Chapter 4

On a Conjecture of C. C. Yang

4.1 Introduction

We say that two polynomials f and g share the value $a \in \mathbb{C}_\infty$ provided that if $f(z) = a$ if, and only if $g(z) = a$. We will state whether a shared value is by CM(counting multiplicities) or by IM(ignoring multiplicities).

For the sharing value problem of polynomials, we have the following simple well-known results [35].

Theorem 4.1.1 *Let f and g be non-constant polynomials and a be a finite complex number. If f and g share a CM, then there exists a non-zero constant K such that $f - a \equiv K(g - a)$.*

Corollary 4.1.2 Let f and g be non-constant polynomials and a be a finite complex number. If f and g share a CM, and if there exists a point z_0 such that $f(z_0) = g(z_0) \neq a$, then $f \equiv g$.

For the case of sharing values IM, Adams-Straus[1] proved the following result.

Theorem 4.1.3 *Let f and g be non-constant polynomials and a, b be distinct finite complex numbers. If f and g share a and b* IM, *then $f \equiv g$.*

Note that the number 2 in Theorem 4.1.3 is sharp. For example, the following polynomials

$$f(z) = (z-1)(z-2)^2, \quad g(z) = (z-1)^2(z-2)$$

share 0 IM, but $f \not\equiv g$.

In [22], C. C. Yang suggested the following problem: Let $p(z)$ and $q(z)$ be two non-constant polynomials of the same degree satisfying

$$p(z)(p(z)-1) = 0 \Leftrightarrow q(z)(q(z)-1) = 0.$$

Prove (or disprove) that either $p(z) \equiv q(z)$ or $p(z) + q(z) \equiv 1$. In [35], C. C. Yang exhibited the following example to show that the problem may not be true.

Let $p(z)$ and $q(z)$ be polynomials defined by

$$p(z) = \frac{1}{2}z(z^2-3), \quad q(z) = \frac{1}{3}(2z^2-5).$$

Then $(p(z)+1)(p(z)-1) = 0 \Leftrightarrow (q(z)+1)(q(z)-1) = 0$. But $p(z) \not\equiv q(z)$ and $p(z) + q(z) \not\equiv -1 + 1 = 0$.

In the above example, the degree of $p(z)$ and $q(z)$ are distinct, therefore, C. C. Yang [22, 36] raised the following conjecture: Let $p(z)$ and $q(z)$ be two non-constant polynomials of the same degree. If there are distinct finite complex numbers α and β satisfying

$$(p(z)-\alpha)(p(z)-\beta) = 0 \Leftrightarrow (q(z)-\alpha)(q(z)-\beta) = 0,$$

then $p(z) \equiv q(z)$ or $p(z) + q(z) \equiv \alpha + \beta$. We will give an elementary proof of this conjecture in section 4.3

4.2 Some Lemmas

We remark that, in [13] and [23], there are some equivalent descriptions and special cases about Yang's conjecture. In [23], Moh gave a proof of the Yang's conjecture by using algebraic method. In order to prove Yang's conjecture, we need some basic properties of polynomials.

Lemma 4.2.1 *Let $p(z)$ be a polynomial of degree n and let α be a nonzero complex number. If $p(z)$ has k distinct zeros and $p(z)-\alpha$ has r distinct zeros, then $k+r \geq n+1$.*

Proof. Write $p(z) = a(z-u_1)^{l_1}\cdots(z-u_k)^{l_k}$ and $p(z)-\alpha = a(z-v_1)^{m_1}\cdots(z-v_r)^{m_r}$, where $u_1,\ldots,u_k,v_1,\ldots,v_r$ are distinct complex numbers and $a \neq 0$. Then u_i and v_j are distinct zeros of $p'(z)$ of order $l_i - 1$ and $m_j - 1$, respectively. Since $p'(z)$ is a polynomial of degree $n-1$, we have

$$\sum_{i=1}^{k}(l_i - 1) + \sum_{j=1}^{r}(m_j - 1) \leq n-1.$$

Since $l_1 + \cdots + l_k = m_1 + \cdots + m_r = n$, we get $k + r \geq n+1$. ❑

Corollary 4.2.2 Let $p(z)$ be a polynomial of degree n, α and β be distinct complex numbers. If $p(z)-\alpha$ has k distinct zeros and $p(z)-\beta$ has r distinct zeros, then $k+r \geq n+1$.

Proof. By Lemma 4.2.1 and replace $p(z)$ by $p(z)-\alpha$, we are done. ❑

Lemma 4.2.3 *Let $n \geq 2$ and $a_1,\ldots,a_n,b_1,\ldots,b_n,c_1,c_2$ be complex numbers. If*

$$\prod_{i=1}^{n}(z-a_i) = \prod_{i=1}^{n}(z-b_i) + c_1 \tag{4.2.1}$$

and

$$\prod_{i=1}^{s}(z-a_i)\prod_{i=s+1}^{n}(z-b_i) = \prod_{i=1}^{s}(z-b_i)\prod_{i=s+1}^{n}(z-a_i) + c_2 \tag{4.2.2}$$

for some $1 \leq s \leq n-1$, then $c_1 = c_2 = 0$.

Proof. By equalities (4.2.1) and (4.2.2), we have

$$\prod_{i=1}^{s}(z-a_i)\left[\prod_{i=s+1}^{n}(z-a_i)-\prod_{i=s+1}^{n}(z-b_i)\right]=\prod_{i=1}^{s}(z-b_i)\left[\prod_{i=s+1}^{n}(z-b_i)-\prod_{i=s+1}^{n}(z-a_i)\right]+c_1-c_2.$$

Hence,

$$\left[\prod_{i=1}^{s}(z-a_i)+\prod_{i=1}^{s}(z-b_i)\right]\left[\prod_{i=s+1}^{n}(z-a_i)-\prod_{i=s+1}^{n}(z-b_i)\right]=c_1-c_2,$$

which implies that

$$\prod_{i=s+1}^{n}(z-a_i)=\prod_{i=s+1}^{n}(z-b_i)$$

and $c_1=c_2=0$. ❑

By Lemma 4.2.3, we have the following consequence.

Corollary 4.2.4 Let $n \geq 2$ and $a_1,\ldots,a_n,b_1,\ldots,b_n,c_1,c_2$ be complex numbers. If $c_1 \neq c_2$ then the two equalities

$$\prod_{i=1}^{n}(z-a_i)=\prod_{i=1}^{n}(z-b_i)+c_1$$

and

$$\prod_{i=1}^{s}(z-a_i)\prod_{i=s+1}^{n}(z-b_i)=\prod_{i=1}^{s}(z-b_i)\prod_{i=s+1}^{n}(z-a_i)+c_2$$

can not hold simultaneously.

Note that, in Lemma 4.2.3 and Corollary 4.2.4, $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ may not be distinct.

4.3 Main Result and Proof

Now, we can prove the C. C. Yang's conjecture.

Theorem 4.3.1 *Let $p(z)$ and $q(z)$ be non-constant polynomials of the same degree. If there exists distinct finite complex numbers α and β satisfying*

$$(p(z)-\alpha)(p(z)-\beta)=0 \Leftrightarrow (q(z)-\alpha)(q(z)-\beta)=0,$$

then either $p(z) \equiv q(z)$ or $p(z)+q(z) \equiv \alpha+\beta$.

Proof. Assume that $\deg(p)=\deg(q)=n$. Write $p(z)-\alpha = a\prod_{i=1}^{k}(z-u_i)^{l_i}$ and $p(z)-\beta = a\prod_{i=1}^{r}(z-v_i)^{m_i}$, where $a \neq 0$, $u_1,\ldots,u_k$ are the distinct roots of $p(z)-\alpha$ with multiplicities $l_1,\ldots,l_k$ and $v_1,\ldots,v_r$ are the distinct roots of $p(z)-\beta$ with multiplicities $m_1,\ldots,m_r$, respectively. We separate the proof into three cases:

Case1. $q(z)-\alpha = b\prod_{i=1}^{k}(z-u_i)^{l'_i}$, where $l'_i \geq 1$ may differ from l_i and $\sum_{i=1}^{k} l'_i = n$.

In this case, we get $q(z)-\beta = b\prod_{i=1}^{r}(z-v_i)^{m'_i}$. Hence, $p(z)$ and $q(z)$ share α and β IM. By Theorem 4.1.3, $p(z) \equiv q(z)$.

Case2. $q(z)-\alpha = b\prod_{i=1}^{r}(z-v_i)^{m'_i}$, where $m'_i \geq 1$ may differ from m_i and $\sum_{i=1}^{r} m'_i = n$.

In this case, we get $q(z)-\beta = b\prod_{i=1}^{k}(z-u_i)^{l'_i}$. Hence, $p(z)+q(z)=\alpha+\beta$ for $z=u_1,\ldots,u_k,v_1,\ldots,v_r$. By Corollary 4.2.2, $k+r \geq n+1$. Since $p(z)+q(z)$ is a polynomial of degree less than or equal to n and $p(z)+q(z)=\alpha+\beta$ has at least $n+1$ distinct roots, it must be the case that $p(z)+q(z) \equiv \alpha+\beta$.

Case3. $q(z)-\alpha = b\prod_{i=1}^{h}(z-u_i)^{l'_i}\prod_{i=t+1}^{r}(z-v_i)^{m'_i}$ for some $1 \leq h \leq k-1$ and $1 \leq t \leq r-1$, where $\sum_{i=1}^{h} l'_i + \sum_{i=t+1}^{r} m'_i = n$.

In this case, we get $q(z)-\beta = b\prod_{i=1}^{t}(z-v_i)^{m'_i}\prod_{i=h+1}^{k}(z-u_i)^{l'_1}$. By assumption,

$$p(z) = a\prod_{i=1}^{k}(z-u_i)^{l_i}+\alpha$$

and

$$p(z) = a\prod_{i=1}^{r}(z-v_i)^{m_i}+\beta,$$

which imply that

$$\prod_{i=1}^{k}(z-u_i)^{l_i}=\prod_{i=1}^{r}(z-v_i)^{m_i}+\frac{\beta-\alpha}{a}.$$

Similarly, we have

$$\prod_{i=1}^{h}(z-u_i)^{l'_i}\prod_{i=t+1}^{r}(z-v_i)^{m'_i}=\prod_{i=1}^{t}(z-v_i)^{m'_i}\prod_{i=h+1}^{k}(z-u_i)^{l'_i}+\frac{\beta-\alpha}{b}.$$

By Lemma 4.2.3, we get $\frac{\beta-\alpha}{a}=\frac{\beta-\alpha}{b}=0$, i.e., $\alpha=\beta$ which is impossible. ❏