Appendix B

Probability-Generating Functions

Let W be a nonnegative integer-valued random variable with probability distribution $Pr\{W = w\}$. Consider now the power series $h(z)$,

$$h(z) = \sum_{w=0}^{\infty} Pr\{W = w\} z^w.$$

Another important use of probability-generating functions is in the analysis of problems concerning sums of independent random variables. Suppose $W = W_1 + W_2$, where W_1 and W_2 are independent, nonnegative, integer-valued random variables. Then $Pr\{W = w\}, w = 0, 1, 2, \ldots$, is given by the convolution

$$Pr\{W = w\} = \sum_{j=0}^{w} Pr\{W_1 = j\} Pr\{W_2 = w - j\}. \tag{B.1}$$

Let W_1 and W_2 have generating functions $h_1(z)$ and $h_2(z)$, respectively:

$$h_1(z) = \sum_{j=0}^{\infty} Pr\{W_1 = j\} z^j$$

and

$$h_2(z) = \sum_{j=0}^{\infty} Pr\{W_2 = j\} z^j.$$

Then term-by-term multiplication shows that the product $h_1(z)h_2(z)$ is given by

$$h_1(z)h_2(z) = \sum_{w=0}^{\infty} \left[\sum_{j=0}^{w} Pr\{W_1 = j\} Pr\{W_2 = w - j\} \right] z^w. \tag{B.2}$$

If W has generating functions $h(z) = \sum_{w=0}^{\infty} Pr\{W = w\} z^w$, then (B.1) and (B.2) show that

$$h(z) = h_1(z)h_2(z).$$

Thus we have the important result: the generating function of a sum of two mutually independent random variables is equal to the product of their respective generating functions.