1 Introduction

Picard's theorem says that any non-constant meromorphic function has at most two Picard exceptional values, and the remarkable result stimulated the study of value distribution of meromorphic functions for decades. In particular, the problem on value distribution of meromorphic functions with their derivatives is an important research field in complex analysis. One important problem in such area is to find the particular form of a meromorphic function $f(z)$ under certain assumptions on the zeros of $f(z)$ and its derivatives. In this aspect, the following results are well-known.

Theorem 1.1 *Let $f(z)$ be a non-constant entire function. Suppose $f(z)$, $f^{(k)}(z)$ and $f^{(l)}(z)$ have no zeros, for some distinct positive integers k and l. Then $f(z)$ is of the form e^{az+b}, where both a and b are complex numbers.*

Saxer [14] proved the theorem for $k = 1, l = 2$ and Csillag [5] proved the general case. Furthermore, Hayman [8] conjectured that the restriction on $f^{(l)}$ may be dropped if $k \geq 2$, and raised the following conjecture and proved the case $k = 2$.

Theorem 1.2 *Let $k \geq 2$ be an integer and let $f(z)$ be a non-constant meromorphic function such that $f(z)$ and $f^{(k)}(z)$ have only finite number of zeros and poles. Then $f(z)$ is of the form*

$$f(z) = R(z)e^{P(z)},$$

where $P(z)$ is a polynomial and $R(z)$ is a rational function. Moreover, if $f(z)$ and $f^{(k)}(z)$ have no zeros, then $f(z)$ is of the form e^{az+b} or $(az+b)^{-m}$, for some complex numbers a, b and positive integer m.

Note that Theorem 1.2 fails for $k = 1$. For example, consider the function $f(z) = \exp(\exp(z))$. It is easy to see that both $f(z)$ and $f'(z)$ have no zeros, but $f(z)$ is not of the form in Theorem 1.2.

For entire function, the case $k \geq 2$ was proved by Clunie [3]. In fact, the condition in Theorem 1.2 that $f(z)$ and $f^{(k)}(z)$ have only a finite number of poles is redundant, which was proved by Frank [6, 7], Hennekemper and Polloczek [7], in the case $k \geq 3$, and by Langley [11] for $k = 2$.

Moreover, Hayman [10] considered the value distribution of $f'(z)f(z)^n$ and raised the following conjecture which is now a well-known result.

Theorem 1.3 *If $f(z)$ is a transcendental meromorphic function, then $f'(z)f(z)^n$ assumes all finite values except possibly zero infinitely often.*

For transcendental entire function, the case $n \geq 2$ was proved by Hayman [8] and the case $n = 1$ was proved by Clunie [4]. For transcendental meromorphic function, the case $n \geq 3$ was proved by Hayman [8], the case $n = 2$ was proved by Mues [12] and the case $n = 1$ was proved by Bergweiler and Eremenko [1], Chen and Fang [2], Zalcman [17] separately.

Hayman [8] also considered the value distribution of functions in the from $f'(z) - af(z)^n$, where $f(z)$ is a meromorphic function and a is a nonzero complex number, and proved the following results.

Theorem 1.4 *Let $f(z)$ be a transcendental entire function and set*

$$\varphi(z) = f'(z) - af(z)^n,$$

where n is a positive integer and a is a nonzero complex number. Then if $n = 2$, $\varphi(z)$ has infinitely many zeros and if $n \geq 3$, $\varphi(z)$ assumes all finite values infinitely often.

Theorem 1.5 *Let $f(z)$ be a non-constant meromorphic function and set*

$$\varphi(z) = f'(z) - af(z)^n,$$

where $n \geq 5$ *is an integer and* a *is a nonzero complex number. Then* $\varphi(z)$ *assumes all finite values infinitely often unless* $f(z)$ *is rational.*

For value distribution of some differential polynomials in a meromorphic function f with coefficients being small functions of f, Hayman [9] proved the following theorem.

Theorem 1.6 *Let* $f(z)$ *be a transcendental meromorphic function and* $a_0(z), \ldots, a_n(z)$ *be small functions of* $f(z)$*. Set*

$$\psi(z) = \sum_{i=0}^{n} a_i(z) f^{(i)}(z).$$

If $f(z)$ *has only a finite number of zeros and poles, then* $\psi(z)$ *assumes all finite values except possibly zero infinitely often or else* $\psi(z)$ *is identically constant.*

In this thesis, we will generalize Theorem 1.3 to the case $f^{(k)}(z)f(z)^n$, Theorem 1.4 and Theorem 1.5 to the case $f^{(k)}(z) - af(z)^n$ and Theorem 1.6 to the case that $f(z)$ may have infinitely many zeros and poles. Also, value distribution of meromorphic functions in class $\mathcal{A}$ with their derivatives are obtained.