5 Main Results in Value Distribution of Meromorphic Functions with Their Derivatives

In this section, we will prove our main results which generalize the Theorems mentioned in the introduction.

Clunie have already proved the result of Therom1.3 for the case $n = 1$. In addition, Clunie mentioned that the same method of proof can be used to prove the result for any case $n \geq 1$. So, here we will complete the proof for the case $n \geq 2$.

Theorem 5.1 *If $f(z)$ is a transcendental entire function, $n \geq 2$ is a positive integer, then $f'(z)f(z)^n$ assumes all finite values except possibly zero infinitely often.*

Proof. Suppose the result is false. There exists $a \in \mathbb{C}\backslash\{0\}$ such that $f'(z)f^n(z) - a = 0$ has only finitely many zeros. Hence, there exists a polynomial p and a non-constant entire function g such that

$$f'f^n - a = pe^g. \tag{5.1}$$

Differentiating (5.1) twice, we obtain

$$f''f^n + n(f')^2f^{n-1} = (p' + pg')e^g. \tag{5.2}$$

and

$$f'''f^n + 3nf''f'f^{n-1} + n(n-1)(f')^3f^{n-2} = [(p' + pg')' + g'(p' + pg')]e^g. \tag{5.3}$$

From (5.2) and (5.3), it follows that

$$f'''f^n + 3nf''f'f^{n-1} + n(n-1)(f')^3f^{n-2} = h[f''f^n + n(f')^2f^{n-1}], \tag{5.4}$$

where

$$h = g' + \frac{(p' + pg')'}{(p' + pg')}. \tag{5.5}$$

From (5.4)

$$\frac{f''' - hf''}{f'} = \frac{nhf'f - 3nf''f - n(n-1)(f')^2}{f^2} \equiv k.$$

So

$$f''' - hf'' = kf', \tag{5.6}$$

$$nhf'f - 3nf''f - n(n-1)(f')^2 = kf^2. \tag{5.7}$$

Differentiating (5.7), we have

$$nh'f'f + nhf''f + nh(f')^2 - 3nf'''f - 3nf''f' - 2n(n-1)f''f' = k'f^2 + 2kf'f. \tag{5.8}$$

From (5.6) and (5.8), eliminating f''', we get

$$nh'f'f + nhf''f + nh(f')^2 - 3n(hf'' + kf')f - 3nf''f' - 2n(n-1)f''f' = k'f^2 + 2kf'f,$$

that is,

$$(nh' - 3nk - 2k)f'f + nh(f')^2 - 2nhf''f - n(2n+1)f''f' - k'f^2 = 0 \tag{5.9}$$

From (5.7) and (5.9), eliminating f'', we get

$$\begin{aligned}(nh' - 3nk - 2k)f'f + nh(f')^2 - \frac{2h}{3}[nhf'f - n(n-1)(f')^2 - kf^2] \\ -\frac{2n+1}{3}\frac{f'}{f}[nhf'f - n(n-1)(f')^2 - kf^2] - k'f^2 = 0,\end{aligned}$$

that is,

$$\begin{aligned}\frac{n(n-1)(2n+1)}{3}\left(\frac{f'}{f}\right)^3 + \left[nh + \frac{2n(n-1)}{3}h - \frac{n(2n+1)}{3}h\right]\left(\frac{f'}{f}\right)^2 \\ + \left[nh' - 3nk - 2k + \frac{2n}{3}h^2 + \frac{2n+1}{3}k\right]\left(\frac{f'}{f}\right) = \left(k' - \frac{2}{3}hk\right).\end{aligned} \tag{5.10}$$

Claim: *All functions $p(z), g(z), h(z), \frac{f'(z)}{f(z)}, \frac{f''(z)}{f'(z)}, \frac{f''(z)}{f(z)}, \frac{1}{f'(z)f(z)^n}$ are small functions of f.*

Assume the claim for a moment, we complete the proof.

$$\begin{aligned}nT(r,f) &= T(r, \frac{1}{f^n}) + O(1) \\ &\le T\left(r, \frac{f'}{f}\right) + T\left(r, \frac{1}{f'f^n}\right) + O(1) \\ &= S(r,f)\end{aligned}$$

which is impossible.Therefore, Theorem 5.1 holds.

Now, to prove the claim. Since f is transcendental entire function and p is a polynomial, so $T(r,p) = S(r,f)$. By (5.1), we have,

$$e^g = \frac{f'f^n - a}{p}$$

Since $T(r,g) = S(r,e^g)$, we get

$$T(r,g) = S\left(r, \frac{f'f^n - a}{p}\right) = S(r,f).$$

By (5.5), we have $T(r,h) = S(r,f)$. By (5.6), we get $k = \frac{f'''}{f''} - \frac{hf''}{f'}$.

Hence,

$$\begin{aligned} m(r,k) &\leq m\left(r, \frac{f'''}{f'}\right) + m(r,h) + m\left(r, \frac{f''}{f'}\right) + O(1) \\ &= S(r,f). \end{aligned}$$

From (5.2), the poles of k occur among the poles of h and the common zeros of f and f'. Consequently,

$$\begin{aligned} N(r,k) &\leq N(r,h) + N\left(r, \frac{1}{p' + pg'}\right) \\ &= S(r,f). \end{aligned}$$

Therefore, $T(r,k) = S(r,f)$. From (5.10), we obtain

$$3T\left(r, \frac{f'}{f}\right) = O(T(r,h) + T(r,k) + O(1)) \tag{5.11}$$

Hence, $T\left(r, \frac{f'}{f}\right) = S(r,f)$. By (5.7),

$$\begin{aligned} T\left(r, \frac{f''}{f}\right) &\leq O\left(T(r,h) + T\left(r, \frac{f'}{f}\right) + O(1)\right) \\ &= S(r,f). \end{aligned} \tag{5.12}$$

From (5.11) and (5.12), we have

$$\begin{aligned} T\left(r, \frac{f''}{f'}\right) &\leq T\left(r, \frac{f''}{f}\right) + T\left(r, \frac{f}{f'}\right) \\ &\leq T\left(r, \frac{f''}{f}\right) + T\left(r, \frac{f'}{f}\right) + O(1) \\ &= S(r,f). \end{aligned} \tag{5.13}$$

From (5.1) and (5.2),

$$f'f^n - a = p\frac{f''f^n + n(f')^2 f^{n-1}}{p' + pg'}.$$

Hence,

$$\begin{aligned}\frac{a}{f'f^n} &= 1 - \left(\frac{p'}{p} + g'\right)^{-1} \left(\frac{f''f^n + n(f')^2 f^{n-1}}{f'f^n}\right) \\ &= 1 - \left(\frac{p'}{p} + g'\right)^{-1} \left(\frac{f''}{f'} + n\frac{f'}{f}\right). \end{aligned} \tag{5.14}$$

Therefore, by (5.14),

$$\begin{aligned} T\left(r, \frac{1}{f'f^n}\right) &= T\left(r, \frac{a}{f'f^n}\right) + O(1) \\ &= O\left(T(r,p) + T(r,g) + T\left(r, \frac{f''}{f'}\right) + T\left(r, \frac{f'}{f}\right)\right) \\ &= S(r,f). \end{aligned}$$

❑

Theorem 5.2 *If $f(z)$ is a transcendental entire function, k and n are two positive integers with $1 \le k \le n$, then $f^{(k)}(z)f(z)^n$ assumes all finite values except possibly zero infinitely often.*

Proof. If $n = 1$, then it follows from Theorem 1.3. Now, we assume that $n \ge 2$. Given a nonzero complex number a and consider the function

$$\psi(z) = \frac{f^{(k)}(z)f(z)^n}{a}.$$

By Nevanlinna's second fundamental theorem,

$$\begin{aligned} T(r,\psi) &\le \overline{N}\left(r, \frac{1}{\psi}\right) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) + \overline{N}(r,\psi) + S(r,\psi) \qquad (5.15) \\ &= \overline{N}\left(r, \frac{1}{\psi}\right) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) + S(r,\psi). \end{aligned}$$

By the definition of ψ and Lemma 3.2, we have

$$\begin{aligned}\overline{N}\left(r,\frac{1}{\psi}\right) &\le \overline{N}\left(r,\frac{1}{f}\right)+N_0\left(r,\frac{1}{f^{(k)}}\right)\\ &\le (k+1)\overline{N}\left(r,\frac{1}{f}\right)+k\overline{N}\left(r,f\right)+S(r,f)\\ &\le (k+1)\overline{N}\left(r,\frac{1}{f}\right)+S(r,f). \qquad (5.16)\end{aligned}$$

At each zero of f, ψ has a zero of order at least n. Thus,

$$(n-1)\overline{N}\left(r,\frac{1}{f}\right)\le N\left(r,\frac{1}{\psi}\right)-\overline{N}\left(r,\frac{1}{\psi}\right). \qquad (5.17)$$

By (5.16) and (5.17), we get

$$\overline{N}\left(r,\frac{1}{\psi}\right)\le \frac{k+1}{n-1}\left[N\left(r,\frac{1}{\psi}\right)-\overline{N}\left(r,\frac{1}{\psi}\right)\right]+S(r,f).$$

Hence,

$$\overline{N}\left(r,\frac{1}{\psi}\right)\le \frac{k+1}{n+k}N\left(r,\frac{1}{\psi}\right)+S(r,f)\le \frac{k+1}{n+k}T\left(r,\psi\right)+S(r,f). \qquad (5.18)$$

Finally, from (5.15), (5.18) and Lemma 3.1, we get

$$\frac{n-1}{n+k}T\left(r,\psi\right)\le \overline{N}\left(r,\frac{1}{\psi-1}\right)+S(r,\psi) \qquad (5.19)$$

Therefore, $\psi(z)=1$, that is, $f^{(k)}(z)f(z)^n=a$, has infinitely many roots. ❑

Remark. From the proof of Theorem 5.2, it is easy to see that, in the case $n\ge 2$, the condition $1\le k\le n$ in Theorem 5.2 is redundant, that is, if $n\ge 2$ and k is an arbitrary positive integer, then $f^{(k)}(z)f(z)^n$ assumes all finite values except possibly zero infinitely often.

Theorem 5.3 *If $f(z)$ is a transcendental meromorphic function, $1\le k\le n^2-n-2$ and $n\ge 3$ are integers, then $f^{(k)}(z)f(z)^n$ assume all finite values except possibly zero infinitely often.*

Proof. Given a nonzero complex number a and consider the function

$$\psi(z)=\frac{f^{(k)}(z)f(z)^n}{a}.$$

As in the proof in Theorem 5.2, we have the following inequalities,

$$T(r,\psi) \le \overline{N}\left(r,\frac{1}{\psi}\right) + \overline{N}\left(r,\frac{1}{\psi-1}\right) + \overline{N}\left(r,\psi\right) + S(r,\psi), \tag{5.20}$$

$$\overline{N}\left(r,\frac{1}{\psi}\right) \le (k+1)\overline{N}\left(r,\frac{1}{f}\right) + k\overline{N}\left(r,f\right) + S(r,f), \tag{5.21}$$

and

$$(n-1)\overline{N}\left(r,\frac{1}{f}\right) \le N\left(r,\frac{1}{\psi}\right) - \overline{N}\left(r,\frac{1}{\psi}\right). \tag{5.22}$$

At each pole of f, ψ has a pole of order at least $n+k+1$. Thus,

$$(n+k)\overline{N}\left(r,f\right) \le N\left(r,\psi\right) - \overline{N}\left(r,\psi\right), \tag{5.23}$$

which implies that

$$(n+k)\overline{N}\left(r,\psi\right) = (n+k)\overline{N}\left(r,f\right) \le N\left(r,\psi\right) - \overline{N}\left(r,\psi\right).$$

Hence,

$$\overline{N}\left(r,\psi\right) \le \frac{1}{n+k+1}N\left(r,\psi\right) \le \frac{1}{n+k+1}T\left(r,\psi\right) \tag{5.24}$$

By (5.21) and (5.22), we obtain

$$\overline{N}\left(r,\frac{1}{\psi}\right) \le \frac{k+1}{n-1}\left[N\left(r,\frac{1}{\psi}\right) - \overline{N}\left(r,\frac{1}{\psi}\right)\right] + k\overline{N}\left(r,f\right) + S(r,f). \tag{5.25}$$

By (5.23) and (5.25), we have

$$\begin{aligned}\overline{N}\left(r,\frac{1}{\psi}\right) &\le \frac{k+1}{n+k}N\left(r,\frac{1}{\psi}\right) + \frac{k(n-1)}{n+k}\overline{N}(r,f) + S(r,f) \\ &\le \frac{k+1}{n+k}N\left(r,\frac{1}{\psi}\right) + \frac{k(n-1)}{(n+k)^2}\left[N\left(r,\psi\right) - \overline{N}\left(r,\psi\right)\right] + S(r,f) \\ &\le \left[\frac{k+1}{n+k} + \frac{k(n-1)}{(n+k)^2}\right]T\left(r,\psi\right) - \frac{k(n-1)}{(n+k)^2}\overline{N}\left(r,\psi\right) + S(r,f).\end{aligned} \tag{5.26}$$

Finally, from (5.20), (5.24), (5.26) and Lemma 3.1, we get

$$\begin{aligned}T\left(r,\psi\right) \le &\left[\frac{k+1}{n+k} + \frac{k(n-1)}{(n+k)^2}\right]T\left(r,\psi\right) + \left[1 - \frac{k(n-1)}{(n+k)^2}\right]\left(\frac{1}{n+k+1}\right)T\left(r,\psi\right) \\ &+ \overline{N}\left(r,\frac{1}{\psi-1}\right) + S(r,\psi),\end{aligned}$$

which implies that

$$\left\{\left[1-\frac{k+1}{n+k}-\frac{k(n-1)}{(n+k)^2}\right]-\frac{1}{n+k+1}\left[1-\frac{k(n-1)}{(n+k)^2}\right]\right\}T(r,\psi)\le \overline{N}\left(r,\frac{1}{\psi-1}\right)+S(r,\psi).$$

By a simple computation, we have

$$\left[1-\frac{k+1}{n+k}-\frac{k(n-1)}{(n+k)^2}\right]-\frac{1}{n+k+1}\left[1-\frac{k(n-1)}{(n+k)^2}\right]>0\Leftrightarrow (n+k)(n^2-n-1-k)>0.$$

Since $n\ge 3$ and $1\le k\le n^2-n-2$, $(n+k)(n^2-n-1-k)>0$. Therefore, $\psi(z)=1$, that is, $f^{(k)}(z)f(z)^n=a$, has infinitely many roots. ❑

Theorem 5.4 *Let $f(z)$ be a transcendental entire function and set*

$$\varphi(z)=f^{(k)}(z)-af(z)^n,$$

where $k\ge 1$, $n\ge 3$ are integers and a is a nonzero complex number. Then $\varphi(z)$ assumes all finite values infinitely often.

Proof. Let b be an arbitrary complex number and consider the function

$$\psi(z)=\frac{f^{(k)}(z)-b}{af(z)^n}.$$

Note that $S(r,\psi)=o(T(r,f))$ as $r\to\infty$ possibly outside a set of finite linear measure. By Nevanlinna's second fundamental theorem,

$$T(r,\psi)\le \overline{N}(r,\psi)+\overline{N}\left(r,\frac{1}{\psi}\right)+\overline{N}\left(r,\frac{1}{\psi-1}\right)+S(r,\psi). \tag{5.27}$$

Since the poles and zeros of $\psi(z)$ occur only at the zeros of $f(z)$ or $f^{(k)}(z)-b$. If $f^{(k)}(z_0)-b=0$, then z_0 is counted in $\overline{n}(r,\psi)+\overline{n}(r,\frac{1}{\psi})$ at most once. If $f^{(k)}(z_0)-b\ne 0$ and $f(z_0)=0$, then z_0 is a pole of ψ of order at least n. Thus, we have

$$\begin{aligned}\overline{N}(r,\psi)+\overline{N}\left(r,\frac{1}{\psi}\right) &\le \frac{1}{n}N(r,\psi)+\overline{N}\left(r,\frac{1}{f^{(k)}-b}\right) \qquad (5.28)\\ &\le \frac{1}{n}T(r,\psi)+T\left(r,f^{(k)}\right)+O(1)\\ &= \frac{1}{n}T(r,\psi)+m\left(r,f^{(k)}\right)+O(1)\\ &\le \frac{1}{n}T(r,\psi)+m(r,f)+m\left(r,\frac{f^{(k)}}{f}\right)\\ &\le \frac{1}{n}T(r,\psi)+T(r,f)+S(r,f).\end{aligned}$$

By (5.27) and (5.28), we get

$$\left(1-\frac{1}{n}\right)T(r,\psi) \le T(r,f)+\overline{N}\left(r,\frac{1}{\psi-1}\right)+S(r,f),$$

which implies that

$$\begin{aligned}
nT(r,f) &= T(r,f^n)\\
&= T\left(r,\frac{f^{(k)}-b}{a\psi}\right)\\
&\le T\left(r,f^{(k)}\right)+T(r,\psi)+O(1)\\
&\le T(r,f)+m\left(r,\frac{f^{(k)}}{f}\right)+T(r,\psi)+O(1)\\
&= T(r,f)+T(r,\psi)+S(r,f)\\
&\le T(r,f)+\frac{1}{1-1/n}T(r,f)+\frac{1}{1-1/n}\overline{N}\left(r,\frac{1}{\psi-1}\right)+S(r,f).
\end{aligned}$$

Therefore, we obtain

$$\left[(n-1)^2-n\right]T(r,f) \le n\overline{N}\left(r,\frac{1}{\psi-1}\right)+S(r,f).$$

Since $n \ge 3$ and $(n-1)^2-n>0$, $\psi(z)=1$, that is, $\varphi(z)=b$, has infinitely many roots. ❑

Theorem 5.5 *Let $f(z)$ be a transcendental meromorphic function and set*

$$\varphi(z)=f^{(k)}(z)-af(z)^n,$$

where n,k are positive integers with $n \ge k+4$ and a is a nonzero complex number. Then $\varphi(z)$ assumes all finite values infinitely often.

Proof. Let b be a complex number and consider the function

$$\psi(z)=\frac{f^{(k)}-b}{af(z)^n}.$$

We now divide the zeros of $f^{(k)}(z)-b$ into four classes. Denote by $n_0(r)$ the number of zeros of $f^{(k)}(z)-b$ in $|z|\le r$ which are not zeros of $f(z)$, $n_1(r)$ the number of zeros of $f^{(k)}(z) - b$ in $|z| \leq r$ which are zeros of $f(z)$ and poles of $\psi(z)$, $n_2(r)$ the number of zeros of $f^{(k)}(z) - b$ in $|z| \leq r$ which are zeros of $f(z)$ and are neither poles nor zeros of $\psi(z)$, and $n_3(r)$ the number of zeros of $f^{(k)}(z) - b$ in $|z| \leq r$ which are zeros of both $f(z)$ and $\psi(z)$. Let $\overline{n}_j(r), 0 \leq j \leq 3$, be their corresponding reduced counting functions and wirte

$$N_j(r) = \int_0^r \frac{n_j(t)}{t} dt, \quad \overline{N}_j(r) = \int_0^r \frac{\overline{n}_j(t)}{t} dt$$

for $0 \leq j \leq 3$, where we may assume that $f^{(k)}(0) \neq b$.

Claim 1 $n\overline{N}(r, \psi) \leq N(r, \psi) + N_1(r)$.

Suppose z_0 is a pole of $\psi(z)$. Since the poles of $\psi(z)$ occur only at the zeros or poles of $f(z)$, if $f(z)$ has a pole at z_0 of order p, then $\psi(z)$ has a zero at z_0 of order $np - (p + k) > 0$. So $\psi(z)$ has no pole at z_0. Therefore, the poles of $\psi(z)$ arise only from the zeros of $f(z)$. If $f(z)$ has a zero at z_0 and $f^{(k)}(z_0) - b \neq 0$, then z_0 is a pole of $\psi(z)$ of order at least n. If $f(z)$ has a zero at z_0 of order p, $f^{(k)}(z) - b$ has a zero at z_0 of order q and z_0 is a pole of $\psi(z)$, then $np > q$, z_0 is counted $np - q$ times in $n(r, \psi)$, and q times in $n_1(r)$. Hence, Claim 1 is proved.

Claim 2 $(n - k - 1)\overline{N}\left(r, \frac{1}{\psi}\right) \leq N\left(r, \frac{1}{\psi}\right) + (n - k - 2)N_0(r) + \frac{n - k - 2}{n} N_3(r)$.

Suppose z_0 is a zero of $\psi(z)$. Since the zeros of $\psi(z)$ occur only at the poles of $f(z)$ or the zeros of $f^{(k)}(z) - b$. As we mention above, if z_0 is a pole of $f(z)$, then z_0 is a zero of $\psi(z)$ of order at least $n - k - 1$. If z_0 is a zero of $f^{(k)}(z) - b$, but not a zero of $f(z)$, then z_0 is counted at least once in both $n(r, \frac{1}{\psi})$ and $n_0(r)$. If $f(z)$ has a zero at z_0 of order p, $f^{(k)}(z) - b$ has a zero at z_0 of order q, then $q > np \geq n$, z_0 is counted $q - np \geq 1$ times in $n(r, \frac{1}{\psi})$ and q times in $n_3(r)$. Thus, Claim 2 is proved.

By Nevanlinna's second fundamental theorem, Claim 1 and Claim 2, we have

$$
\begin{aligned}
T(r,\psi) \leq & \overline{N}(r,\psi) + \overline{N}\left(r,\frac{1}{\psi}\right) + \overline{N}\left(r,\frac{1}{\psi-1}\right) + S(r,\psi) \\
\leq & \frac{1}{n}N(r,\psi) + \frac{1}{n}N_1(r) + \frac{1}{n-k-1}N\left(r,\frac{1}{\psi}\right) + \frac{n-k-2}{n(n-k-1)}N_3(r) \\
& + \frac{n-k-2}{n-k-1}N_0(r) + \overline{N}\left(r,\frac{1}{\psi-1}\right) + S(r,f) \\
\leq & \left(\frac{1}{n} + \frac{1}{n-k-1}\right)T(r,\psi) + \frac{1}{n}\left[N_1(r) + N_3(r)\right] + \frac{n-k-2}{n-k-1}N_0(r) \\
& + \overline{N}\left(r,\frac{1}{\psi-1}\right) + S(r,f),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\left(1 - \frac{1}{n} - \frac{1}{n-k-1}\right)T(r,\psi) \leq & \frac{1}{n}\left[N_1(r) + N_3(r)\right] + \frac{n-k-2}{n-k-1}N_0(r) \\
& + \overline{N}\left(r,\frac{1}{\psi-1}\right) + S(r,f).
\end{aligned}
\tag{5.29}
$$

Note that if z_0 is a pole of $f(z)$ of order p, then z_0 is a zero of $\psi(z)$ of order $np-(p+k) \geq (n-k-1)p$. Also, if $f^{(k)}(z) - b$ has a zero at z_0 of order p and $f(z_0) \neq 0$, then $\psi(z)$ has a zero at z_0 of order p and we have

$$
(n-k-1)N(r,f) + N_0(r) \leq N(r,\frac{1}{\psi}). \tag{5.30}
$$

Again, we have

$$
\begin{aligned}
nm(r,f) &= m(r,f^n) \\
&= m\left(r,\frac{af^n}{f^{(k)}-b}\right) + m(r,f^{(k)}-b) + O(1) \\
&\leq m\left(r,\frac{1}{\psi}\right) + m(r,f^{(k)}) + O(1) \\
&\leq m\left(r,\frac{1}{\psi}\right) + m(r,f) + S(r,f),
\end{aligned}
$$

which implies that

$$
(n-1)m(r,f) \leq m\left(r,\frac{1}{\psi}\right) + S(r,f). \tag{5.31}
$$

By (5.30) and (5.31), we get

$$
(n-k-1)T(r,f) \leq T(r,\psi) - N_0(r) + S(r,f). \tag{5.32}
$$

From (5.29) and (5.32), we have

$$\begin{aligned}
&(n-k-1)\left(1-\frac{1}{n}-\frac{1}{n-k-1}\right)T(r,f)\\
\leq&\left(1-\frac{1}{n}-\frac{1}{n-k-1}\right)T(r,\psi)-\left(1-\frac{1}{n}-\frac{1}{n-k-1}\right)N_0(r)+S(r,f)\\
\leq&\frac{n-k-2}{n-k-1}N_0(r)+\frac{1}{n}\left[N_1(r)+N_3(r)\right]+\overline{N}\left(r,\frac{1}{\psi-1}\right)\\
&-\left(1-\frac{1}{n}-\frac{1}{n-k-1}\right)N_0(r)+S(r,f)\\
=&\frac{1}{n}\left[N_0(r)+N_1(r)+N_3(r)\right]+\overline{N}\left(r,\frac{1}{\psi-1}\right)+S(r,f)\\
\leq&\frac{1}{n}N(r,\frac{1}{f^{(k)}-b})+\overline{N}\left(r,\frac{1}{\psi-1}\right)+S(r,f)\\
\leq&\frac{1}{n}T(r,f^{(k)})+\overline{N}\left(r,\frac{1}{\psi-1}\right)+S(r,f)\\
\leq&\frac{k+1}{n}T(r,f)+\overline{N}\left(r,\frac{1}{\psi-1}\right)+S(r,f).
\end{aligned}$$

With a simple calculation, we deduce that

$$(n-k-3)T(r,f)\leq\overline{N}(r,\frac{1}{\psi-1})+S(r,f).$$

Since $n\geq k+4$, $\psi(z)=1$, that is, $\varphi(z)=b$, has infinitely many roots. ❑

Theorem 5.6 *Let $f(z)$ be a transcendental meromorphic function and $a_0(z),\ldots,a_n(z)$ be small functions of f. Set*

$$\psi(z)=\sum_{i=0}^{n}a_i(z)f^{(i)}(z).$$

If $\delta(0,f)+\Theta(\infty,f)>1$, then $\psi(z)$ assumes all finite values except possibly zero infinitely often or else $\psi(z)$ is identically constant.

Proof. We may assume that $\psi(z)$ is non-constant, for otherwise, the theorem is obvious. By hypothesis,

$$\begin{aligned}
1&<\delta(0,f)+\Theta(\infty,f)\\
&=1-\limsup_{r\to\infty}\frac{N(r,\frac{1}{f})}{T(r,f)}+1-\limsup_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)},
\end{aligned}$$

which implies that

$$\limsup_{r\to\infty}\frac{N(r,\frac{1}{f})+\overline{N}(r,f)}{T(r,f)}\le\limsup_{r\to\infty}\frac{N(r,\frac{1}{f})}{T(r,f)}+\limsup_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)}<1.$$

Given $\varepsilon>0$, choose $R>0$ such that, for all $r\ge R$,

$$N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)<(1-\varepsilon)T(r,f). \tag{5.33}$$

By (5.33) and Theorem 3.3, we get

$$\varepsilon T(r,f)<\overline{N}\left(r,\frac{1}{\psi-1}\right)+S(r,f).$$

Hence, $\psi(z)=1$ has infinitely many roots. For arbitrary nonzero complex number w, the above proof shows that $\psi(z)/w=1$ has infinitely many roots, that is, $\psi(z)=w$ has infinitely many roots. ❑

Clearly, if $\delta(0,f)+\delta(\infty,f)>1$, then Theorem 5.6 also holds. Moreover, the following example shows that the condition $\delta(0,f)+\Theta(\infty,f)>1$ in Theorem 5.6 is sharp.

Example 5.7 *Let $f(z)=e^z+p(z)$, where $p(z)=a_nz^n+\cdots+a_0$, $a_n\neq 0$, is a polynomial of degree n and $\psi(z)=f^{(n)}(z)$. Since f is entire, $\Theta(\infty,f)=1$. By Theorem 2.12, the Nevanlinna's second fundamental theorem for three small functions, we have $\delta(0,f)=0$. Therefore,*

$$\delta(0,f)+\Theta(\infty,f)=1,$$

but $\psi(z)=e^z+n!a_n$ does not take the value $n!a_n$, which is a non-zero finite value.