{n} = \frac{k}{n}$ and $\delta(\infty, f) = \frac{\nu(\infty)}{n} = 1$. In particular, 0 and ∞ are the only possible deficient values of f.

(ii) If $k = 0$, then $\delta(a_0, f) \leq \frac{\nu(0)}{n}$ and $\delta(\infty, f) = \frac{\nu(\infty)}{n} = 1$. In particular, a_0 and ∞ are the only possible deficient values of f.

5.4 The Proof of Theorem B

Since 0 and ∞ are the Picard exceptional values of e^h, Theorem A says that $P(0)$ and $P(\infty)$ are the only deficient values of $f = P(e^h)$. Hence, it is reasonable to conjecture that if e^h is replaced by any meromorphic function g with two Picard exceptional values a and b, then $P(a)$ and $P(b)$ are the only deficient values of $P(g)$. Indeed, it is true. First, we need some lemmas.

Lemma 5.4.1 *Let g be a non-constant meromorphic function with two Picard exceptional values a and b. Then*

$$m(r, \frac{1}{g - \alpha}) = S(r, g)$$

for any $\alpha \in \mathbb{C}_\infty \setminus \{a, b\}$.

Proof. Given $\alpha \in \mathbb{C}_\infty \setminus \{a, b\}$. We may assume that α, a and b are finite. By the Nevanlinna's second fundamental theorem,

$$\begin{aligned} T(r, \tfrac{1}{g-\alpha}) &= T(r, g) + O(1) \\ &\leq \overline{N}(r, \tfrac{1}{g-\alpha}) + \overline{N}(r, \tfrac{1}{g-a}) + \overline{N}(r, \tfrac{1}{g-b}) + S(r, g) \\ &\leq \overline{N}(r, \tfrac{1}{g-\alpha}) + S(r, g) \\ &\leq N(r, \tfrac{1}{g-\alpha}) + S(r, g). \end{aligned}$$

Hence,

$$m(r, \frac{1}{g-\alpha}) = T(r, \frac{1}{g-\alpha}) - N(r, \frac{1}{g-\alpha}) = S(r, g).$$

❑

The following theorem is fundamental in finding the deficiency of $P(g)$.

Theorem 5.4.2 *Let g be a non-constant meromorphic function with two finite Picard exceptional values a, b and let $P(z)$ be a non-constant polynomial of degree n. We have*

(i) *If $P(a) \neq P(b)$, then*

$$m(r, \frac{1}{P(g) - P(a)}) = \nu(a) m(r, \frac{1}{g-a}) + S(r, g), \text{ and}$$

$$m(r, \frac{1}{P(g) - P(b)}) = \nu(b) m(r, \frac{1}{g-b}) + S(r, g).$$

(ii) *If $P(a) = P(b)$, then*

$$m(r, \frac{1}{P(g) - P(a)}) = (\nu(a) + \nu(b)) m(r, \frac{1}{g-a}) + S(r, g).$$

Proof. Denote $k_1 = \nu(a)$ and $k_2 = \nu(b)$. Then we can write

$$P(z) - P(a) = c(z-a)^{k_1} \prod_{i=1}^{n-k_1} (z - \alpha_i)$$

and

$$P(z) - P(b) = c(z-b)^{k_2} \prod_{j=1}^{n-k_2} (z - \beta_j),$$

where $\alpha_i \neq a$ for all $1 \leq i \leq n-k_1$, and $\beta_j \neq b$ for all $1 \leq j \leq n-k_2$.

Note that if $P(a) \neq P(b)$, then $\alpha_i \neq a, b$ for all $1 \leq i \leq n-k_1$ and $\beta_j \neq a, b$ for all $1 \leq j \leq n-k_2$. By Lemma 5.4.1 and Theorem 5.3.4, we have

$$\begin{aligned}
T(r, P(g)) &= T(r, \tfrac{1}{P(g)-P(a)}) + O(1) \\
&= N(r, \tfrac{1}{P(g)-P(a)}) + m(r, \tfrac{1}{P(g)-P(a)}) + O(1) \\
&\leq \textstyle\sum_{i=1}^{n-k_1} N(r, \tfrac{1}{g-\alpha_i}) + m(r, \tfrac{1}{P(g)-P(a)}) + O(1) \\
&\leq \textstyle\sum_{i=1}^{n-k_1} N(r, \tfrac{1}{g-\alpha_i}) + k_1\, m(r, \tfrac{1}{g-a}) + \sum_{i=1}^{n-k_1} m(r, \tfrac{1}{g-\alpha_i}) + O(1) \\
&\leq \textstyle\sum_{i=1}^{n-k_1} T(r, \tfrac{1}{g-\alpha_i}) + k_1\, T(r, \tfrac{1}{g-a}) + S(r, g) \\
&= nT(r, g) + S(r, g) \\
&= T(r, P(g)) + S(r, g).
\end{aligned}$$

Hence,

$$m(r, \frac{1}{P(g)-P(a)}) = k_1\, m(r, \frac{1}{g-a}) + S(r, g).$$

Similarly, we have

$$m(r, \frac{1}{P(g)-P(b)}) = k_2\, m(r, \frac{1}{g-b}) + S(r, g).$$

This proves (i).

If $P(a) = P(b)$, then we can write

$$P(z) - P(a) = c(z-a)^{k_1}(z-b)^{k_2} \prod_{j=1}^{n-k_1-k_2} (z-\gamma_j),$$

where $\gamma_j \neq a, b$ for all $1 \leq j \leq n-k_1-k_2$. As in the proof of (i), we still get

$$m(r, \frac{1}{P(g)-P(a)}) = (k_1+k_2) m(r, \frac{1}{g-a}) + S(r, g),$$

which proves (ii). ❑

In Theorem 5.4.2, we assume that both a and b are finite values. If one of a and b is ∞, say $b = \infty$, then $P(a) \neq P(b)$ and $P(g)$ is entire. So, as in the proof of Theorem 5.4.2, we have

Theorem 5.4.2′ Let g be a non-constant meromorphic function with two Picard exceptional values a and ∞ and let $P(z)$ be a non-constant polynomial of degree n. Then we have

$$m(r, \frac{1}{P(g) - P(a)}) = \nu(a)m(r, \frac{1}{g-a}) + S(r, g)$$

and

$$m(r, P(g)) = \nu(\infty)m(r, g) + S(r, g) = T(r, P(g)).$$

Now, we are in the position to prove Theorem B.

Theorem B Let g be a non-constant meromorphic function of finite order, such that g has two Picard exceptional values a and b. Let $P(z)$ be a non-constant polynomial of degree n. We have

(i) If $P(a) = P(b)$, then $P(a)$ is the only deficient value of $P(g)$. Moreover, $\delta(P(a), P(g)) = \frac{\nu(a)+\nu(b)}{n}$.

(ii) If $P(a) \neq P(b)$ and a, b are finite, then $P(a)$ and $P(b)$ are the only two deficient values of $P(g)$. Moreover, $\delta(P(a), P(g)) = \frac{\nu(a)}{n}$ and $\delta(P(b), P(g)) = \frac{\nu(b)}{n}$.

(iii) If a is finite and $b = \infty$, then $P(a)$ and ∞ are the only two deficient values of $P(g)$. Moreover, $\delta(P(a), P(g)) = \frac{\nu(a)}{n}$ and $\delta(\infty, P(g)) = \frac{\nu(\infty)}{n} = 1$.

Proof. Note that g is of finite order, so is $P(g)$ by Theorem 5.3.3. We have $S(r, g) = o(T(r, g))$ as $r \to \infty$.

If $P(a) = P(b)$, then a and b must be finite values. By Theorem 5.4.2, we get

$$\delta(P(a), P(g)) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{P(g)-P(a)})}{T(r, P(g))} = \frac{\nu(a) + \nu(b)}{n}.$$

On the other hand, for any $\alpha \neq P(a)$, we can write $P(z) - \alpha = c\prod_{j=1}^{n}(z - \alpha_j)$, where $\alpha_j \neq a, b$ for all $1 \leq j \leq n$. Then, by Lemma 5.4.1, we have

$$\delta(\alpha, P(g)) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{P(g)-\alpha})}{T(r, P(g))} = \liminf_{r\to\infty} \frac{S(r, g)}{T(r, P(g))} = 0.$$

Therefore, $P(a)$ is the only deficient value of $P(g)$ and $\delta(P(a), P(g)) = \frac{\nu(a)+\nu(b)}{n}$. This proves (i).

If $P(a) \neq P(b)$ and a, b are finite, then, by Theorem 5.4.2, we have

$$\delta(P(a), P(g)) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{P(g)-P(a)})}{T(r, P(g))} = \frac{\nu(a)}{n}$$

and

$$\delta(P(b), P(g)) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{P(g)-P(b)})}{T(r, P(g))} = \frac{\nu(b)}{n}.$$

Moreover, as in the proof of (i), for any $\alpha \neq P(a), P(b)$, we have

$$\delta(\alpha, P(g)) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{P(g)-\alpha})}{T(r, P(g))} = \liminf_{r\to\infty} \frac{S(r, g)}{T(r, P(g))} = 0.$$

Therefore, $P(a)$ and $P(b)$ are the only deficient values of $P(g)$ and $\delta(P(a), P(g)) = \frac{\nu(a)}{n}$, $\delta(P(b), P(g)) = \frac{\nu(b)}{n}$. This proves (ii).

Finally, if a is finite and $b = \infty$, then, by Theorem 5.4.2′, we have

$$\delta(P(a), P(g)) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{P(g)-P(a)})}{T(r, P(g))} = \frac{\nu(a)}{n}$$

and

$$\delta(\infty, P(g)) = \liminf_{r\to\infty} \frac{m(r, P(g))}{T(r, P(g))} = \frac{\nu(\infty)}{n} = 1.$$

Moreover, as in the proof of (i), for any $\alpha \neq P(a), P(b)$, we have

$$\delta(\alpha, P(g)) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{P(g)-\alpha})}{T(r, P(g))} = \liminf_{r\to\infty} \frac{S(r, g)}{T(r, P(g))} = 0.$$

Therefore, $P(a)$ and ∞ are the only deficient values of $P(g)$ and $\delta(P(a), P(g)) = \frac{\nu(a)}{n}$, $\delta(P(a), P(g)) = 1$, which proves (iii). ❑

For arbitrary meromorphic function g with two Picard exceptional values, as the reasoning in the end of section 5.3, we have the following result.

Theorem B′ Let g be a non-constant meromorphic function of infinite order, such that g has two Picard exceptional values a and b. Let $P(z)$ be a non-constant polynomial of degree n. We have

(i) If $P(a) = P(b)$, then $\delta(P(a), P(g)) \leq \frac{\nu(a)+\nu(b)}{n}$. In particular, $P(a)$ is the only possible deficient value of $P(g)$.

(ii) If $P(a) \neq P(b)$ and a, b are finite, then $\delta(P(a), P(g)) \leq \frac{\nu(a)}{n}$ and $\delta(P(b), P(g)) \leq \frac{\nu(b)}{n}$. In particular, $P(a)$ and $P(b)$ are the only possible deficient values of $P(g)$.

(iii) If a is finite and $b = \infty$, then $\delta(P(a), P(g)) \leq \frac{\nu(a)}{n}$ and $\delta(\infty, P(g)) = \frac{\nu(\infty)}{n} = 1$. In particular, $P(a)$ and ∞ are the only possible deficient values of $P(g)$.