Chapter 7

On the Uniqueness of Entire Functions and Their Derivatives

7.1 Introduction

The uniqueness problems on entire functions that share a finite non-zero value a with their derivatives has been broadly studied. In 1977, Rubel and C. C. Yang [29] proved that if an entire function f share two finite, distinct values CM with f', then $f \equiv f'$. This result has been generalized to the case that f and f' share two values IM by Gundersen [9] and by Mues-Steinmetz [20]. In 1986, Jank, Mues and Volkmann [19] proved the following theorem.

Theorem A Let f be a non-constant entire function. If f and f' share the value a $(a \neq 0)$ IM, and $f''(z) = a$ whenever $f(z) = a$, then $f \equiv f'$.

Gundersen and L. Z. Yang [11, 39] considered the sharing value problems of f and its nth-order derivative $f^{(n)}$, and obtained the following result.

Theorem B Let f be a non-constant entire function of finite order, let a be a finite non-zero value and let n be a positive integer. If f and $f^{(n)}$ share a CM, then

$(f^{(n)} - a)/(f - a) = c$, where c is a non-zero constant.

The next two theorems are proved by Li and C. C. Yang [16].

Theorem C Let f be a non-constant entire function, let a be a finite non-zero value and let n be a positive integer. If f, $f^{(n)}$ and $f^{(n+1)}$ share a CM, then $f = f'$.

Theorem D Let f be a non-constant entire function, let a be a finite non-zero value and let $n \geq 2$ be a positive integer. If f, f' and $f^{(n)}$ share a CM, then

$$f(z) = be^{cz} - \frac{a(1-c)}{c},$$

where b, c are non-zero constants and $c^{n-1} = 1$.

In this chapter, we study non-constant entire functions which share opposite values with their derivatives. To explain more precisely, we use a terminology introduced by Frank and Ohlenroth in [6]. Let f and g be two non-constant entire functions, a_1 and a_2 be two arbitrary complex numbers. We say that f and g share the pair (a_1, a_2) CM (IM) if $f - a_1$ and $g - a_2$ share 0 CM (IM). Under this terminology, given an entire function f and a positive integer n, f and $f^{(n)}$ share the pair $(a, -a)$ CM (IM) means that $f - a$ and $f^{(n)} + a$ have the same zeros with the same multiplicities (without counting multiplicities).

The main results of this chapter is to reformulate all theorems A-D in the sense of non-constant entire functions sharing opposite values with their derivatives, and give their proofs by using similar techniques in the original proofs in [19, 11, 16, 39].

7.2 Lemmas and Known Results

In this section, we summarize some facts that we need in the proofs of our main theorems.

Lemma 7.2.1 [24] *Let $g_1, \ldots, g_p$ be transcendental entire functions and $a_1, \ldots, a_p$ be non-zero constants. If $\sum_{i=1}^{p} a_i g_i(z) = 1$, then $\sum_{i=1}^{p} \delta(0, g_i) \leq p - 1$.*

Lemma 7.2.2 *Let f be a non-constant entire function, and let $a \neq 0$ be a finite value. If f and f' share the pair $(a, -a)$ IM, then a is not a Picard exceptional value of f.*

Proof. If a is a Picard exceptional value of f, then $-a$ is a Picard exceptional value of f'. Write $f - a = e^{\alpha}$ and $f' + a = e^{\beta}$ for some non-constant entire functions α and β, we have

$$-\frac{1}{a}\alpha' e^{\alpha} + \frac{1}{a} e^{\beta} = 1.$$

By Lemma 7.2.1, we get a contradiction. Hence, a is not a Picard exceptional value of f. ❑

Lemma 7.2.3 *Let f be a non-constant entire function, and let $a \neq 0$ be a finite value. If f and f' share the pair $(a, -a)$ IM and $f''(z) = a$ whenever $f(z) = a$, then f and f' share $(a, -a)$ CM, and all zeros of $f - a$ and $f' + a$ are simple.*

Proof. By Lemma 7.2.2, a is not a Picard exceptional value of f. Let z_0 be a zero of $f - a$, then $f'(z_0) = -a \neq 0$ and $f''(z_0) = a \neq 0$. Hence, all the zeros of $f - a$ and $f' + a$ are simple. In particularly, $f - a$ and $f' + a$ share 0 CM, that is, f and f' share $(a, -a)$ CM. ❑

Lemma 7.2.4 [12] *Let f be a meromorphic function, let n be a positive integer and let F be a function of the form $F = f^n + Q(f)$, where $Q(f)$ is a differential polynomial in f with degree less or equal to $n - 1$. If*

$$N(r, f) + N\left(r, \frac{1}{F}\right) = S(r, f),$$

then

$$F = \left(f + \frac{g}{n}\right)^n,$$

where g is a meromorphic function satisfying $T(r, g) = S(r, f)$.

Lemma 7.2.5 [3, 5] *Let f be a non-constant meromorphic function and let $P(f)$, $Q(f)$ be differential polynomials in f with $P(f) \not\equiv 0$. Let n be a positive integer and*

$$f^n Q(f) = P(f).$$

If the degree of $P(f)$ is not greater than n, then

$$m(r, Q(f)) = S(r, f).$$

Lemma 7.2.6 [11, 39] *Let h be a non-constant polynomial and n be a positive integer. Then every solution F of the differential equation*

$$F^{(n)} - e^h F = 1$$

is an entire function of infinite order.

Lemma 7.2.7 [15] *Let φ be a non-zero entire function. If $\varphi^n + P(\varphi) \equiv 0$, where $P(\varphi)$ is a differential polynomial in φ with constant coefficients, and the degree of $P(\varphi)$ is at most $n-1$, then φ is a constant.*

7.3 Main Results and Proofs

Theorem 7.3.1 *Let f be a non-constant entire function, and let $a \neq 0$ be a finite value. If f and f' share the pair $(a, -a)$ IM, and $f''(z) = a$ whenever $f(z) = a$, then $f \equiv -f'$.*

Proof. By Lemma 7.2.2, a is not a Picard exceptional value of f. Let z_0 be a zero of $f - a$ and write

$$f(z) = a + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots . \tag{7.3.1}$$

Then

$$f'(z) = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots , \tag{7.3.2}$$

and

$$f''(z) = 2a_2 + 6a_3(z - z_0) + \cdots . \tag{7.3.3}$$

From (7.3.1), (7.3.2), (7.3.3) and the assumptions, we get $a_1 = -a$ and $2a_2 = a$. Therefore, z_0 is a multiple zero of $f + f'$. If $f \not\equiv -f'$, then, by Lemma 7.2.3, we have

$$\begin{aligned}
2N\left(r, \frac{1}{f-a}\right) &\le N\left(r, \frac{1}{f+f'}\right) \\
&\le T(r, f+f') + O(1) \\
&= m\left(r, f\left(1 + \frac{f'}{f}\right)\right) + O(1) \\
&\le m(r,f) + S(r,f) \\
&= T(r,f) + S(r,f).
\end{aligned}$$

Note that

$$\begin{aligned}
m\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{1}{f'+a}\right) &\le m\left(r, \frac{1}{f'}\right) + m\left(r, \frac{1}{f'+a}\right) + S(r,f) \\
&= m\left(r, \frac{1}{f'} + \frac{1}{f'+a}\right) + S(r,f) \\
&\le m\left(r, \frac{1}{f''}\right) + S(r,f).
\end{aligned}$$

From above inequalities, we have

$$\begin{aligned}
T(r,f) + T(r,f') &= N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f'+a}\right) + m\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{1}{f'+a}\right) + O(1) \\
&\le N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f'+a}\right) + m\left(r, \frac{1}{f''}\right) + S(r,f) \\
&= 2N\left(r, \frac{1}{f-a}\right) + T(r, f'') - N\left(r, \frac{1}{f''}\right) + S(r,f) \\
&\le 2N\left(r, \frac{1}{f-a}\right) + T(r, f') - N\left(r, \frac{1}{f''}\right) + S(r,f) \\
&\le T(r,f) + T(r,f') - N\left(r, \frac{1}{f''}\right) + S(r,f),
\end{aligned}$$

which implies that

$$N\left(r, \frac{1}{f''}\right) = S(r,f) \tag{7.3.4}$$

and

$$T(r,f) \le 2N\left(r, \frac{1}{f-a}\right) + S(r,f). \tag{7.3.5}$$

By assumption, we have

$$\begin{aligned} N\left(r, \frac{1}{f-a}\right) &\le N\left(r, \frac{1}{\frac{f''}{f'}+1}\right) \\ &\le T\left(r, \frac{f''}{f'}\right) + O(1) \\ &= N\left(r, \frac{f''}{f'}\right) + S(r,f) \\ &= \overline{N}\left(r, \frac{1}{f'}\right) + S(r,f). \end{aligned}$$

From this and (7.3.5), we obtain

$$T(r,f) \le 2\overline{N}\left(r, \frac{1}{f'}\right) + S(r,f). \tag{7.3.6}$$

Set

$$\phi = \frac{f'}{f-a} - \frac{f''}{f'+a}, \qquad \psi = \frac{f''+f'}{f-a}. \tag{7.3.7}$$

By assumption, we know that ϕ and ψ are entire functions satisfying

$$T(r,\phi) = m(r,\phi) = S(r,f)$$

and

$$T(r,\psi) = m(r,\psi) = S(r,f).$$

Let z_0 be a zero of $f-a$. From (7.3.1) and (7.3.7), we get

$$\phi(z_0) = \frac{1}{2}\left(-1 - \frac{1}{a}f'''(z_0)\right)$$

and

$$\psi(z_0) = -\frac{1}{a}f'''(z_0) - 1.$$

Therefore,

$$2\phi(z_0) - \psi(z_0) = 0. \tag{7.3.8}$$

If $2\phi - \psi \not\equiv 0$, then (7.3.8) implies that

$$N\left(r, \frac{1}{f-a}\right) \le N\left(r, \frac{1}{2\phi-\psi}\right) \le T(r,\phi) + T(r,\psi) + O(1) = S(r,f),$$

which contradicts to (7.3.5). Hence

$$2\phi - \psi \equiv 0. \tag{7.3.9}$$

Let z_1 be a zero of f' but $f''(z_1) \neq 0$. Such z_1 does exist by (7.3.4) and (7.3.6). From (7.3.7) and (7.3.9), we have

$$0 = 2\phi(z_1) - \psi(z_1) = -f''(z_1)\left(\frac{2}{a} + \frac{1}{f(z_1) - a}\right).$$

Hence

$$f(z_1) = \frac{a}{2}. \tag{7.3.10}$$

From (7.3.9), we get

$$2\phi' - \psi' \equiv 0.$$

From (7.3.7) and (7.3.10), we obtain

$$0 = 2\phi'(z_1) - \psi'(z_1) = \frac{2}{a} f''(z_1)\left(\frac{f''(z_1)}{a} - 1\right),$$

Hence

$$f''(z_1) = a \text{ and } \phi(z_1) = -1. \tag{7.3.11}$$

If $\phi \not\equiv -1$, then, from (7.3.11), we have

$$\begin{aligned}\overline{N}\left(r, \frac{1}{f'}\right) - N\left(r, \frac{1}{f''}\right) &\leq N\left(r, \frac{1}{\phi + 1}\right) \\ &\leq T(r, \phi) + O(1) = S(r, f). \end{aligned} \tag{7.3.12}$$

On the other hand, from (7.3.4) and (7.3.6), we obtain

$$T(r, f) \leq 2\overline{N}\left(r, \frac{1}{f'}\right) - 2N\left(r, \frac{1}{f''}\right) + S(r, f).$$

Combining this with (7.3.12), we get

$$T(r, f) = S(r, f).$$

This is a contradiction. Therefore, $\phi \equiv -1$ and, then, $\psi \equiv -2$ by (7.3.9). Substituting this into (7.3.7), we obtain the second order differential equation

$$f'' + f' + 2(f - a) = 0, \tag{7.3.13}$$

which has solutions

$$f(z) = c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + a, \tag{7.3.14}$$

where λ_1 and λ_2 are the distinct roots of the equation $\lambda^2 + \lambda + 2 = 0$, c_1 and c_2 are constants. Hence

$$f''(z) = c_1 \lambda_1^2 e^{\lambda_1 z} + c_2 \lambda_2^2 e^{\lambda_2 z},$$

which implies that one of c_1 and c_2 must be zero by (7.3.4). Without loss of generality, we assume that $c_2 = 0$. Then

$$f(z) = c_1 e^{\lambda_1 z} + a.$$

Obviously, $c_1 \neq 0$, so a is a Picard exceptional value of f, which is impossible by Lemma 7.2.2. Hence, $f \equiv -f'$, and the theorem is proved. ❑

From Theorem 7.3.1, we immediately have the following consequence.

Corollary 7.3.2 *Let f be a non-constant entire function, and let $a \neq 0$ be a finite value. If f, f' share the pair $(a, -a)$ CM and f, f'' share a CM, then $f \equiv -f'$.*

Now, we extend the corollary to the case of higher order derivatives of f.

Theorem 7.3.3 *Let f be a non-constant entire function, $a \neq 0$ be a finite value, and let n be a positive integer. If f, $f^{(n)}$ share the pair $(a, -a)$ CM and f, $f^{(n+1)}$ share a CM, then $f^{(n)} \equiv -f^{(n+1)}$. More precisely,*

$$f(z) = \begin{cases} ce^{-z} + 2a & \text{if } n \text{ is even} \\ ce^{-z} & \text{if } n \text{ is odd,} \end{cases}$$

where c is a non-zero constant.

Proof. If $f^{(n)} \not\equiv -f^{(n+1)}$, then, by the hypothesis,

$$\frac{f^{(n)} + a}{f - a} = e^{\alpha}, \quad \frac{f^{(n+1)} - a}{f - a} = e^{\beta}, \tag{7.3.15}$$

where α and β are entire functions. Set

$$\varphi = \frac{f^{(n)} + f^{(n+1)}}{f - a}. \tag{7.3.16}$$

Then φ is a non-zero entire function satisfying $T(r, \varphi) = S(r, f)$. Note that $\dfrac{e^{\alpha}}{\varphi} + \dfrac{e^{\beta}}{\varphi} = 1$. By Nevanlinna's second fundamental theorem, we get

$$\begin{aligned}
T\left(r, \frac{e^{\alpha}}{\varphi}\right) &\leq \overline{N}\left(r, \frac{e^{\alpha}}{\varphi}\right) + \overline{N}\left(r, \frac{\varphi}{e^{\alpha}}\right) + \overline{N}\left(r, \frac{1}{\frac{e^{\alpha}}{\varphi} - 1}\right) + S\left(r, \frac{e^{\alpha}}{\varphi}\right) \\
&= \overline{N}\left(r, \frac{e^{\alpha}}{\varphi}\right) + \overline{N}\left(r, \frac{\varphi}{e^{\alpha}}\right) + \overline{N}\left(r, \frac{\varphi}{e^{\beta}}\right) + S\left(r, \frac{e^{\alpha}}{\varphi}\right) \\
&= \overline{N}\left(r, \frac{1}{\varphi}\right) + \overline{N}(r, \varphi) + \overline{N}(r, \varphi) + S\left(r, \frac{e^{\alpha}}{\varphi}\right) \\
&\leq 3T(r, \varphi) + S\left(r, \frac{e^{\alpha}}{\varphi}\right) \\
&= S(r, f).
\end{aligned}$$

Hence

$$T(r, e^{\alpha}) \leq T\left(r, \frac{e^{\alpha}}{\varphi}\right) + T(r, \varphi) = S(r, f).$$

Similarly, we have $T(r, e^{\beta}) = S(r, f)$. From (7.3.15), we have

$$e^{\alpha}(f - a) = f^{(n)} + a, \quad f^{(n+1)} = e^{\beta}(f - a) + a.$$

We deduce that

$$\alpha' e^{\alpha}(f - a) + e^{\alpha} f' = f^{(n+1)} = e^{\beta}(f - a) + a.$$

Hence

$$f' = \alpha_1 f + \beta_1, \tag{7.3.17}$$

where $\alpha_1 = e^{\beta - \alpha} - \alpha'$ and $\beta_1 = ae^{-\alpha} - ae^{\beta - \alpha} + a\alpha'$. Note that α_1 and β_1 are entire functions satisfying $T(r, \alpha_1) = S(r, f)$ and $T(r, \beta_1) = S(r, f)$. Taking derivatives on equation (7.3.17), we get

$$f^{(k)} = \alpha_k f + \beta_k \tag{7.3.18}$$

for $k = 1, 2, \ldots$, where α_k and β_k are entire functions satisfying the recursive formulas

$$\alpha_{k+1} = \alpha_k' + \alpha_1\alpha_k, \quad \beta_{k+1} = \beta_k' + \beta_1\alpha_k \tag{7.3.19}$$

for $k = 1, 2, \ldots$. Clearly, $T(r, \alpha_k) = S(r, f)$ and $T(r, \beta_k) = S(r, f)$ for $k = 1, 2, \ldots$. From (7.3.15), we have

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) &= m\left(r, \frac{1}{a}\left(e^{\alpha} - \frac{f^{(n)}}{f-a}\right)\right) \\ &\le m(r, e^{\alpha}) + m\left(r, \frac{f^{(n)}}{f-a}\right) + O(1) \\ &= S(r, f). \end{aligned}$$

Therefore,

$$m\left(r, \frac{1}{f-a}\right) = S(r, f). \tag{7.3.20}$$

By hypothesis, we know that the multiplicities of the zeros of $f - a$ are at most n. So, by (7.3.20), we have

$$\begin{aligned} T(r, f) &= N\left(r, \frac{1}{f-a}\right) + S(r, f) \\ &\le n\overline{N}\left(r, \frac{1}{f-a}\right) + S(r, f). \end{aligned}$$

Again, by hypothesis, we know that the zeros of $f - a$ are the zeros of $a\alpha_n + \beta_n + a$ and $a\alpha_{n+1} + \beta_{n+1} - a$. If $a\alpha_n + \beta_n + a \not\equiv 0$, then

$$\begin{aligned} T(r, f) &\le n\overline{N}\left(r, \frac{1}{f-a}\right) + S(r, f) \\ &\le nN\left(r, \frac{1}{a\alpha_n + \beta_n + a}\right) \\ &= S(r, f), \end{aligned}$$

which is a contradiction. Hence

$$a\alpha_n + \beta_n + a \equiv 0. \tag{7.3.21}$$

Similarly, we have

$$a\alpha_{n+1} + \beta_{n+1} - a \equiv 0. \tag{7.3.22}$$

Combine (7.3.15), (7.3.18), (7.3.21) and (7.3.22), we easily get

$$\alpha_n = e^{\alpha}, \quad \alpha_{n+1} = e^{\beta}.$$

If α_1 is constant, then, by (7.3.19), we get $\alpha_k = \alpha_1^k$ for $k = 1, 2, \ldots$. Hence, e^{α} and e^{β} are constants. It follows that β_1 is also a constant. Again, by (7.3.19), we get $\beta_k = \beta_1\alpha_1^{k-1}$. In this case, (7.3.21) and (7.3.22) become

$$\alpha_1^{n-1}(a\alpha_1 + \beta_1) = -a, \quad \alpha_1^n(a\alpha_1 + \beta_1) = a.$$

From which we deduce that $\alpha_1 = -1, \beta_1 = 2a$ if n is even and $\alpha_1 = -1, \beta_1 = 0$ if n is odd. In any case, from (7.3.17), we conclude that $f^{(n)} \equiv -f^{(n+1)}$ which is impossible by the assumption.

Now, we assume that α_1 is not a constant. By (7.3.19) and by induction, we get

$$\alpha_k = \alpha_1^k + \frac{k(k-1)}{2}\alpha_1^{k-2}\alpha_1' + P_{k-2}, \tag{7.3.23}$$

where P_{k-2} is a differential polynomial in α_1 with degree not greater than $k-2$ for $k \geq 2$. Note that

$$N(r, \alpha_1) + N\left(r, \frac{1}{\alpha_n}\right) + N\left(r, \frac{1}{\alpha_{n+1}}\right) = 0.$$

By Lemma 7.2.4, there exists two entire functions g_1 and g_2 such that

$$\alpha_n = \left(\alpha_1 + \frac{g_1}{n}\right)^n, \quad \alpha_{n+1} = \left(\alpha_1 + \frac{g_2}{n+1}\right)^{n+1}, \tag{7.3.24}$$

and $T(r, g_1) + T(r, g_2) = S(r, \alpha_1)$. Therefore, $\alpha_1 + \frac{g_1}{n}$ and $\alpha_1 + \frac{g_2}{n+1}$ have no zeros. If $\frac{g_1}{n} \not\equiv \frac{g_2}{n+1}$, then, by Nevanlinna's second fundamental theorem for small functions, we have

$$\begin{aligned} T(r, \alpha_1) &\leq \overline{N}\left(r, \frac{1}{\alpha_1 + \frac{g_1}{n}}\right) + \overline{N}\left(r, \frac{1}{\alpha_1 + \frac{g_2}{n+1}}\right) + \overline{N}(r, \alpha_1) + S(r, \alpha_1) \\ &= S(r, \alpha_1), \end{aligned}$$

which is impossible. Hence, $\frac{g_1}{n} \equiv \frac{g_2}{n+1}$. By (7.3.24), we get

$$\alpha_n^{n+1} \equiv \alpha_{n+1}^n.$$

Therefore, from (7.3.23), we obtain

$$\alpha_n^{n+1} = \alpha_1^{n(n+1)} + \frac{n(n-1)(n+1)}{2}\alpha_1^{n^2+n-2}\alpha_1' + Q_1,$$

and

$$\alpha_{n+1}^{n} = \alpha_1^{n(n+1)} + \frac{n^2(n+1)}{2}\alpha_1^{n^2+n-2}\alpha_1' + Q_2,$$

where Q_1 and Q_2 are differential polynomials in α_1 with constant coefficients and degrees not greater than n^2+n-2. Hence

$$\frac{n^2+n}{2}\alpha_1^{n^2+n-2}\alpha_1' = Q_1 - Q_2.$$

By Lemma 7.2.5, we get $m(r,\alpha_1') = S(r,\alpha_1)$. Thus

$$T(r,\alpha_1') = S(r,\alpha_1). \tag{7.3.25}$$

From (7.3.19) and (7.3.24), and $\dfrac{g_2}{n+1} = \dfrac{g_1}{n}$, we get

$$\begin{aligned}\left(\alpha_1 + \frac{g_1}{n}\right)^{n+1} &= \left(\alpha_1 + \frac{g_2}{n+1}\right)^{n+1} \\ &= \alpha_{n+1} \\ &= \alpha_n' + \alpha_1\alpha_n \\ &= n\left(\alpha_1 + \frac{g_1}{n}\right)^{n-1}\left(\alpha_1' + \frac{g_1'}{n}\right) + \alpha_1\left(\alpha_1 + \frac{g_1}{n}\right)^{n}.\end{aligned}$$

Therefore

$$\left(\alpha_1 + \frac{g_1}{n}\right)^{n-1}\left(\frac{g_1}{n}\alpha_1 - n\alpha_1' - g_1' + \frac{g_1^2}{n^2}\right) = 0.$$

Since $\alpha_1 + \dfrac{g_1}{n}$ has no zeros, we conclude that

$$\frac{g_1}{n}\alpha_1 - n\alpha_1' - g_1' + \frac{g_1^2}{n^2} \equiv 0,$$

that is,

$$\alpha_1 = n^2\frac{\alpha'}{g_1} + n\frac{g'}{g_1} - \frac{1}{n}g_1.$$

From $T(r,g_1) = S(r,\alpha_1)$ and (7.3.25), we get $T(r,\alpha_1) = S(r,\alpha_1)$, which is also impossible. Therefore, $f^{(n)} \equiv -f^{(n+1)}$, that is, $f^{(n)}(z) = ce^{-z}$ and $f(z) =$

$(-1)^n ce^{-z} + P(z)$, where c is a non-zero constant and $P(z)$ is a polynomial with degree not greater than $n-1$. Choose n distinct zeros z_i of $f^{(n)}(z)+a$, $1 \le i \le n$. It is easy to see that if n is even, then $P(z_i) = 2a$ for all $1 \le i \le n$, and if n is odd, then $P(z_i) = 0$ for all $1 \le i \le n$. Since the degree of $P(z)$ is at most n, we conclude that $P(z) \equiv 2a$ when n is even, and $P(z) \equiv 0$ when n is odd. Therefore,

$$f(z) = \begin{cases} ce^{-z} + 2a & \text{if } n \text{ is even,} \\ ce^{-z} & \text{if } n \text{ is odd,} \end{cases}$$

and the theorem is proved. ❑

Theorem 7.3.4 *Let f be a non-constant entire function of finite order and let a be a finite value. If f and f' share the pair $(a, -a)$ CM, then*

$$\frac{f'+a}{f-a} = c,$$

where c is a non-zero constant.

Proof. Case 1. $a \neq 0$. Since f and f' share $(a,-a)$ CM and f is of finite order, there exists a polynomial h such that

$$\frac{f'+a}{f-a} = e^h.$$

Set $F = 1 - \dfrac{f}{a}$. Then

$$F' - e^h F = 1.$$

If h is non-constant, then, by Lemma 7.2.6 ($n = 1$), we obtain that F is of infinite order. Since f is of finite order, it is a contradiction. Therefore, h is a constant and

$$\frac{f'+a}{f-a} = c,$$

where c is a non-zero constant.

Case 2. $a = 0$. In this case, the hypothesis says that f and f' share 0 CM, so 0 must be a Picard exceptional value of f and f', and $f = e^h$ for some non-constant entire function h. Since f is of finite order, h must be a polynomial. Since $f' = h'e^h$

has no zeros, $h' \equiv c$ for some constant $c \neq 0$. Hence, $f' = cf$, and the theorem is proved. ❑

If we replace f' by $f^{(n)}$ in the proof of Theorem 7.3.4, then we can easily conclude the following theorem.

Theorem 7.3.5 *Let f be a non-constant entire function of finite order, let a be a finite non-zero value and let n be a positive integer. If f and $f^{(n)}$ share the pair $(a, -a)$ CM, then*

$$\frac{f^{(n)} + a}{f - a} = c,$$

where c is a non-zero constant.

Theorem 7.3.6 *Let f be an entire function, let a be a finite non-zero value and let $n \geq 2$ be a positive integer. If f and f' share the pair $(a, -a)$ CM and f, $f^{(n)}$ share a CM, then $f' \equiv -f^{(n)}$. More precisely,*

$$f(z) = be^{cz} + \frac{a(1+c)}{c},$$

where b, c are non-zero constants and $c^{n-1} = -1$.

Proof. Suppose $f' \not\equiv -f^{(n)}$. Since f, f' share $(a, -a)$ CM and f, $f^{(n)}$ share a CM, similar to the proof of Theorem 7.3.3, there exists an entire function α such that

$$e^{\alpha} = \frac{f' + a}{f - a} \tag{7.3.26}$$

and $T(r, e^{\alpha}) = S(r, f)$. Rewriting (7.3.26) as

$$f' = e^{\alpha} f - a - ae^{\alpha}$$

and taking the derivatives, we get

$$f^{(k)} = \alpha_k f + \beta_k, \tag{7.3.27}$$

$k = 1, 2, \ldots,$ where $\alpha_1 = e^{\alpha}$, $\beta_1 = -(a + ae^{\alpha})$ and we can get the the recursive formulas

$$\alpha_{k+1} = \alpha_k' + \alpha_1\alpha_k, \quad \beta_{k+1} = \beta_k' + \beta_1\alpha_k \tag{7.3.28}$$

as in the proof of Theorem 7.3.3. Clearly, α_k and β_k are entire functions satisfying $T(r, \alpha_k) = S(r, f)$ and $T(r, \beta_k) = S(r, f)$. By the hypothesis and (7.3.27), all zeros of $f - a$ are simple and are also zeros of $a\alpha_n + \beta_n - a$. As in the proof of Theorem 7.3.3, (7.3.26) and $T(r, e^{\alpha}) = S(r, f)$ implies $m\left(r, \dfrac{1}{f-a}\right) = S(r, f)$. Therefore, $N\left(r, \dfrac{1}{f-a}\right) \neq S(r, f)$. If $a\alpha_n + \beta_n - a \not\equiv 0$, then we have

$$\begin{aligned} N\left(r, \frac{1}{f-a}\right) &\leq N\left(r, \frac{1}{a\alpha_n + \beta_n - a}\right) \\ &\leq T(r, \alpha_n) + T(r, \beta_n) + O(1) \\ &= S(r, f), \end{aligned}$$

which is impossible. So $a\alpha_n + \beta_n - a \equiv 0$. On the other hand, by the recursive formulas (7.3.28) and by induction, it follows that

$$a\alpha_k + \beta_k = -a\left(\alpha_{k-1} + \alpha_{k-2}' + \cdots + \alpha_1^{(k-2)}\right),$$

$k = 2, 3, \ldots$. In particular, we have

$$\alpha_{n-1} + \alpha_{n-2}' + \cdots + \alpha_1^{(n-2)} \equiv -1. \tag{7.3.29}$$

It follows from (7.3.28), the equation (7.3.29) can be expressed as

$$\alpha_1^{n-1} + P(\alpha_1) \equiv 0,$$

where $P(\alpha_1)$ is a differential polynomial in α_1 with degree not greater than $n - 2$. By Lemma 7.2.7, we conclude that α_1 is a constant. Therefore, from (7.3.28) and (7.3.29), we obtain $\alpha_1^{n-1} = -1$ and $\alpha_k = \alpha_1^k$, for $k = 1, 2, \ldots$, which imply $\alpha_n = -\alpha_1$ and $\beta_n = -\beta_1$. Again, from (7.3.27), we have

$$\begin{aligned} f^{(n)} &= \alpha_n f + \beta_n \\ &= -\alpha_1 f - \beta_1 \\ &= -f', \end{aligned}$$

which contradicts to the assumption. Therefore, $f' \equiv -f^{(n)}$. Hence

$$f(z) = \sum_{j=1}^{n} c_j e^{\lambda_j} z,$$

where $\lambda_1, \ldots, \lambda_n$ are distinct roots of $\lambda^n + \lambda = 0$ and $c_1, \ldots, c_n$ are constants. It follows that f is of finite order. Since f and f' share the pair $(a, -a)$ CM, by Theorem 7.3.4, there exists a non-zero constant c such that $f' + a = c(f - a)$. Hence

$$f(z) = be^{cz} + \frac{a(1+c)}{c},$$

and

$$f^{(n)}(z) = bc^n e^{cz} = c^n \left(f - \frac{a(1+c)}{c} \right),$$

where b is a non-zero constant. Let z_0 be a zero of $f^{(n)} - a$. Then

$$a = f^{(n)}(z_0) = bc^n e^{cz_0} = c^n \left(f(z_0) - \frac{a(1+c)}{c} \right) = -c^{n-1}a.$$

So $c^{n-1} = -1$, and the theorem is proved. ❑