3 Some Lemmas

In order to prove our main theorems in section 4 and 5, we need some preliminaries in the theory of value distribution.

Lemma 3.1 *Let $f(z)$ be a meromorphic function and set*

$$\psi(z) = \frac{f^{(k)}(z)f(z)^n}{a},$$

where n, k are positive integers and a is a finite non-zero complex number, then $S(r,\psi) = S(r,f)$.

Proof. By the definition of ψ, we have

$$\begin{aligned} T(r,\psi) &\leq T(r,f^{(k)}) + T(r,f^n) + O(1) \\ &\leq (k+1)T(r,f) + nT(r,f) + O(1) \\ &= (n+k+1)T(r,f) + O(1). \end{aligned}$$

On the other hand,

$$\begin{aligned} (n+1)T(r,f) &= T(r,f^{n+1}) \\ &= T\left(r,\frac{1}{f^{n+1}}\right) + O(1) \\ &= N\left(r,\frac{1}{f^{n+1}}\right) + m\left(r,\frac{1}{f^{n+1}}\right) + O(1) \\ &\leq \frac{n+1}{n}N\left(r,\frac{1}{f^n}\right) + m\left(r,\frac{1}{\psi}\cdot\frac{f^{(k)}}{f}\right) + O(1) \\ &\leq \frac{n+1}{n}N\left(r,\frac{1}{\psi}\right) + m\left(r,\frac{1}{\psi}\right) + S(r,f) \\ &\leq \frac{n+1}{n}T\left(r,\psi\right) + S(r,f). \end{aligned}$$

Hence, $S(r,\psi) = S(r,f)$. ❑

Lemma 3.2 *Let $f(z)$ be a non-constant meromorphic function and $N_0\left(r,\frac{1}{f^{(k)}}\right)$ be the counting function of the zeros of $f^{(k)}(z)$ but not the zeros of $f(z)$. Then*

$$N_0\left(r,\frac{1}{f^{(k)}}\right) \leq k\overline{N}\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

Proof. By the definition of $N_0\left(r, \frac{1}{f^{(k)}}\right)$, we have

$$N_0\left(r, \frac{1}{f^{(k)}}\right) \le N\left(r, \frac{f}{f^{(k)}}\right) \le T\left(r, \frac{f}{f^{(k)}}\right) = T\left(r, \frac{f^{(k)}}{f}\right) + O(1).$$

Also,

$$\begin{aligned} T\left(r, \frac{f^{(k)}}{f}\right) &= N\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{f^{(k)}}{f}\right) \\ &\le k\overline{N}\left(r, \frac{1}{f}\right) + k\overline{N}\left(r, f\right) + S(r, f). \end{aligned}$$

Hence,

$$N_0\left(r, \frac{1}{f^{(k)}}\right) \le k\overline{N}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

❑

Theorem 3.3 ([9]) *Let $f(z)$ be a non-constant meromorphic function and $a_0(z), \ldots, a_n(z)$ be small functions of f. Set*

$$\psi(z) = \sum_{i=0}^{n} a_i(z) f^{(i)}(z).$$

If $\psi(z)$ is a non-constant meromorphic function, then

$$T(r, f) < \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) - N_0\left(r, \frac{1}{\psi'}\right) + S(r, f),$$

where in $N_0(r, \frac{1}{\psi'})$ only zeros of ψ' not corresponding to the repeated roots of $\psi(z) = 1$ are to be considered.